Contents

Chapter 1

Symmetric Rational Inequalities

1.1 Applications

1.1. If a, b are nonnegative real numbers, then

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} \geq \frac{1}{1+ab}.$$

1.2. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

(a)
$$\frac{1}{2a+1} + \frac{1}{2b+1} + \frac{1}{2c+1} \geq 1;$$

(b)
$$\frac{1}{a+2} + \frac{1}{b+2} + \frac{1}{c+2} \leq 1.$$

1.3. If a, b, c are nonnegative real numbers such that $ab + bc + ca = 3$, then

$$\frac{1}{1+b^2+c^2} + \frac{1}{1+c^2+a^2} + \frac{1}{1+a^2+b^2} \leq 1.$$

1.4. If $0 \leq a, b, c \leq 1$, then

$$2\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) \geq 3\left(\frac{1}{2a+1} + \frac{1}{2b+1} + \frac{1}{2c+1}\right).$$

1.5. If a, b, c are nonnegative real numbers such that $a + b + c \leq 3$, then

$$2\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) \geq 5\left(\frac{1}{2a+3} + \frac{1}{2b+3} + \frac{1}{2c+3}\right).$$

1.6. If a, b, c are nonnegative real numbers, then

$$\frac{a^2-bc}{3a+b+c}+\frac{b^2-ca}{3b+c+a}+\frac{c^2-ab}{3c+a+b}\geq 0.$$

1.7. If a, b, c are positive real numbers, then

$$\frac{4a^2-b^2-c^2}{a(b+c)}+\frac{4b^2-c^2-a^2}{b(c+a)}+\frac{4c^2-a^2-b^2}{c(a+b)}\leq 3.$$

1.8. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

(a) $$\frac{1}{a^2+bc}+\frac{1}{b^2+ca}+\frac{1}{c^2+ab}\geq\frac{3}{ab+bc+ca};$$

(b) $$\frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}+\frac{1}{2c^2+ab}\geq\frac{2}{ab+bc+ca}.$$

(c) $$\frac{1}{a^2+2bc}+\frac{1}{b^2+2ca}+\frac{1}{c^2+2ab}>\frac{2}{ab+bc+ca}.$$

1.9. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2}\geq\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}.$$

1.10. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\geq\frac{a}{a^2+bc}+\frac{b}{b^2+ca}+\frac{c}{c^2+ab}.$$

1.11. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\geq\frac{2a}{3a^2+bc}+\frac{2b}{3b^2+ca}+\frac{2c}{3c^2+ab}.$$

1.12. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

(a) $$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{13}{6}-\frac{2(ab+bc+ca)}{3(a^2+b^2+c^2)};$$

(b) $$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}-\frac{3}{2}\geq(\sqrt{3}-1)\left(1-\frac{ab+bc+ca}{a^2+b^2+c^2}\right).$$

1.13. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{a^2+2bc}+\frac{1}{b^2+2ca}+\frac{1}{c^2+2ab} \le \left(\frac{a+b+c}{ab+bc+ca}\right)^2.$$

1.14. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a^2(b+c)}{b^2+c^2}+\frac{b^2(c+a)}{c^2+a^2}+\frac{c^2(a+b)}{a^2+b^2} \ge a+b+c.$$

1.15. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a} \le \frac{3(a^2+b^2+c^2)}{a+b+c)}.$$

1.16. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{a^2+ab+b^2}+\frac{1}{b^2+bc+c^2}+\frac{1}{c^2+ca+a^2} \ge \frac{9}{(a+b+c)^2}.$$

1.17. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a^2}{(2a+b)(2a+c)}+\frac{b^2}{(2b+c)(2b+a)}+\frac{c^2}{(2c+a)(2c+b)} \le \frac{1}{3}.$$

1.18. Let a, b, c be positive real numbers. Prove that

(a) $$\sum \frac{a}{(2a+b)(2a+c)} \le \frac{1}{a+b+c};$$

(b) $$\sum \frac{a^3}{(2a^2+b^2)(2a^2+c^2)} \le \frac{1}{a+b+c}.$$

1.19. If a, b, c are positive real numbers, then

$$\sum \frac{1}{(a+2b)(a+2c)} \ge \frac{1}{(a+b+c)^2}+\frac{2}{3(ab+bc+ca)}.$$

1.20. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

(a)
$$\frac{1}{(a-b)^2} + \frac{1}{(b-c)^2} + \frac{1}{(c-a)^2} \ge \frac{4}{ab+bc+ca};$$

(b)
$$\frac{1}{a^2-ab+b^2} + \frac{1}{b^2-bc+c^2} + \frac{1}{c^2-ca+a^2} \ge \frac{3}{ab+bc+ca};$$

(c)
$$\frac{1}{a^2+b^2} + \frac{1}{b^2+c^2} + \frac{1}{c^2+a^2} \ge \frac{5}{2(ab+bc+ca)}.$$

1.21. If a, b, c are positive real numbers, then

$$\frac{(a^2+b^2)(a^2+c^2)}{(a+b)(a+c)} + \frac{(b^2+c^2)(b^2+a^2)}{(b+c)(b+a)} + \frac{(c^2+a^2)(c^2+b^2)}{(c+a)(c+b)} \ge a^2+b^2+c^2.$$

1.22. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\frac{1}{a^2+b+c} + \frac{1}{b^2+c+a} + \frac{1}{c^2+a+b} \le 1.$$

1.23. Let a, b, c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$\frac{a^2-bc}{a^2+3} + \frac{b^2-ca}{b^2+3} + \frac{c^2-ab}{c^2+3} \ge 0.$$

1.24. Let a, b, c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$\frac{1-bc}{5+2a} + \frac{1-ca}{5+2b} + \frac{1-ab}{5+2c} \ge 0.$$

1.25. Let a, b, c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$\frac{1}{a^2+b^2+2} + \frac{1}{b^2+c^2+2} + \frac{1}{c^2+a^2+2} \le \frac{3}{4}.$$

1.26. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\frac{1}{4a^2+b^2+c^2} + \frac{1}{4b^2+c^2+a^2} + \frac{1}{4c^2+a^2+b^2} \le \frac{1}{2}.$$

1.27. Let a, b, c be nonnegative real numbers such that $a + b + c = 2$. Prove that

$$\frac{bc}{a^2+1} + \frac{ca}{b^2+1} + \frac{ab}{c^2+1} \le 1.$$

1.28. Let a, b, c be nonnegative real numbers such that $a + b + c = 1$. Prove that

$$\frac{bc}{a+1} + \frac{ca}{b+1} + \frac{ab}{c+1} \le \frac{1}{4}.$$

1.29. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$\frac{1}{a(2a^2+1)} + \frac{1}{b(2b^2+1)} + \frac{1}{c(2c^2+1)} \le \frac{3}{11abc}.$$

1.30. Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that

$$\frac{1}{a^3+b+c} + \frac{1}{b^3+c+a} + \frac{1}{c^3+a+b} \le 1.$$

1.31. Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that

$$\frac{a^2}{1+b^3+c^3} + \frac{b^2}{1+c^3+a^3} + \frac{c^2}{1+a^3+b^3} \ge 1.$$

1.32. Let a, b, c be nonnegative real numbers such that $a + b + c = 3$. Prove that

$$\frac{1}{6-ab} + \frac{1}{6-bc} + \frac{1}{6-ca} \le \frac{3}{5}.$$

1.33. Let a, b, c be nonnegative real numbers such that $a + b + c = 3$. Prove that

$$\frac{1}{2a^2+7} + \frac{1}{2b^2+7} + \frac{1}{2c^2+7} \le \frac{1}{3}.$$

1.34. Let a, b, c be nonnegative real numbers such that $a + b + c = 3$. Prove that

$$\frac{1}{2a^2+3} + \frac{1}{2b^2+3} + \frac{1}{2c^2+3} \ge \frac{3}{5}.$$

1.35. Let a, b, c be nonnegative real numbers such that $ab+bc+ca=3$. Prove that

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \geq \frac{a+b+c}{6}+\frac{3}{a+b+c}.$$

1.36. If a, b, c are nonnegative real numbers such that $ab+bc+ca=3$, then

(a)
$$\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1} \geq \frac{3}{2};$$

(b)
$$\frac{1}{a^2+2}+\frac{1}{b^2+2}+\frac{1}{c^2+2} \geq \frac{9}{10};$$

(c)
$$\frac{a(b+c)}{a^2+1}+\frac{b(c+a)}{b^2+1}+\frac{c(a+b)}{c^2+1} \leq 3.$$

1.37. Let a, b, c be positive real numbers such that $ab+bc+ca=3$. Prove that

$$\frac{a^2}{a^2+b+c}+\frac{b^2}{b^2+c+a}+\frac{c^2}{c^2+a+b} \geq 1.$$

1.38. Let a, b, c be positive real numbers such that $ab+bc+ca=3$. Prove that

$$\frac{bc+4}{a^2+4}+\frac{ca+4}{b^2+4}+\frac{ab+4}{c^2+4} \leq 3 \leq \frac{bc+2}{a^2+2}+\frac{ca+2}{b^2+2}+\frac{ab+2}{c^2+2}.$$

1.39. Let a, b, c be nonnegative real numbers such that $ab+bc+ca=3$. If

$$k \geq 2+\sqrt{3},$$

then

$$\frac{1}{a+k}+\frac{1}{b+k}+\frac{1}{c+k} \leq \frac{3}{1+k}.$$

1.40. Let a, b, c be nonnegative real numbers such that $a^2+b^2+c^2=3$. Prove that

$$\frac{a(b+c)}{1+bc}+\frac{b(c+a)}{1+ca}+\frac{c(a+b)}{1+ab} \leq 3.$$

1.41. Let a, b, c be positive real numbers such that $a^2+b^2+c^2=3$. Prove that

$$\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a} \geq 3.$$

1.42. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that

$$\frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a} + 2 \le \frac{7}{6}(a+b+c).$$

1.43. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that

(a) $$\frac{1}{3-ab} + \frac{1}{3-bc} + \frac{1}{3-ca} \le \frac{3}{2};$$

(b) $$\frac{1}{5-2ab} + \frac{1}{5-2bc} + \frac{1}{5-2ca} \le 1;$$

(c) $$\frac{1}{\sqrt{6}-ab} + \frac{1}{\sqrt{6}-bc} + \frac{1}{\sqrt{6}-ca} \le \frac{3}{\sqrt{6}-1}.$$

1.44. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that

$$\frac{1}{1+a^5} + \frac{1}{1+b^5} + \frac{1}{1+c^5} \ge \frac{3}{2}.$$

1.45. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^2+a+1} + \frac{1}{b^2+b+1} + \frac{1}{c^2+c+1} \ge 1.$$

1.46. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^2-a+1} + \frac{1}{b^2-b+1} + \frac{1}{c^2-c+1} \le 3.$$

1.47. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{3+a}{(1+a)^2} + \frac{3+b}{(1+b)^2} + \frac{3+c}{(1+c)^2} \ge 3.$$

1.48. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{7-6a}{2+a^2} + \frac{7-6b}{2+b^2} + \frac{7-6c}{2+c^2} \ge 1.$$

1.49. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{a^6}{1+2a^5} + \frac{b^6}{1+2b^5} + \frac{c^6}{1+2c^5} \ge 1.$$

1.50. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{a}{a^2+5} + \frac{b}{b^2+5} + \frac{c}{c^2+5} \le \frac{1}{2}.$$

1.51. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{2}{(1+a)(1+b)(1+c)} \ge 1.$$

1.52. Let a, b, c be nonnegative real numbers such that

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} = \frac{3}{2}.$$

Prove that

$$\frac{3}{a+b+c} \ge \frac{2}{ab+bc+ca} + \frac{1}{a^2+b^2+c^2}.$$

1.53. Let a, b, c be nonnegative real numbers such that

$$7(a^2+b^2+c^2) = 11(ab+bc+ca).$$

Prove that

$$\frac{51}{28} \le \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \le 2.$$

1.54. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{1}{a^2+b^2} + \frac{1}{b^2+c^2} + \frac{1}{c^2+a^2} \ge \frac{10}{(a+b+c)^2}.$$

1.55. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{1}{a^2-ab+b^2} + \frac{1}{b^2-bc+c^2} + \frac{1}{c^2-ca+a^2} \ge \frac{3}{\max\{ab, bc, ca\}}.$$

1.56. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a(2a+b+c)}{b^2+c^2} + \frac{b(2b+c+a)}{c^2+a^2} + \frac{c(2c+a+b)}{a^2+b^2} \geq 6.$$

1.57. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a^2(b+c)^2}{b^2+c^2} + \frac{b^2(c+a)^2}{c^2+a^2} + \frac{c^2(a+b)^2}{a^2+b^2} \geq 2(ab+bc+ca).$$

1.58. If a, b, c are positive real numbers, then

$$3\sum \frac{a}{b^2-bc+c^2} + 5\left(\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}\right) \geq 8\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

1.59. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

(a) $$2abc\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) + a^2 + b^2 + c^2 \geq 2(ab+bc+ca);$$

(b) $$\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+a} \leq \frac{3(a^2+b^2+c^2)}{2(a+b+c)}.$$

1.60. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

(a) $$\frac{a^2-bc}{b^2+c^2} + \frac{b^2-ca}{c^2+a^2} + \frac{c^2-ab}{a^2+b^2} + \frac{3(ab+bc+ca)}{a^2+b^2+c^2} \geq 3;$$

(b) $$\frac{a^2}{b^2+c^2} + \frac{b^2}{c^2+a^2} + \frac{c^2}{a^2+b^2} + \frac{ab+bc+ca}{a^2+b^2+c^2} \geq \frac{5}{2};$$

(c) $$\frac{a^2+bc}{b^2+c^2} + \frac{b^2+ca}{c^2+a^2} + \frac{c^2+ab}{a^2+b^2} \geq \frac{ab+bc+ca}{a^2+b^2+c^2} + 2.$$

1.61. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a^2}{b^2+c^2} + \frac{b^2}{c^2+a^2} + \frac{c^2}{a^2+b^2} \geq \frac{(a+b+c)^2}{2(ab+bc+ca)}.$$

1.62. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{2ab}{(a+b)^2} + \frac{2bc}{(b+c)^2} + \frac{2ca}{(c+a)^2} + \frac{a^2+b^2+c^2}{ab+bc+ca} \ge \frac{5}{2}.$$

1.63. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{ab}{(a+b)^2} + \frac{bc}{(b+c)^2} + \frac{ca}{(c+a)^2} + \frac{1}{4} \ge \frac{ab+bc+ca}{a^2+b^2+c^2}.$$

1.64. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{3ab}{(a+b)^2} + \frac{3bc}{(b+c)^2} + \frac{3ca}{(c+a)^2} \le \frac{ab+bc+ca}{a^2+b^2+c^2} + \frac{5}{4}.$$

1.65. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

(a) $$\frac{a^3+abc}{b+c} + \frac{b^3+abc}{c+a} + \frac{c^3+abc}{a+b} \ge a^2+b^2+c^2;$$

(b) $$\frac{a^3+2abc}{b+c} + \frac{b^3+2abc}{c+a} + \frac{c^3+2abc}{a+b} \ge \frac{1}{2}(a+b+c)^2;$$

(c) $$\frac{a^3+3abc}{b+c} + \frac{b^3+3abc}{c+a} + \frac{c^3+3abc}{a+b} \ge 2(ab+bc+ca);$$

(d) $$\frac{a^2+2bc}{b+c} + \frac{b^2+2ca}{c+a} + \frac{c^2+2ab}{a+b} \ge \frac{3}{2}(ab+bc+ca).$$

1.66. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a^3+3abc}{(b+c)^2} + \frac{b^3+3abc}{(c+a)^2} + \frac{c^3+3abc}{(a+b)^2} \ge a+b+c.$$

1.67. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

(a) $$\frac{a^3+3abc}{(b+c)^3} + \frac{b^3+3abc}{(c+a)^3} + \frac{c^3+3abc}{(a+b)^3} \ge \frac{3}{2};$$

(b) $$\frac{3a^3+13abc}{(b+c)^3} + \frac{3b^3+13abc}{(c+a)^3} + \frac{3c^3+13abc}{(a+b)^3} \ge 6.$$

1.68. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

(a) $$\frac{a^3}{b+c}+\frac{b^3}{c+a}+\frac{c^3}{a+b}+ab+bc+ca \geq \frac{3}{2}(a^2+b^2+c^2);$$

(b) $$\frac{2a^2+bc}{b+c}+\frac{2b^2+ca}{c+a}+\frac{2c^2+ab}{a+b} \geq \frac{9(a^2+b^2+c^2)}{2(a+b+c)}.$$

1.69. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a(b+c)}{b^2+bc+c^2}+\frac{b(c+a)}{c^2+ca+a^2}+\frac{c(a+b)}{a^2+ab+b^2} \geq 2.$$

1.70. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a(b+c)}{b^2+bc+c^2}+\frac{b(c+a)}{c^2+ca+a^2}+\frac{c(a+b)}{a^2+ab+b^2} \geq 2+4\prod\left(\frac{a-b}{a+b}\right)^2.$$

1.71. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{ab-bc+ca}{b^2+c^2}+\frac{bc-ca+ab}{c^2+a^2}+\frac{ca-ab+bc}{a^2+b^2} \geq \frac{3}{2}.$$

1.72. Let a, b, c be nonnegative real numbers, no two of which are zero. If $k > -2$, then

$$\sum \frac{ab+(k-1)bc+ca}{b^2+kbc+c^2} \geq \frac{3(k+1)}{k+2}.$$

1.73. Let a, b, c be nonnegative real numbers, no two of which are zero. If $k > -2$, then

$$\sum \frac{3bc-a(b+c)}{b^2+kbc+c^2} \leq \frac{3}{k+2}.$$

1.74. Let a, b, c be nonnegative real numbers such that $ab+bc+ca=3$. Prove that

$$\frac{ab+1}{a^2+b^2}+\frac{bc+1}{b^2+c^2}+\frac{ca+1}{c^2+a^2} \geq \frac{4}{3}.$$

1.75. Let a, b, c be nonnegative real numbers such that $ab + bc + ca = 3$. Prove that

$$\frac{5ab+1}{(a+b)^2} + \frac{5bc+1}{(b+c)^2} + \frac{5ca+1}{(c+a)^2} \geq 2.$$

1.76. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a^2-bc}{2b^2-3bc+2c^2} + \frac{b^2-ca}{2c^2-3ca+2a^2} + \frac{c^2-ab}{2a^2-3ab+2b^2} \geq 0.$$

1.77. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{2a^2-bc}{b^2-bc+c^2} + \frac{2b^2-ca}{c^2-ca+a^2} + \frac{2c^2-ab}{a^2-ab+b^2} \geq 3.$$

1.78. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a^2}{2b^2-bc+2c^2} + \frac{b^2}{2c^2-ca+2a^2} + \frac{c^2}{2a^2-ab+2b^2} \geq 1.$$

1.79. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{1}{4b^2-bc+4c^2} + \frac{1}{4c^2-ca+4a^2} + \frac{1}{4a^2-ab+4b^2} \geq \frac{9}{7(a^2+b^2+c^2)}.$$

1.80. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{2a^2+bc}{b^2+c^2} + \frac{2b^2+ca}{c^2+a^2} + \frac{2c^2+ab}{a^2+b^2} \geq \frac{9}{2}.$$

1.81. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{2a^2+3bc}{b^2+bc+c^2} + \frac{2b^2+3ca}{c^2+ca+a^2} + \frac{2c^2+3ab}{a^2+ab+b^2} \geq 5.$$

1.82. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{2a^2+5bc}{(b+c)^2} + \frac{2b^2+5ca}{(c+a)^2} + \frac{2c^2+5ab}{(a+b)^2} \geq \frac{21}{4}.$$

1.83. Let a, b, c be nonnegative real numbers, no two of which are zero. If $k > -2$, then

$$\sum \frac{2a^2 + (2k+1)bc}{b^2 + kbc + c^2} \ge \frac{3(2k+3)}{k+2}.$$

1.84. Let a, b, c be nonnegative real numbers, no two of which are zero. If $k > -2$, then

$$\sum \frac{3bc - 2a^2}{b^2 + kbc + c^2} \le \frac{3}{k+2}.$$

1.85. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{a^2 + 16bc}{b^2 + c^2} + \frac{b^2 + 16ca}{c^2 + a^2} + \frac{c^2 + 16ab}{a^2 + b^2} \ge 10.$$

1.86. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{a^2 + 128bc}{b^2 + c^2} + \frac{b^2 + 128ca}{c^2 + a^2} + \frac{c^2 + 128ab}{a^2 + b^2} \ge 46.$$

1.87. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{a^2 + 64bc}{(b+c)^2} + \frac{b^2 + 64ca}{(c+a)^2} + \frac{c^2 + 64ab}{(a+b)^2} \ge 18.$$

1.88. Let a, b, c be nonnegative real numbers, no two of which are zero. If $k \ge -1$, then

$$\sum \frac{a^2(b+c) + kabc}{b^2 + kbc + c^2} \ge a + b + c.$$

1.89. Let a, b, c be nonnegative real numbers, no two of which are zero. If $k \ge \frac{-3}{2}$, then

$$\sum \frac{a^3 + (k+1)abc}{b^2 + kbc + c^2} \ge a + b + c.$$

1.90. Prove that $\frac{7}{8}$ is the least positive value of k such that

$$\sum \frac{1}{b^2+kbc+c^2} \ge \frac{3}{k+2}$$

for any nonnegative real numbers a,b,c, no two of which are zero, with $a+b+c=3$.

1.91. If a,b,c are the lengths of the sides of a triangle, then

(a) $$\frac{b+c-a}{b^2-bc+c^2}+\frac{c+a-b}{c^2-ca+a^2}+\frac{a+b-c}{a^2-ab+b^2} \ge \frac{2(a+b+c)}{a^2+b^2+c^2};$$

(b) $$\frac{2bc-a^2}{b^2-bc+c^2}+\frac{2ca-b^2}{c^2-ca+a^2}+\frac{2ab-c^2}{a^2-ab+b^2} \ge 0.$$

1.92. If a,b,c are nonnegative real numbers, then

(a) $$\frac{a^2}{5a^2+(b+c)^2}+\frac{b^2}{5b^2+(c+a)^2}+\frac{c^2}{5c^2+(a+b)^2} \le \frac{1}{3};$$

(b) $$\frac{a^3}{13a^3+(b+c)^3}+\frac{b^3}{13b^3+(c+a)^3}+\frac{c^3}{13c^3+(a+b)^3} \le \frac{1}{7}.$$

1.93. If a,b,c are nonnegative real numbers, then

$$\frac{b^2+c^2-a^2}{2a^2+(b+c)^2}+\frac{c^2+a^2-b^2}{2b^2+(c+a)^2}+\frac{a^2+b^2-c^2}{2c^2+(a+b)^2} \ge \frac{1}{2}.$$

1.94. If a,b,c are nonnegative real numbers, no two of which are zero, such that $a+b+c=3$, then

$$\frac{a+b}{2a^2+3ab+2b^2}+\frac{b+c}{2b^2+3bc+2c^2}+\frac{c+a}{2c^2+3ca+2a^2} \ge \frac{6}{7}.$$

1.95. Let a,b,c be nonnegative real numbers, no two of which are zero. If $k \ge 3+\sqrt{7}$, then

(a) $$\frac{a}{a^2+kbc}+\frac{b}{b^2+kca}+\frac{c}{c^2+kab} \ge \frac{9}{(1+k)(a+b+c)};$$

(b) $$\frac{1}{ka^2+bc}+\frac{1}{kb^2+ca}+\frac{1}{kc^2+ab} \ge \frac{9}{(k+1)(ab+bc+ca)}.$$

1.96. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}+\frac{1}{2c^2+ab} \ge \frac{6}{a^2+b^2+c^2+ab+bc+ca}.$$

1.97. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{1}{22a^2+5bc}+\frac{1}{22b^2+5ca}+\frac{1}{22c^2+5ab} \ge \frac{1}{(a+b+c)^2}.$$

1.98. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}+\frac{1}{2c^2+ab} \ge \frac{8}{(a+b+c)^2}.$$

1.99. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{1}{a^2+bc}+\frac{1}{b^2+ca}+\frac{1}{c^2+ab} \ge \frac{12}{(a+b+c)^2}.$$

1.100. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

(a) $$\frac{1}{a^2+2bc}+\frac{1}{b^2+2ca}+\frac{1}{c^2+2ab} \ge \frac{1}{a^2+b^2+c^2}+\frac{2}{ab+bc+ca};$$

(b) $$\frac{a(b+c)}{a^2+2bc}+\frac{b(c+a)}{b^2+2ca}+\frac{c(a+b)}{c^2+2ab} \ge 1+\frac{ab+bc+ca}{a^2+b^2+c^2}.$$

1.101. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

(a) $$\frac{a}{a^2+2bc}+\frac{b}{b^2+2ca}+\frac{c}{c^2+2ab} \le \frac{a+b+c}{ab+bc+ca};$$

(b) $$\frac{a(b+c)}{a^2+2bc}+\frac{b(c+a)}{b^2+2ca}+\frac{c(a+b)}{c^2+2ab} \le 1+\frac{a^2+b^2+c^2}{ab+bc+ca}.$$

1.102. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

(a) $$\frac{a}{2a^2+bc}+\frac{b}{2b^2+ca}+\frac{c}{2c^2+ab} \ge \frac{a+b+c}{a^2+b^2+c^2};$$

(b) $$\frac{b+c}{2a^2+bc}+\frac{c+a}{2b^2+ca}+\frac{a+b}{2c^2+ab} \ge \frac{6}{a+b+c}.$$

1.103. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a(b+c)}{a^2+bc}+\frac{b(c+a)}{b^2+ca}+\frac{c(a+b)}{c^2+ab}\ge\frac{(a+b+c)^2}{a^2+b^2+c^2}.$$

1.104. Let a, b, c be nonnegative real numbers, no two of which are zero. If $k > 0$, then

$$\frac{b^2+c^2+\sqrt{3}bc}{a^2+kbc}+\frac{c^2+a^2+\sqrt{3}ca}{b^2+kca}+\frac{a^2+b^2+\sqrt{3}ab}{c^2+kab}\ge\frac{3(2+\sqrt{3})}{1+k}.$$

1.105. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{1}{a^2+b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}+\frac{8}{a^2+b^2+c^2}\ge\frac{6}{ab+bc+ca}.$$

1.106. If a, b, c are the lengths of the sides of a triangle, then

$$\frac{a(b+c)}{a^2+2bc}+\frac{b(c+a)}{b^2+2ca}+\frac{c(a+b)}{c^2+2ab}\le 2.$$

1.107. If a, b, c are real numbers, then

$$\frac{a^2-bc}{2a^2+b^2+c^2}+\frac{b^2-ca}{2b^2+c^2+a^2}+\frac{c^2-ab}{2c^2+a^2+b^2}\ge 0.$$

1.108. If a, b, c are nonnegative real numbers, then

$$\frac{3a^2-bc}{2a^2+b^2+c^2}+\frac{3b^2-ca}{2b^2+c^2+a^2}+\frac{3c^2-ab}{2c^2+a^2+b^2}\le\frac{3}{2}.$$

1.109. If a, b, c are nonnegative real numbers, then

$$\frac{(b+c)^2}{4a^2+b^2+c^2}+\frac{(c+a)^2}{4b^2+c^2+a^2}+\frac{(a+b)^2}{4c^2+a^2+b^2}\ge 2.$$

1.110. If a, b, c are positive real numbers, then

(a) $$\sum\frac{1}{11a^2+2b^2+2c^2}\le\frac{3}{5(ab+bc+ca)};$$

(b) $$\sum\frac{1}{4a^2+b^2+c^2}\le\frac{1}{2(a^2+b^2+c^2)}+\frac{1}{ab+bc+ca}.$$

1.111. If a, b, c are nonnegative real numbers such that $ab + bc + ca = 3$, then

$$\frac{\sqrt{a}}{b+c} + \frac{\sqrt{b}}{c+a} + \frac{\sqrt{c}}{a+b} \geq \frac{3}{2}.$$

1.112. If a, b, c are nonnegative real numbers such that $ab + bc + ca \geq 3$, then

$$\frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c} \geq \frac{1}{1+b+c} + \frac{1}{1+c+a} + \frac{1}{1+a+b}.$$

1.113. If a, b, c are the lengths of the sides of a triangle, then

(a) $$\frac{a^2 - bc}{3a^2 + b^2 + c^2} + \frac{b^2 - ca}{3b^2 + c^2 + a^2} + \frac{c^2 - ab}{3c^2 + a^2 + b^2} \leq 0;$$

(b) $$\frac{a^4 - b^2c^2}{3a^4 + b^4 + c^4} + \frac{b^4 - c^2a^2}{3b^4 + c^4 + a^4} + \frac{c^4 - a^2b^2}{3c^4 + a^4 + b^4} \leq 0.$$

1.114. If a, b, c are the lengths of the sides of a triangle, then

$$\frac{bc}{4a^2 + b^2 + c^2} + \frac{ca}{4b^2 + c^2 + a^2} + \frac{ab}{4c^2 + a^2 + b^2} \geq \frac{1}{2}.$$

1.115. If a, b, c are the lengths of the sides of a triangle, then

$$\frac{1}{b^2 + c^2} + \frac{1}{c^2 + a^2} + \frac{1}{a^2 + b^2} \leq \frac{9}{2(ab + bc + ca)}.$$

1.116. If a, b, c are the lengths of the sides of a triangle, then

(a) $$\left|\frac{a+b}{a-b} + \frac{b+c}{b-c} + \frac{c+a}{c-a}\right| > 5;$$

(b) $$\left|\frac{a^2+b^2}{a^2-b^2} + \frac{b^2+c^2}{b^2-c^2} + \frac{c^2+a^2}{c^2-a^2}\right| \geq 3.$$

1.117. If a, b, c are the lengths of the sides of a triangle, then

$$\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} + 3 \geq 6\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right).$$

1.118. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\sum \frac{3a(b+c)-2bc}{(b+c)(2a+b+c)} \ge \frac{3}{2}.$$

1.119. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\sum \frac{a(b+c)-2bc}{(b+c)(3a+b+c)} \ge 0.$$

1.120. Let a, b, c be positive real numbers such that $a^2+b^2+c^2 \ge 3$. Prove that

$$\frac{a^5-a^2}{a^5+b^2+c^2} + \frac{b^5-b^2}{b^5+c^2+a^2} + \frac{c^5-c^2}{c^5+a^2+b^2} \ge 0.$$

1.121. Let a, b, c be positive real numbers such that $a^2+b^2+c^2 = a^3+b^3+c^3$. Prove that

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \ge \frac{3}{2}.$$

1.122. If $a, b, c \in [0, 1]$, then

$$\frac{a}{bc+2} + \frac{b}{ca+2} + \frac{c}{ab+2} \le 1.$$

1.123. Let a, b, c be positive real numbers such that $a+b+c = 2$. Prove that

$$5(1-ab-bc-ca)\left(\frac{1}{1-ab} + \frac{1}{1-bc} + \frac{1}{1-ca}\right) + 9 \ge 0.$$

1.124. Let a, b, c be nonnegative real numbers such that $a+b+c = 2$. Prove that

$$\frac{2-a^2}{2-bc} + \frac{2-b^2}{2-ca} + \frac{2-c^2}{2-ab} \le 3.$$

1.125. Let a, b, c be nonnegative real numbers such that $a+b+c = 3$. Prove that

$$\frac{3+5a^2}{3-bc} + \frac{3+5b^2}{3-ca} + \frac{3+5c^2}{3-ab} \ge 12.$$

1.126. Let a, b, c be nonnegative real numbers such that $a+b+c=2$. If

$$\frac{-1}{7} \le m \le \frac{7}{8},$$

then

$$\frac{a^2+m}{3-2bc}+\frac{b^2+m}{3-2ca}+\frac{c^2+m}{3-2ab} \ge \frac{3(4+9m)}{19}.$$

1.127. Let a, b, c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$\frac{47-7a^2}{1+bc}+\frac{47-7b^2}{1+ca}+\frac{47-7c^2}{1+ab} \ge 60.$$

1.128. Let a, b, c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$\frac{26-7a^2}{1+bc}+\frac{26-7b^2}{1+ca}+\frac{26-7c^2}{1+ab} \le \frac{57}{2}.$$

1.129. If a, b, c are nonnegative real numbers, then

$$\sum \frac{5a(b+c)-6bc}{a^2+b^2+c^2+bc} \le 3.$$

1.130. Let a, b, c be nonnegative real numbers, no two of which are zero, and let

$$x = \frac{a^2+b^2+c^2}{ab+bc+ca}.$$

Prove that

(a) $$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{1}{2} \ge x+\frac{1}{x};$$

(b) $$6\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right) \ge 5x+\frac{4}{x};$$

(c) $$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}-\frac{3}{2} \ge \frac{1}{3}\left(x-\frac{1}{x}\right).$$

1.131. If a, b, c are real numbers, then

$$\frac{1}{a^2+7(b^2+c^2)}+\frac{1}{b^2+7(c^2+a^2)}+\frac{1}{c^2+7(a^2+b^2)} \le \frac{9}{5(a+b+c)^2}.$$

1.132. If a, b, c are real numbers, then

$$\frac{bc}{3a^2+b^2+c^2}+\frac{ca}{3b^2+c^2+a^2}+\frac{ab}{3c^2+a^2+b^2}\le\frac{3}{5}.$$

1.133. If a, b, c are real numbers such that $a+b+c=3$, then

$$\frac{1}{8+5(b^2+c^2)}+\frac{1}{8+5(c^2+a^2)}+\frac{1}{8+5(a^2+b^2)}\le\frac{1}{6}.$$

1.134. If a, b, c are real numbers, then

$$\frac{(a+b)(a+c)}{a^2+4(b^2+c^2)}+\frac{(b+c)(b+a)}{b^2+4(c^2+a^2)}+\frac{(c+a)(c+b)}{c^2+4(a^2+b^2)}\le\frac{4}{3}.$$

1.135. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\sum\frac{1}{(b+c)(7a+b+c)}\le\frac{1}{2(ab+bc+ca)}.$$

1.136. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\sum\frac{1}{b^2+c^2+4a(b+c)}\le\frac{9}{10(ab+bc+ca)}.$$

1.137. Let a, b, c be nonnegative real numbers, no two of which are zero. If $a+b+c=3$, then

$$\frac{1}{3-ab}+\frac{1}{3-bc}+\frac{1}{3-ca}\le\frac{9}{2(ab+bc+ca)}.$$

1.138. If a, b, c are nonnegative real numbers such that $a+b+c=3$, then

$$\frac{bc}{a^2+a+6}+\frac{ca}{b^2+b+6}+\frac{ab}{c^2+c+6}\le\frac{3}{8}.$$

1.139. If a, b, c are nonnegative real numbers such that $ab+bc+ca=3$, then

$$\frac{1}{8a^2-2bc+21}+\frac{1}{8b^2-2ca+21}+\frac{1}{8c^2-2ab+21}\ge\frac{1}{9}.$$

1.140. Let a, b, c be real numbers, no two of which are zero. Prove that

(a) $$\frac{a^2+bc}{b^2+c^2}+\frac{b^2+ca}{c^2+a^2}+\frac{c^2+ab}{a^2+b^2}\ge\frac{(a+b+c)^2}{a^2+b^2+c^2};$$

(b) $$\frac{a^2+3bc}{b^2+c^2}+\frac{b^2+3ca}{c^2+a^2}+\frac{c^2+3ab}{a^2+b^2}\ge\frac{6(ab+bc+ca)}{a^2+b^2+c^2}.$$

1.141. Let a, b, c be real numbers, no two of which are zero. If $ab+bc+ca\ge 0$, then

$$\frac{a(b+c)}{b^2+c^2}+\frac{b(c+a)}{c^2+a^2}+\frac{c(a+b)}{a^2+b^2}\ge\frac{3}{10}.$$

1.142. If a, b, c are positive real numbers such that $abc>1$, then

$$\frac{1}{a+b+c-3}+\frac{1}{abc-1}\ge\frac{4}{ab+bc+ca-3}.$$

1.143. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\sum\frac{(4b^2-ac)(4c^2-ab)}{b+c}\le\frac{27}{2}abc.$$

1.144. Let a, b, c be nonnegative real numbers, no two of which are zero, such that

$$a+b+c=2.$$

Prove that

$$\frac{a}{2a+bc}+\frac{b}{2b+ca}+\frac{c}{2c+ab}\ge 1.$$

1.145. Let a, b, c be positive real numbers such that

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=10.$$

Prove that

$$\frac{19}{12}\le\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\le\frac{5}{3}.$$

1.146. Let a,b,c be nonnegative real numbers, no two of which are zero, such that $a+b+c=3$. Prove that

$$\frac{9}{10}<\frac{a}{2a+bc}+\frac{b}{2b+ca}+\frac{c}{2c+ab}\le 1.$$

1.147. Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a^3}{2a^2+bc}+\frac{b^3}{2b^2+ca}+\frac{c^3}{2c^2+ab}\le\frac{a^3+b^3+c^3}{a^2+b^2+c^2}.$$

1.148. If a,b,c are positive real numbers, then

$$\frac{a^3}{4a^2+bc}+\frac{b^3}{4b^2+ca}+\frac{c^3}{4c^2+ab}\ge\frac{a+b+c}{5}.$$

1.149. If a,b,c are positive real numbers, then

$$\frac{1}{(2+a)^2}+\frac{1}{(2+b)^2}+\frac{1}{(2+c)^2}\ge\frac{3}{6+ab+bc+ca}.$$

1.150. If a,b,c are positive real numbers, then

$$\frac{1}{1+3a}+\frac{1}{1+3b}+\frac{1}{1+3c}\ge\frac{3}{3+abc}.$$

1.151. Let a,b,c be real numbers, no two of which are zero. If $1<k\le 3$, then

$$\left(k+\frac{2ab}{a^2+b^2}\right)\left(k+\frac{2bc}{b^2+c^2}\right)\left(k+\frac{2ca}{c^2+a^2}\right)\ge(k-1)(k^2-1).$$

1.152. If a,b,c are non-zero and distinct real numbers, then

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+3\left[\frac{1}{(a-b)^2}+\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}\right]\ge 4\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right).$$

1.153. Let a,b,c be positive real numbers, and let

$$A=\frac{a}{b}+\frac{b}{a}+k,\quad B=\frac{b}{c}+\frac{c}{b}+k,\quad C=\frac{c}{a}+\frac{a}{b}+k,$$

where $-2<k\le 4$. Prove that

$$\frac{1}{A}+\frac{1}{B}+\frac{1}{C}\le\frac{1}{k+2}+\frac{4}{A+B+C-k-2}.$$

1.154. If a,b,c are nonnegative real numbers, no two of which are zero, then

$$\frac{1}{b^2+bc+c^2}+\frac{1}{c^2+ca+a^2}+\frac{1}{a^2+ab+b^2}\ge\frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}+\frac{1}{2c^2+ab}.$$

1.155. If a,b,c are nonnegative real numbers such that $a+b+c\le 3$, then

(a) $$\frac{1}{2a+1}+\frac{1}{2b+1}+\frac{1}{2c+1}\ge\frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2};$$

(b) $$\frac{1}{2ab+1}+\frac{1}{2bc+1}+\frac{1}{2ca+1}\ge\frac{1}{a^2+2}+\frac{1}{b^2+2}+\frac{1}{c^2+2}.$$

1.156. If a,b,c are nonnegative real numbers such that $a+b+c=4$, then

$$\frac{1}{ab+2}+\frac{1}{bc+2}+\frac{1}{ca+2}\ge\frac{1}{a^2+2}+\frac{1}{b^2+2}+\frac{1}{c^2+2}.$$

1.157. If a,b,c are nonnegative real numbers, no two of which are zero, then

(a) $$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{(a-b)^2(b-c)^2(c-a)^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}\le 1;$$

(b) $$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{(a-b)^2(b-c)^2(c-a)^2}{(a^2-ab+b^2)(b^2-bc+c^2)(c^2-ca+a^2)}\le 1.$$

1.158. If a,b,c are nonnegative real numbers, no two of which are zero, then

$$\frac{a^2+b^2+c^2}{ab+bc+ca}\ge 1+\frac{9(a-b)^2(b-c)^2(c-a)^2}{(a+b)^2(b+c)^2(c+a)^2}.$$

1.159. If a,b,c are nonnegative real numbers, no two of which are zero, then

$$\frac{a^2+b^2+c^2}{ab+bc+ca}\ge 1+(1+\sqrt{2})^2\frac{(a-b)^2(b-c)^2(c-a)^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}.$$

1.160. If a,b,c are nonnegative real numbers, no two of which are zero, then

$$\frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{c+a}\ge\frac{5}{3a+b+c}+\frac{5}{3b+c+a}+\frac{5}{3c+a+b}.$$

1.161. If a, b, c are real numbers, no two of which are zero, then

(a) $$\frac{8a^2+3bc}{b^2+bc+c^2}+\frac{8b^2+3ca}{c^2+ca+a^2}+\frac{8c^2+3ab}{a^2+ab+b^2}\ge 11;$$

(b) $$\frac{8a^2-5bc}{b^2-bc+c^2}+\frac{8b^2-5ca}{c^2-ca+a^2}+\frac{8c^2-5ab}{a^2-ab+b^2}\ge 9.$$

1.162. If a, b, c are real numbers, no two of which are zero, then

$$\frac{4a^2+bc}{4b^2+7bc+4c^2}+\frac{4b^2+ca}{4c^2+7ca+4a^2}+\frac{4c^2+ab}{4a^2+7ab+4b^2}\ge 1.$$

1.163. If a, b, c are real numbers, no two of which are equal, then

$$\frac{1}{(a-b)^2}+\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}\ge \frac{27}{4(a^2+b^2+c^2-ab-bc-ca)}.$$

1.164. If a, b, c are real numbers, no two of which are zero, then

$$\frac{1}{a^2-ab+b^2}+\frac{1}{b^2-bc+c^2}+\frac{1}{c^2-ca+a^2}\ge \frac{14}{3(a^2+b^2+c^2)}.$$

1.165. If a, b, c are real numbers, then

$$\frac{a^2+bc}{2a^2+b^2+c^2}+\frac{b^2+ca}{a^2+2b^2+c^2}+\frac{c^2+ab}{a^2+b^2+2c^2}\ge \frac{1}{6}.$$

1.166. If a, b, c are real numbers, then

$$\frac{2b^2+2c^2+3bc}{(a+3b+3c)^2}+\frac{2c^2+2a^2+3ca}{(b+3c+3a)^2}+\frac{2a^2+2b^2+3ab}{(c+3a+3b)^2}\ge \frac{3}{7}.$$

1.167. If a, b, c are nonnegative real numbers, then

$$\frac{6b^2+6c^2+13bc}{(a+2b+2c)^2}+\frac{6c^2+6a^2+13ca}{(b+2c+2a)^2}+\frac{6a^2+6b^2+13ab}{(c+2a+2b)^2}\le 3.$$

1.168. If a,b,c are nonnegative real numbers such that $a+b+c=3$, then

$$\frac{3a^2+8bc}{9+b^2+c^2}+\frac{3b^2+8ca}{9+c^2+a^2}+\frac{3c^2+8ab}{9+a^2+b^2}\le 3.$$

1.169. If a,b,c are nonnegative real numbers such that $a+b+c=3$, then

$$\frac{5a^2+6bc}{9+b^2+c^2}+\frac{5b^2+6ca}{9+c^2+a^2}+\frac{5c^2+6ab}{9+a^2+b^2}\ge 3.$$

1.170. If a,b,c are nonnegative real numbers such that $a+b+c=3$, then

$$\frac{1}{a^2+bc+12}+\frac{1}{b^2+ca+12}+\frac{1}{c^2+ab+12}\le \frac{3}{14}.$$

1.171. If a,b,c are nonnegative real numbers, no two of which are zero, then

$$\frac{1}{a^2+b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}\ge \frac{45}{8(a^2+b^2+c^2)+2(ab+bc+ca)}.$$

1.172. If a,b,c are real numbers, no two of which are zero, then

$$\frac{a^2-7bc}{b^2+c^2}+\frac{b^2-7ca}{a^2+b^2}+\frac{c^2-7ab}{a^2+b^2}+\frac{9(ab+bc+ca)}{a^2+b^2+c^2}\ge 0.$$

1.173. If a,b,c are nonnegative real numbers, no two of which are zero, then

$$\frac{a^2-4bc}{b^2+c^2}+\frac{b^2-4ca}{c^2+a^2}+\frac{c^2-4ab}{a^2+b^2}+\frac{9(ab+bc+ca)}{a^2+b^2+c^2}\ge \frac{9}{2}.$$

1.174. If a,b,c are real numbers such that $abc\ne 0$, then

$$\frac{(b+c)^2}{a^2}+\frac{(c+a)^2}{b^2}+\frac{(a+b)^2}{c^2}\ge 2+\frac{10(a+b+c)^2}{3(a^2+b^2+c^2)}.$$

1.175. Let a,b,c be real numbers, no two of which are zero. If $ab+bc+ca\ge 0$, then

(a)
$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\ge \frac{3}{2};$$

(b) *if* $ab\le 0$, then
$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\ge 2.$$

1.176. If a,b,c are nonnegative real numbers, then

$$\frac{a}{7a+b+c}+\frac{b}{7b+c+a}+\frac{c}{7c+a+b}\ge\frac{ab+bc+ca}{(a+b+c)^2}.$$

1.177. If a,b,c are positive real numbers such that $abc=1$, then

$$\frac{a+b+c}{30}+\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}\ge\frac{8}{5}.$$

1.178. Let a,b,c be positive real numbers such that at most one of them is larger that 1 and $abc=1$. Prove that

$$\frac{11(b^2+c^2)-10a^2}{b+c}+\frac{11(c^2+a^2)-10b^2}{c+a}+\frac{11(a^2+b^2)-10c^2}{a+b}\le 18.$$

1.179. Let $a,b,c\le 8$ be real numbers such that $a+b+c=3$. Prove that

$$\frac{13a-1}{a^2+23}+\frac{13b-1}{b^2+23}+\frac{13c-1}{c^2+23}\le\frac{3}{2}.$$

1.180. Let $a,b,c\ne\frac{3}{4}$ be nonnegative real numbers such that $a+b+c=3$. Prove that

$$\frac{1-a}{(4a-3)^2}+\frac{1-b}{(4b-3)^2}+\frac{1-c}{(4c-3)^2}\ge 0.$$

1.181. If a,b,c are the lengths of the sides of a triangle, then

$$\frac{a^2}{4a^2+5bc}+\frac{b^2}{4b^2+5ca}+\frac{c^2}{4c^2+5ab}\ge\frac{1}{3}.$$

1.182. If a,b,c are the lengths of the sides of a triangle, then

$$\frac{1}{7a^2+b^2+c^2}+\frac{1}{7b^2+c^2+a^2}+\frac{1}{7c^2+a^2+b^2}\ge\frac{3}{(a+b+c)^2}.$$

1.183. Let a,b,c be the lengths of the sides of a triangle. If $k>-2$, then

$$\sum\frac{a(b+c)+(k+1)bc}{b^2+kbc+c^2}\le\frac{3(k+3)}{k+2}.$$

1.184. Let a, b, c be the lengths of the sides of a triangle. If $k > -2$, then

$$\sum \frac{2a^2 + (4k+9)bc}{b^2 + kbc + c^2} \le \frac{3(4k+11)}{k+2}.$$

1.185. If a, b, c are nnonnegative numbers such that $abc = 1$, then

$$\frac{1}{(a+1)^2} + \frac{1}{(b+1)^2} + \frac{1}{(c+1)^2} + \frac{1}{2(a+b+c-1)} \ge 1.$$

1.186. If a, b, c are positive real numbers such that

$$a \le b \le c, \qquad a^2bc \ge 1,$$

then

$$\frac{1}{1+a^3} + \frac{1}{1+b^3} + \frac{1}{1+c^3} \ge \frac{3}{1+abc}.$$

1.187. If a, b, c are positive real numbers such that

$$a \le b \le c, \qquad a^2c \ge 1,$$

then

$$\frac{1}{1+a^3} + \frac{1}{1+b^3} + \frac{1}{1+c^3} \ge \frac{3}{1+abc}.$$

1.188. If a, b, c are positive real numbers such that

$$a \le b \le c, \qquad 2a + c \ge 3,$$

then

$$\frac{1}{3+a^2} + \frac{1}{3+b^2} + \frac{1}{3+c^2} \ge \frac{3}{3 + \left(\frac{a+b+c}{3}\right)^2}.$$

1.189. If a, b, c are positive real numbers such that

$$a \le b \le c, \qquad 9a + 8b \ge 17,$$

then

$$\frac{1}{3+a^2} + \frac{1}{3+b^2} + \frac{1}{3+c^2} \ge \frac{3}{3 + \left(\frac{a+b+c}{3}\right)^2}.$$

1.190. Let a, b, c, d be positive real numbers such that $abcd = 1$. Prove that

$$\sum \frac{1}{1+ab+bc+ca} \le 1.$$

1.191. Let a, b, c, d be positive real numbers such that $abcd = 1$. Prove that

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{1}{(1+d)^2} \ge 1.$$

1.192. Let $a, b, c, d \ne \frac{1}{3}$ be positive real numbers such that $abcd = 1$. Prove that

$$\frac{1}{(3a-1)^2} + \frac{1}{(3b-1)^2} + \frac{1}{(3c-1)^2} + \frac{1}{(3d-1)^2} \ge 1.$$

1.193. Let a, b, c, d be positive real numbers such that $abcd = 1$. Prove that

$$\frac{1}{1+a+a^2+a^3} + \frac{1}{1+b+b^2+b^3} + \frac{1}{1+c+c^2+c^3} + \frac{1}{1+d+d^2+d^3} \ge 1.$$

1.194. Let a, b, c, d be positive real numbers such that $abcd = 1$. Prove that

$$\frac{1}{1+a+2a^2} + \frac{1}{1+b+2b^2} + \frac{1}{1+c+2c^2} + \frac{1}{1+d+2d^2} \ge 1.$$

1.195. Let a, b, c, d be positive real numbers such that $abcd = 1$. Prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{9}{a+b+c+d} \ge \frac{25}{4}.$$

1.196. If a, b, c, d are real numbers such that $a + b + c + d = 0$, then

$$\frac{(a-1)^2}{3a^2+1} + \frac{(b-1)^2}{3b^2+1} + \frac{(c-1)^2}{3c^2+1} + \frac{(d-1)^2}{3d^2+1} \le 4.$$

1.197. If $a, b, c, d \ge -5$ such that $a + b + c + d = 4$, then

$$\frac{1-a}{(1+a)^2} + \frac{1-b}{(1+b)^2} + \frac{1-c}{(1+c)^2} + \frac{1-d}{(1+d)^2} \ge 0.$$

1.198. If a, b, c, d are nonnegative real numbers such that

$$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}+\frac{1}{d+1}=3,$$

then

$$3(ab+ac+ad+bc+bd+cd)+\frac{4}{a+b+c+d}\le 5.$$

1.199. If a, b, c, d are nonnegative real numbers such that

$$ab+ac+ad+bc+bd+cd=9,$$

then

$$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}+\frac{1}{d+1}\le 1+\frac{a+b+c+d}{4}.$$

1.200. If a, b, c, d are nonnegative real numbers such that

$$ab+ac+ad+bc+bd+cd=6,$$

then

$$\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}+\frac{1}{d^2+1}\ge 2.$$

1.201. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1+a_2+\cdots+a_n=n$. Prove that

$$\sum \frac{1}{(n+1)a_1^2+a_2^2+\cdots+a_n^2}\le \frac{1}{2}.$$

1.202. Let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1+a_2+\cdots+a_n=n$. Prove that

$$\frac{a_1^2}{a_1^2-2a_1+n}+\frac{a_2^2}{a_2^2-2a_2+n}+\cdots+\frac{a_n^2}{a_n^2-2a_n+n}\ge \frac{n}{n-1}.$$

1.203. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1a_2\cdots a_n=1$. Prove that

(a) $$\frac{1}{1+(n-1)a_1}+\frac{1}{1+(n-1)a_2}+\cdots+\frac{1}{1+(n-1)a_n}\ge 1;$$

(b) $$\frac{1}{a_1+n-1}+\frac{1}{a_2+n-1}+\cdots+\frac{1}{a_n+n-1}\le 1.$$

1.204. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1a_2\cdots a_n = 1$. Prove that

$$\frac{1}{1-a_1+na_1^2}+\frac{1}{1-a_2+na_2^2}+\cdots+\frac{1}{1-a_n+na_n^2}\ge 1.$$

1.205. Let $n \ge 3$ and $a_1, a_2, \ldots, a_n \ge \dfrac{2(n-3)}{2n-3}$ such that $a_1a_2\cdots a_n = 1$. Prove that

$$\frac{1}{a_1+2}+\frac{1}{a_2+2}+\cdots+\frac{1}{a_n+2}\le\frac{n}{3}.$$

1.206. If $a_1, a_2, \ldots, a_n \ge 0$, then

$$\frac{1}{1+na_1}+\frac{1}{1+na_2}+\cdots+\frac{1}{1+na_n}\ge\frac{n}{n+a_1a_2\cdots a_n}.$$

1.207. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that

$$a_1+a_2+\cdots+a_n=\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n},$$

then

$$\frac{1}{(n-1)a_1+1}+\frac{1}{(n-1)a_2+1}+\cdots+\frac{1}{(n-1)a_n+1}\ge 1.$$

1.208. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that

$$\sum_{1\le i<j\le n} a_ia_j=\frac{n(n-1)}{2}\ ,$$

then

$$\frac{1}{a_1+n-1}+\frac{1}{a_2+n-1}+\cdots+\frac{1}{a_n+n-1}\ge 1.$$

1.209. Let a, b, c be nonnegative real numbers such that

$$a\ge b\ge 1\ge c,\qquad a+b+c=3.$$

Prove that

$$\frac{1}{a^2+3}+\frac{1}{b^2+3}+\frac{1}{c^2+3}\le\frac{3}{4}.$$

1.210. Let a, b, c be nonnegative real numbers such that

$$a \geq 1 \geq b \geq c, \qquad a+b+c=3.$$

Prove that

$$\frac{1}{a^2+2}+\frac{1}{b^2+2}+\frac{1}{c^2+2} \geq 1.$$

1.211. If $a \geq 1 \geq b \geq c > -3$ such that $ab+bc+ca=3$, then

$$\frac{1}{a^2+ab+b^2}+\frac{1}{b^2+bc+c^2}+\frac{1}{c^2+ca+a^2} \geq 1.$$

1.212. If $a \geq b \geq 1 \geq c \geq 0$ such that $a+b+c=3$, then

$$\frac{1}{a^2+ab+b^2}+\frac{1}{b^2+bc+c^2}+\frac{1}{c^2+ca+a^2} \leq \frac{3}{ab+bc+ca}.$$

1.213. If a, b, c are positive real numbers such that

$$a \geq 1 \geq b \geq c, \qquad abc=1,$$

then

$$\frac{1-a}{3+a^2}+\frac{1-b}{3+b^2}+\frac{1-c}{3+c^2} \geq 0.$$

1.214. If a, b, c are positive real numbers such that

$$a \geq 1 \geq b \geq c, \qquad abc=1,$$

then

$$\frac{1}{a^2+4ab+b^2}+\frac{1}{b^2+4bc+c^2}+\frac{1}{c^2+4ca+a^2} \geq \frac{1}{2}.$$

1.215. If $a_1, a_2, \ldots, a_n$ are real number such that at most one of them is less than 1 and $a_1+a_2+\cdots+a_n=n$, then

(a)
$$\frac{a_1+1}{a_1^2+1}+\frac{a_2+1}{a_2^2+1}+\cdots+\frac{a_n+1}{a_n^2+1} \leq n;$$

(b)
$$\frac{1}{a_1^2+3}+\frac{1}{a_2^2+3}+\cdots+\frac{1}{a_1^2+3} \leq \frac{n}{4}.$$

1.216. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that at most one of them is less than 1 and $a_1 + a_2 + \cdots + a_n = n$, then

$$\frac{a_1^2 - 1}{(a_1+3)^2} + \frac{a_2^2 - 1}{(a_2+3)^2} + \cdots + \frac{a_n^2 - 1}{(a_n+3)^2} \ge 0.$$

1.217. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that at most one of them is larger than 1 and $a_1 + a_2 + \cdots + a_n = n$, then

$$\frac{1}{3a_1^3 + 4} + \frac{1}{3a_2^3 + 4} + \cdots + \frac{1}{3a_n^3 + 4} \ge \frac{n}{7}.$$

1.218. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that at most one of them is less than 1 and $a_1^2 + a_2^2 + \cdots + a_n^2 = n$, then

$$\frac{1}{3 - a_1} + \frac{1}{3 - a_2} + \cdots + \frac{1}{3 - a_n} \le \frac{n}{2}.$$

1.219. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that at most one of them is larger than 1 and $a_1 a_2 \cdots a_n = 1$, then

$$\frac{1}{1 + a_1} + \frac{1}{1 + a_2} + \cdots + \frac{1}{1 + a_n} \ge \frac{n}{2}.$$

1.220. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that at most one of them is larger than 1 and $a_1 a_2 \cdots a_n = 1$, then

$$\frac{1}{(a_1+2)^2} + \frac{1}{(a_2+2)^2} + \cdots + \frac{1}{(a_n+2)^2} \ge \frac{n}{9}.$$

1.221. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that at most one of them is larger than 1 and $a_1 a_2 \cdots a_n = 1$, then

$$\frac{1 - a_1}{3 + a_1^2} + \frac{1 - a_2}{3 + a_2^2} + \cdots + \frac{1 - a_n}{3 + a_n^2} \ge 0.$$

1.222. Let a, b, c be nonnegative real numbers (no two of which are zero) such that $a+b+c = 3$.

(a) If $k \geq 3$, then

$$\frac{(k-2)a^2+3a}{bc+ka}+\frac{(k-2)b^2+3b}{ca+kb}+\frac{(k-2)c^2+3c}{ab+kc} \geq 3;$$

(b) If $0 < k \leq 3$, then

$$\frac{(k-2)a^2+3a}{bc+ka}+\frac{(k-2)b^2+3b}{ca+kb}+\frac{(k-2)c^2+3c}{ab+kc} \leq 3.$$

1.223. Let a, b, c be nonnegative real numbers (no two of which are zero) such that $a+b+c = 3$.

(a) If $0 < k \leq 3$, then

$$\frac{(k-1)a^2+(k+3)a}{bc+ka}+\frac{(k-1)b^2+(k+3)b}{ca+kb}+\frac{(k-1)c^2+(k+3)c}{ab+kc} \geq 6;$$

(b) If $k \geq 3$, then

$$\frac{(k-1)a^2+(k+3)a}{bc+ka}+\frac{(k-1)b^2+(k+3)b}{ca+kb}+\frac{(k-1)c^2+(k+3)c}{ab+kc} \leq 6.$$

1.224. If a, b, c are positive real numbers such that $abc = 1$, then

$$\frac{1}{(4a+b+c)^2}+\frac{1}{(4b+c+a)^2}+\frac{1}{(4c+a+b)^2} \leq \frac{1}{12}.$$

1.225. Let a, b, c be nonnegative real numbers satisfying

$$a^2+b^2+c^2 = 3.$$

If $\frac{1}{3} \leq k \leq 11$, then

$$\frac{15-7a^2}{1+kbc}+\frac{15-7b^2}{1+kca}+\frac{15-7c^2}{1+kab} \geq \frac{24}{1+k}.$$

1.226. If a, b, c are nonnegative real numbers such that $ab+bc+ca = 3$, then

$$\frac{1}{(ka+b+c)^2}+\frac{1}{(a+kb+c)^2}+\frac{1}{(a+b+kc)^2} \leq \frac{3}{(k+2)^2}$$

for $1 \leq k \leq k_0$, where $k_0 \approx 2.82374$ is the positive root of the equation

$$k^3+6k^2-15k-28 = 0.$$

1.227. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = 1$. Prove that

$$\frac{a_1^2}{(1-a_1)^2} + \frac{a_2^2}{(1-a_2)^2} + \cdots + \frac{a_n^2}{(1-a_n)^2} \ge \left(\frac{a_1}{1-a_1} + \frac{a_2}{1-a_2} + \cdots + \frac{a_n}{1-a_n} - \frac{\sqrt{n}}{\sqrt{n}+1} \right)^2.$$

1.228. If a, b, c, d are nonnegative real numbers such that at most one of them is less than 1 and $ab + ac + ad + bc + bd + cd = 6$, then

$$\frac{1}{a+5} + \frac{1}{b+5} + \frac{1}{c+5} + \frac{1}{d+5} \ge \frac{2}{3}.$$

1.229. If $a_1, a_2, \dots, a_n$ $(n \ge 3)$ are nonnegative real number such that at most one of them is less than 1 and

$$\sum_{1 \le i < j \le n} a_i a_j = \frac{n(n-1)}{2},$$

then

$$\frac{1}{a_1 + 2n - 3} + \frac{1}{a_2 + 2n - 3} + \cdots + \frac{1}{a_n + 2n - 3} \ge \frac{n}{2(n-1)}.$$

1.230. If a, b, c, d are nonnegative real numbers such that at most one of them is larger than 1 and $ab + ac + ad + bc + bd + cd = 6$, then

$$\frac{1}{a+5} + \frac{1}{b+5} + \frac{1}{c+5} + \frac{1}{d+5} \le \frac{2}{3}.$$

1.231. If $a_1, a_2, \dots, a_n$ $(n \ge 3)$ are nonnegative real number such that at most one of them is larger than 1 and

$$\sum_{1 \le i < j \le n} a_i a_j = \frac{n(n-1)}{2},$$

then

$$\frac{1}{a_1 + 2n - 3} + \frac{1}{a_2 + 2n - 3} + \cdots + \frac{1}{a_n + 2n - 3} \le \frac{n}{2(n-1)}.$$

1.232. Let a, b, c be the lengths of the sides of a triangle such that $a + b + c = 2$. Prove that

$$\frac{a}{a^2 + bc} + \frac{b}{b^2 + ca} + \frac{c}{c^2 + ab} \ge 2.$$

1.233. Let a, b, c be the lengths of the sides of a triangle such that $a+b+c=\sqrt{2}$. Prove that

$$\frac{a}{a+bc}+\frac{b}{b+ca}+\frac{c}{c+ab} \geq 2.$$

1.234. Let

$$x=\frac{2a}{b+c}, \qquad y=\frac{2b}{c+a}, \qquad z=\frac{2c}{a+b},$$

where a, b, c are positive real numbers such that at most one of them is less than 1 and $abc=1$. Prove that

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3 \geq 2(x+y+z).$$

1.235. If $a_1 \geq a_2 \geq \cdots \geq a_n > 0$, then

$$\frac{1}{n}\left(\sqrt{a_1}-\sqrt{a_n}\right)^2 \leq \frac{a_1+a_2+\cdots+a_n}{n}-\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}} \leq \left(\sqrt{a_1}-\sqrt{a_n}\right)^2.$$

1.236. Prove that 25/17 is the largest positive value of k such that

$$\frac{1}{a^2+k}+\frac{1}{b^2+k}+\frac{1}{c^2+k}+\frac{1}{d^2+k} \geq \frac{n}{1+k}$$

for any nonnegative real numbers a, b, c, d with $ab+ac+ad+bc+bd+cd=6$ and at most one of them larger than 1.

1.237. Let a, b, c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$\frac{1}{32a^2-48a+25}+\frac{1}{32b^2-48b+25}+\frac{1}{32c^2-48c+25} \leq \frac{1}{3}.$$

1.238. If a, b, c are nonnegative real numbers such that at most one of them is less than 1 and $ab+bc+ca=3$, then

$$\frac{1}{3a^2+5}+\frac{1}{3b^2+5}+\frac{1}{3c^2+5} \leq \frac{3}{8}.$$

1.239. Let a, b, c, d be nonnegative real numbers such that $ab+ac+ad+bc+bd+cd=1$. Prove that

$$\frac{1}{a+b+c}+\frac{1}{b+c+d}+\frac{1}{c+d+a}+\frac{1}{d+a+b} \geq 3.$$

1.240. Let a, b, c be nonnegative real numbers such that $a^2 + b^2 + c^2 + ab + bc + ca = 6$. Prove that

$$\frac{1}{a^2+5} + \frac{1}{b^2+5} + \frac{1}{c^2+5} \le \frac{1}{2}.$$

1.241. If $a_1, a_2, \ldots, a_{13}$ are real numbers such that

$$a_1 + a_2 + \cdots + a_{13} = 13,$$

then

$$\frac{4a_1+7}{a_1^2-2a_1+4} + \frac{4a_2+7}{a_2^2-2a_2+4} + \cdots + \frac{4a_{13}+7}{a_{13}^2-2a_{13}+4} \le \frac{143}{3}.$$

1.2 Solutions

P 1.1. *If a, b are nonnegative real numbers, then*

$$\frac{1}{(1+a)^2}+\frac{1}{(1+b)^2}\geq\frac{1}{1+ab}.$$

First Solution. Use the Cauchy-Schwarz inequality as follows:

$$\begin{aligned}\frac{1}{(1+a)^2}+\frac{1}{(1+b)^2}-\frac{1}{1+ab}&\geq\frac{(b+a)^2}{b^2(1+a)^2+a^2(1+b)^2}-\frac{1}{1+ab}\\&=\frac{ab[a^2+b^2-2(a+b)+2]}{(1+ab)[b^2(1+a)^2+a^2(1+b)^2]}\\&=\frac{ab[(a-1)^2+(b-1)^2]}{(1+ab)[b^2(1+a)^2+a^2(1+b)^2]}\geq 0.\end{aligned}$$

The equality holds for $a=b=1$.

Second Solution. By the Cauchy-Schwarz inequality, we have

$$(a+b)\left(a+\frac{1}{b}\right)\geq(a+1)^2,\qquad(a+b)\left(\frac{1}{a}+b\right)\geq(1+b)^2,$$

hence

$$\frac{1}{(1+a)^2}+\frac{1}{(1+b)^2}\geq\frac{1}{(a+b)(a+1/b)}+\frac{1}{(a+b)(1/a+b)}=\frac{1}{1+ab}.$$

Third Solution. The desired inequality follows from the identity

$$\frac{1}{(1+a)^2}+\frac{1}{(1+b)^2}-\frac{1}{1+ab}=\frac{ab(a-b)^2+(1-ab)^2}{(1+a)^2(1+b)^2(1+ab)}.$$

Remark. Replacing a by a/x and b by and b/x, where x is a positive number, we get the inequality

$$\frac{1}{(x+a)^2}+\frac{1}{(x+b)^2}\geq\frac{1}{x^2+ab},$$

which is valid for any $x, a, b\geq 0$.

□

P 1.2. *Let a, b, c be positive real numbers such that $abc=1$. Prove that*

(a)
$$\frac{1}{2a+1}+\frac{1}{2b+1}+\frac{1}{2c+1}\geq 1;$$

(b)
$$\frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2}\leq 1.$$

Solution. (a) Use the substitution

$$a = \frac{yz}{x^2}, \qquad b = \frac{zx}{y^2}, \qquad c = \frac{xy}{z^2},$$

where $x, y, z > 0$. Applying the Cauchy-Schwarz inequality, we have

$$\sum \frac{1}{2a+1} = \sum \frac{x^2}{2yz + x^2} \ge \frac{\left(\sum x\right)^2}{\sum(2yz + x^2)} = 1.$$

The equality occurs for $a = b = c = 1$.

(b) The desired inequality follows from the inequality in (a) by replacing a, b, c with $1/a, 1/b, 1/c$, respectively. The equality holds for $a = b = c = 1$.

□

P 1.3. *If a, b, c are nonnegative real numbers such that $ab + bc + ca = 3$, then*

$$\frac{1}{1+b^2+c^2} + \frac{1}{1+c^2+a^2} + \frac{1}{1+a^2+b^2} \le 1.$$

Solution. Let us write the inequality in the form

$$\sum \frac{b^2+c^2}{1+b^2+c^2} \ge 2.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{b^2+c^2}{1+b^2+c^2} \ge \frac{(\sum \sqrt{b^2+c^2})^2}{\sum(1+b^2+c^2)} = \frac{2\sum a^2 + 2\sum \sqrt{(a^2+b^2)(a^2+c^2)}}{2\sum a^2 + 3}.$$

Since $\sqrt{(a^2+b^2)(a^2+c^2)} \ge a^2 + bc$, we get

$$\sum \frac{b^2+c^2}{1+b^2+c^2} \ge \frac{4\sum a^2 + 6}{2\sum a^2 + 3} = 2.$$

The equality occurs for $a = b = c = 1$.

□

P 1.4. *If $0 \le a, b, c \le 1$, then*

$$2\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) \ge 3\left(\frac{1}{2a+1} + \frac{1}{2b+1} + \frac{1}{2c+1}\right).$$

Solution. Write the inequality as $E(a,b,c) \ge 0$, assume that $0 \le a \le b \le c \le 1$ and show that

$$E(a,b,c) \ge E(a,b,1) \ge E(a,1,1) \ge 0.$$

The inequality $E(a,b,c) \ge E(a,b,1)$ is equivalent to

$$2\left(\frac{1}{b+c}-\frac{1}{b+1}\right)+2\left(\frac{1}{c+a}-\frac{1}{1+a}\right)-3\left(\frac{1}{2c+1}-\frac{1}{3}\right)\ge 0,$$

$$(1-c)\left[\frac{1}{(b+c)(b+1)}+\frac{1}{(c+a)(1+a)}-\frac{1}{2c+1}\right]\ge 0.$$

We have

$$\begin{aligned}\frac{1}{(b+c)(b+1)}+\frac{1}{(c+a)(1+a)}-\frac{1}{2c+1} &\ge \frac{1}{(1+c)(1+1)}+\frac{1}{(c+1)(1+1)}-\frac{1}{2c+1}\\ &= \frac{c}{(c+1)(2c+1)}>0.\end{aligned}$$

The inequality $E(a,b,1) \ge E(a,1,1)$ is equivalent to

$$2\left(\frac{1}{a+b}-\frac{1}{a+1}\right)+2\left(\frac{1}{1+b}-\frac{1}{2}\right)-3\left(\frac{1}{2b+1}-\frac{1}{3}\right),$$

$$(1-b)\left[\frac{2}{(a+b)(a+1)}+\frac{1}{1+b}-\frac{2}{2b+1}\right]\ge 0.$$

We have

$$\begin{aligned}\frac{2}{(a+b)(a+1)}+\frac{1}{1+b}-\frac{2}{2b+1} &\ge \frac{2}{(1+b)(1+1)}+\frac{1}{1+b}-\frac{2}{2b+1}\\ &= \frac{2b}{(1+b)(2b+1)}>0.\end{aligned}$$

Finally,

$$E(a,1,1)=\frac{2a(1-a)}{(a+1)(2a+1)}\ge 0.$$

The equality holds for $a=b=c=1$, and also for $a=0$ and $b=c=1$ (or any cyclic permutation).

□

P 1.5. *If a,b,c are nonnegative real numbers such that $a+b+c \le 3$, then*

$$2\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)\ge 5\left(\frac{1}{2a+3}+\frac{1}{2b+3}+\frac{1}{2c+3}\right).$$

Solution. It suffices to prove the homogeneous inequality

$$\sum \left(\frac{2}{b+c} - \frac{5}{3a+b+c} \right) \ge 0.$$

We use the SOS (sum-of-squares) method. Without loss of generality, assume that

$$a \ge b \ge c.$$

Write the inequality as follows:

$$\sum \frac{2a-b-c}{(b+c)(3a+b+c)} \ge 0,$$

$$\sum \frac{a-b}{(b+c)(3a+b+c)} + \sum \frac{a-c}{(b+c)(3a+b+c)} \ge 0,$$

$$\sum \frac{a-b}{(b+c)(3a+b+c)} + \sum \frac{b-a}{(c+a)(3b+c+a)} \ge 0,$$

$$\sum (a-b) \left(\frac{1}{(b+c)(3a+b+c)} - \frac{1}{(c+a)(3b+c+a)} \right) \ge 0,$$

$$\sum (a-b)^2(a+b-c)(a+b)(3c+a+b) \ge 0.$$

Consider the nontrivial case $a > b + c$. Since $a + b - c > 0$, it suffices to show that

$$(a-c)^2(a+c-b)(a+c)(3b+c+a) \ge (b-c)^2(a-b-c)(b+c)(3a+b+c).$$

This inequality is true since

$$(a-c)^2 \ge (b-c)^2, \qquad a+c-b \ge a-b-c$$

and

$$(a+c)(3b+c+a) \ge (b+c)(3a+b+c).$$

The last inequality is equivalent to

$$(a-b)(a+b-c) \ge 0.$$

The equality holds for $a = b = c = 1$, and also for $a = b = 3/2$ and $c = 0$ (or any cyclic permutation).

□

P 1.6. *If a, b, c are nonnegative real numbers, then*

$$\frac{a^2-bc}{3a+b+c} + \frac{b^2-ca}{3b+c+a} + \frac{c^2-ab}{3c+a+b} \ge 0.$$

Solution. We use the SOS method. Without loss of generality, assume that

$$a \geq b \geq c.$$

We have

$$\begin{aligned} 2\sum \frac{a^2-bc}{3a+b+c} &= \sum \frac{(a-b)(a+c)+(a-c)(a+b)}{3a+b+c} \\ &= \sum \frac{(a-b)(a+c)}{3a+b+c} + \sum \frac{(b-a)(b+c)}{3b+c+a} \\ &= \sum \frac{(a-b)^2(a+b-c)}{(3a+b+c)(3b+c+a)} \end{aligned}$$

Since $a+b-c \geq 0$, it suffices to show that

$$(b-c)^2(b+c-a)(3a+b+c)+(c-a)^2(c+a-b)(3b+c+a) \geq 0;$$

that is,

$$(a-c)^2(c+a-b)(3b+c+a) \geq (b-c)^2(a-b-c)(3a+b+c).$$

For the nontrivial case $a > b+c$, we can get this inequality by multiplying the obvious inequalities

$$c+a-b \geq a-b-c,$$

$$b^2(a-c)^2 \geq a^2(b-c)^2,$$

$$a(3b+c+a) \geq b(3a+b+c),$$

$$a \geq b.$$

The equality holds for $a=b=c$, and also for $a=b$ and $c=0$ (or any cyclic permutation).

□

P 1.7. *If a,b,c are positive real numbers, then*

$$\frac{4a^2-b^2-c^2}{a(b+c)} + \frac{4b^2-c^2-a^2}{b(c+a)} + \frac{4c^2-a^2-b^2}{c(a+b)} \leq 3.$$

(Vasile Cîrtoaje, 2006)

Solution *(by Boris Colakovic)*. Write the inequality as $A \geq B$, where

$$A = \left[1+\frac{b^2+c^2}{a(b+c)}\right] + \left[1+\frac{c^2+a^2}{b(c+a)}\right] + \left[1+\frac{a^2+b^2}{c(a+b)}\right],$$

$$B = \frac{4a}{b+c} + \frac{4b}{c+a} + \frac{4c}{a+b}.$$

Using the AM-GM inequality, we have

$$A = \frac{b(a+b)+c(c+a)}{a(b+c)} + \frac{c(b+c)+a(a+b)}{b(c+a)} + \frac{a(c+a)+b(b+c)}{c(a+b)}$$

$$= \left[\frac{a(c+a)}{c(a+b)} + \frac{a(a+b)}{b(c+a)}\right] + \left[\frac{b(a+b)}{a(b+c)} + \frac{b(b+c)}{c(a+b)}\right] + \left[\frac{c(b+c)}{b(c+a)} + \frac{c(c+a)}{a(b+c)}\right]$$

$$\geq \frac{2a}{\sqrt{bc}} + \frac{2b}{\sqrt{ca}} + \frac{2c}{\sqrt{ab}}$$

and

$$B \leq \frac{2a}{\sqrt{bc}} + \frac{2b}{\sqrt{ca}} + \frac{2c}{\sqrt{ab}}$$

. So, $A \geq B$. The equality occurs for $a = b = c$.

□

P 1.8. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

(a) $$\frac{1}{a^2+bc} + \frac{1}{b^2+ca} + \frac{1}{c^2+ab} \geq \frac{3}{ab+bc+ca};$$

(b) $$\frac{1}{2a^2+bc} + \frac{1}{2b^2+ca} + \frac{1}{2c^2+ab} \geq \frac{2}{ab+bc+ca}.$$

(c) $$\frac{1}{a^2+2bc} + \frac{1}{b^2+2ca} + \frac{1}{c^2+2ab} > \frac{2}{ab+bc+ca}.$$

(Vasile Cîrtoaje, 2005)

Solution. (a) Since

$$\frac{ab+bc+ca}{a^2+bc} = 1 + \frac{a(b+c-a)}{a^2+bc},$$

we can write the inequality as

$$\frac{a(b+c-a)}{a^2+bc} + \frac{b(c+a-b)}{b^2+ca} + \frac{c(a+b-c)}{c^2+ab} \geq 0.$$

Without loss of generality, assume that

$$a = \min\{a, b, c\}.$$

Since $b + c - a > 0$, it suffices to show that

$$\frac{b(c+a-b)}{b^2+ca} + \frac{c(a+b-c)}{c^2+ab} \geq 0.$$

This is equivalent to each of the following inequalities

$$(b^2+c^2)a^2 - (b+c)(b^2-3bc+c^2)a + bc(b-c)^2 \geq 0,$$

$$(b-c)^2a^2-(b+c)(b-c)^2a+bc(b-c)^2+abc(2a+b+c)\ge 0,$$
$$(b-c)^2(a-b)(a-c)+abc(2a+b+c)\ge 0.$$

The last inequality is obviously true. The equality holds for $a=0$ and $b=c$ (or any cyclic permutation thereof).

(b) Using the identities

$$2a^2+bc=a(2a-b-c)+ab+bc+ca,$$
$$2b^2+ca=b(2b-c-a)+ab+bc+ca,$$
$$2c^2+ab=c(2c-a-b)+ab+bc+ca,$$

we can write the inequality as

$$\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+z}\ge 2,$$

where

$$x=\frac{a(2a-b-c)}{ab+bc+ca},\quad y=\frac{b(2b-c-a)}{ab+bc+ca},\quad z=\frac{c(2c-a-b)}{ab+bc+ca}.$$

Without loss of generality, assume that $a=\min\{a,b,c\}$. Since

$$x\le 0,\qquad \frac{1}{1+x}\ge 1,$$

it suffices to show that

$$\frac{1}{1+y}+\frac{1}{1+z}\ge 1.$$

This is equivalent to

$$1\ge yz,$$
$$(ab+bc+ca)^2\ge bc(2b-c-a)(2c-a-b),$$
$$a^2(b^2+bc+c^2)+3abc(b+c)+2bc(b-c)^2\ge 0.$$

The last inequality is obviously true. The equality holds for $a=0$ and $b=c$ (or any cyclic permutation thereof).

(c) According to the identities

$$a^2+2bc=(a-b)(a-c)+ab+bc+ca,$$
$$b^2+2ca=(b-c)(b-a)+ab+bc+ca,$$
$$c^2+2ab=(c-a)(c-b)+ab+bc+ca,$$

we can write the inequality as

$$\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+z}>2,$$

where

$$x = \frac{(a-b)(a-c)}{ab+bc+ca}, \quad y = \frac{(b-c)(b-a)}{ab+bc+ca}, \quad z = \frac{(c-a)(c-b)}{ab+bc+ca}.$$

Since

$$xy + yz + zx = 0$$

and

$$xyz = \frac{-(a-b)^2(b-c)^2(c-a)^2}{(ab+bc+ca)^3} \le 0,$$

we have

$$\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z} - 2 = \frac{1-2xyz}{(1+x)(1+y)(1+z)} > 0.$$

□

P 1.9. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{a^2}{b^2+c^2} + \frac{b^2}{c^2+a^2} + \frac{c^2}{a^2+b^2} \ge \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}.$$

(Vasile Cîrtoaje, 2002)

Solution. Use the SOS method. We have

$$\sum \left(\frac{a^2}{b^2+c^2} - \frac{a}{b+c} \right) = \sum \frac{ab(a-b)+ac(a-c)}{(b^2+c^2)(b+c)}$$

$$= \sum \frac{ab(a-b)}{(b^2+c^2)(b+c)} + \sum \frac{ba(b-a)}{(c^2+a^2)(c+a)}$$

$$= (a^2+b^2+c^2+ab+bc+ca) \sum \frac{ab(a-b)^2}{(b^2+c^2)(c^2+a^2)(b+c)(c+a)} \ge 0.$$

The equality holds for $a = b = c$, and also for $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 1.10. *Let a, b, c be positive real numbers. Prove that*

$$\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \ge \frac{a}{a^2+bc} + \frac{b}{b^2+ca} + \frac{c}{c^2+ab}.$$

First Solution. Without loss of generality, assume that $a = \min\{a, b, c\}$. Since

$$\sum \frac{1}{b+c} - \sum \frac{a}{a^2+bc} = \sum \left(\frac{1}{b+c} - \frac{a}{a^2+bc} \right)$$
$$= \sum \frac{(a-b)(a-c)}{(b+c)(a^2+bc)}$$

and $(a-b)(a-c) \geq 0$, it suffices to show that

$$\frac{(b-c)(b-a)}{(c+a)(b^2+ca)} + \frac{(c-a)(c-b)}{(a+b)(c^2+ab)} \geq 0.$$

This inequality is equivalent to

$$(b-c)[(b^2-a^2)(c^2+ab)+(a^2-c^2)(b^2+ca)] \geq 0,$$

$$a(b-c)^2(b^2+c^2-a^2+ab+bc+ca) \geq 0.$$

The last inequality is clearly true. The equality holds for $a = b = c$.

Second Solution. Since

$$\sum \frac{1}{b+c} = \sum \left[\frac{b}{(b+c)^2} + \frac{c}{(b+c)^2} \right] = \sum a \left[\frac{1}{(a+b)^2} + \frac{1}{(a+c)^2} \right],$$

we can write the inequality as

$$\sum a \left[\frac{1}{(a+b)^2} + \frac{1}{(a+c)^2} - \frac{1}{a^2+bc} \right] \geq 0.$$

This is true since, according to Remark from P 1.1, we have

$$\frac{1}{(a+b)^2} + \frac{1}{(a+c)^2} - \frac{1}{a^2+bc} \geq 0.$$

□

P 1.11. *Let a, b, c be positive real numbers. Prove that*

$$\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \geq \frac{2a}{3a^2+bc} + \frac{2b}{3b^2+ca} + \frac{2c}{3c^2+ab}.$$

(Vasile Cîrtoaje, 2005)

Solution. Since

$$\sum \frac{1}{b+c} - \sum \frac{2a}{3a^2+bc} = \sum \left(\frac{1}{b+c} - \frac{2a}{3a^2+bc} \right)$$
$$= \sum \frac{(a-b)(a-c)+a(2a-b-c)}{(b+c)(3a^2+bc)},$$

it suffices to show that

$$\sum \frac{(a-b)(a-c)}{(b+c)(3a^2+bc)} \ge 0$$

and

$$\sum \frac{a(2a-b-c)}{(b+c)(3a^2+bc)} \ge 0.$$

In order to prove the first inequality, assume that $a = \min\{a, b, c\}$. Since

$$(a-b)(a-c) \ge 0,$$

it is enough to show that

$$\frac{(b-c)(b-a)}{(c+a)(3b^2+ca)} + \frac{(c-a)(c-b)}{(a+b)(3c^2+ab)} \ge 0.$$

This is equivalent to the obvious inequality

$$a(b-c)^2(b^2+c^2-a^2+3ab+bc+3ca) \ge 0.$$

The second inequality can be proved by the SOS method. We have

$$\begin{aligned}\sum \frac{a(2a-b-c)}{(b+c)(3a^2+bc)} &= \sum \frac{a(a-b)+a(a-c)}{(b+c)(3a^2+bc)} \\ &= \sum \frac{a(a-b)}{(b+c)(3a^2+bc)} + \sum \frac{b(b-a)}{(c+a)(3b^2+ca)} \\ &= \sum (a-b)\left[\frac{a}{(b+c)(3a^2+bc)} - \frac{b}{(c+a)(3b^2+ca)}\right] \\ &= \sum \frac{c(a-b)^2[(a-b)^2+c(a+b)]}{(b+c)(c+a)(3a^2+bc)(3b^2+ca)} \ge 0.\end{aligned}$$

The equality holds for $a = b = c$.

□

P 1.12. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

(a)
$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{13}{6} - \frac{2(ab+bc+ca)}{3(a^2+b^2+c^2)};$$

(b)
$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} - \frac{3}{2} \ge (\sqrt{3}-1)\left(1 - \frac{ab+bc+ca}{a^2+b^2+c^2}\right).$$

(Vasile Cîrtoaje, 2006)

Solution. (a) We use the SOS method. Rewrite the inequality as

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}-\frac{3}{2}\geq\frac{2}{3}\left(1-\frac{ab+bc+ca}{a^2+b^2+c^2}\right).$$

Since

$$\begin{aligned}\sum\left(\frac{a}{b+c}-\frac{1}{2}\right)&=\sum\frac{(a-b)+(a-c)}{2(b+c)}\\&=\sum\frac{a-b}{2(b+c)}+\sum\frac{b-a}{2(c+a)}\\&=\sum\frac{a-b}{2}\left(\frac{1}{b+c}-\frac{1}{c+a}\right)\\&=\sum\frac{(a-b)^2}{2(b+c)(c+a)}\end{aligned}$$

and

$$\frac{2}{3}\left(1-\frac{ab+bc+ca}{a^2+b^2+c^2}\right)=\sum\frac{(a-b)^2}{3(a^2+b^2+c^2)},$$

the inequality can be restated as

$$\sum(a-b)^2\left[\frac{1}{2(b+c)(c+a)}-\frac{1}{3(a^2+b^2+c^2)}\right]\geq 0.$$

This is true since

$$3(a^2+b^2+c^2)-2(b+c)(c+a)=(a+b-c)^2+2(a-b)^2\geq 0.$$

The equality holds for $a=b=c$.

(b) Let

$$p=a+b+c,\qquad q=ab+bc+ca,\qquad r=abc.$$

We have

$$\begin{aligned}\sum\frac{a}{b+c}&=\sum\left(\frac{a}{b+c}+1\right)-3=p\sum\frac{1}{b+c}-3\\&=\frac{p(p^2+q)}{pq-r}-3.\end{aligned}$$

According to P 3.57-(a) in Volume 1, for fixed p and q, the product r is minimum when $a=0$ or $b=c$. Therefore, it suffices to prove the inequality for $a=0$ and for $b=c=1$.

Case 1: $a=0$. The original inequality can be written as

$$\frac{b}{c}+\frac{c}{b}-\frac{3}{2}\geq(\sqrt{3}-1)\left(1-\frac{bc}{b^2+c^2}\right).$$

It suffices to show that

$$\frac{b}{c}+\frac{c}{b}-\frac{3}{2} \geq 1-\frac{bc}{b^2+c^2}.$$

Denoting

$$t=\frac{b^2+c^2}{bc}, \quad t \geq 2,$$

this inequality becomes

$$t-\frac{3}{2} \geq 1-\frac{1}{t},$$

$$(t-2)(2t-1) \geq 0.$$

Case 2: $b=c=1$. The original inequality becomes as follows:

$$\frac{a}{2}+\frac{2}{a+1}-\frac{3}{2} \geq (\sqrt{3}-1)\left(1-\frac{2a+1}{a^2+2}\right),$$

$$\frac{(a-1)^2}{2(a+1)} \geq \frac{(\sqrt{3}-1)(a-1)^2}{a^2+2},$$

$$(a-1)^2(a-\sqrt{3}+1)^2 \geq 0.$$

The equality holds for $a=b=c$, and for $\frac{a}{\sqrt{3}-1}=b=c$ (or any cyclic permutation).

□

P 1.13. *Let a,b,c be positive real numbers. Prove that*

$$\frac{1}{a^2+2bc}+\frac{1}{b^2+2ca}+\frac{1}{c^2+2ab} \leq \left(\frac{a+b+c}{ab+bc+ca}\right)^2.$$

(Vasile Cîrtoaje, 2006)

First Solution. Assume that $a \geq b \geq c$ and write the inequality as

$$\frac{(a+b+c)^2}{ab+bc+ca}-3 \geq \sum\left(\frac{ab+bc+ca}{a^2+2bc}-1\right),$$

$$\frac{(a-b)^2+(b-c)^2+(a-b)(b-c)}{ab+bc+ca}+\sum\frac{(a-b)(a-c)}{a^2+2bc} \geq 0.$$

Since

$$(a-b)(a-c) \geq 0, \qquad (c-a)(c-b) \geq 0,$$

it suffices to show that

$$(a-b)^2+(b-c)^2+(a-b)(b-c)-\frac{(ab+bc+ca)(a-b)(b-c)}{b^2+2ca} \geq 0.$$

This inequality is equivalent to

$$(a-b)^2+(b-c)^2-\frac{(a-b)^2(b-c)^2}{b^2+2ca}\ge 0,$$

$$(b-c)^2+\frac{c(a-b)^2(2a+2b-c)}{b^2+2ca}\ge 0.$$

Clearly, the last inequality is true. The equality holds for $a=b=c$.
Second Solution. Assume that $a\ge b\ge c$ and write the desired inequality as

$$\frac{(a+b+c)^2}{ab+bc+ca}-3\ge\sum\left(\frac{ab+bc+ca}{a^2+2bc}-1\right),$$

$$\frac{1}{ab+bc+ca}\sum(a-b)(a-c)+\sum\frac{(a-b)(a-c)}{a^2+2bc}\ge 0,$$

$$\sum\left(1+\frac{ab+bc+ca}{a^2+2bc}\right)(a-b)(a-c)\ge 0.$$

Since $(c-a)(c-b)\ge 0$ and $a-b\ge 0$, it suffices to prove that

$$\left(1+\frac{ab+bc+ca}{a^2+2bc}\right)(a-c)+\left(1+\frac{ab+bc+ca}{b^2+2ca}\right)(c-b)\ge 0.$$

Write this inequality as

$$a-b+(ab+bc+ca)\left(\frac{a-c}{a^2+2bc}+\frac{c-b}{b^2+2ca}\right)\ge 0,$$

$$(a-b)\left[1+\frac{(ab+bc+ca)(3ac+3bc-ab-2c^2}{(a^2+2bc)(b^2+2ca)}\right]\ge 0.$$

Since $a-b\ge 0$ and $2ac+3bc-2c^2>0$, it is enough to show that

$$1+\frac{(ab+bc+ca)(ac-ab)}{(a^2+2bc)(b^2+2ca)}\ge 0.$$

We have

$$1+\frac{(ab+bc+ca)(ac-ab)}{(a^2+2bc)(b^2+2ca)}\ge 1+\frac{(ab+bc+ca)(ac-ab)}{a^2(b^2+ca)}$$
$$=\frac{(a+b)c^2+(a^2-b^2)c}{a(b^2+ca)}>0.$$

□

P 1.14. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{a^2(b+c)}{b^2+c^2}+\frac{b^2(c+a)}{c^2+a^2}+\frac{c^2(a+b)}{a^2+b^2}\ge a+b+c.$$

(Darij Grinberg, 2004)

First Solution. Use the SOS method. We have

$$\begin{aligned}\sum\frac{a^2(b+c)}{b^2+c^2}-\sum a&=\sum\left[\frac{a^2(b+c)}{b^2+c^2}-a\right]\\&=\sum\frac{ab(a-b)+ac(a-c)}{b^2+c^2}\\&=\sum\frac{ab(a-b)}{b^2+c^2}+\sum\frac{ba(b-a)}{c^2+a^2}\\&=\sum\frac{ab(a+b)(a-b)^2}{(b^2+c^2)(c^2+a^2)}\ge 0.\end{aligned}$$

The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

Second Solution. By virtue of the Cauchy-Schwarz inequality, we have

$$\sum\frac{a^2(b+c)}{b^2+c^2}\ge\frac{\left[\sum a^2(b+c)\right]^2}{\sum a^2(b+c)(b^2+c^2)}.$$

Then, it suffices to show that

$$\left[\sum a^2(b+c)\right]^2\ge\left(\sum a\right)\left[\sum a^2(b+c)(b^2+c^2)\right].$$

Let $p=a+b+c$ and $q=ab+bc+ca$. Since

$$\begin{aligned}\left[\sum a^2(b+c)\right]^2&=(pq-3abc)^2\\&=p^2q^2-6abcpq+9a^2b^2c^2\end{aligned}$$

and

$$\begin{aligned}\sum a^2(b+c)(b^2+c^2)&=\sum(b+c)[(a^2b^2+b^2c^2+c^2a^2)-b^2c^2]\\&=2p(a^2b^2+b^2c^2+c^2a^2)-\sum b^2c^2(p-a)\\&=p(a^2b^2+b^2c^2+c^2a^2)+abcq=p(q^2-2abcp)+abcq,\end{aligned}$$

the inequality can be written as

$$p^2q^2-6abcpq+9a^2b^2c^2\ge p^2(q^2-2abcp)+abcpq,$$

$$abc(2p^3+9abc-7pq)\ge 0.$$

Using Schur's inequality

$$p^3+9abc-4pq\ge 0,$$

we have

$$2p^3+9abc-7pq\ge p(p^2-3q)\ge 0.$$

□

P 1.15. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a} \le \frac{3(a^2+b^2+c^2)}{a+b+c}.$$

Solution. Use the SOS method.

First Solution. Multiplying by $2(a+b+c)$, the inequality successively becomes:

$$\sum\left(1+\frac{a}{b+c}\right)(b^2+c^2) \le 3(a^2+b^2+c^2),$$

$$\sum \frac{a}{b+c}(b^2+c^2) \le \sum a^2,$$

$$\sum a\left(a-\frac{b^2+c^2}{b+c}\right) \ge 0,$$

$$\sum \frac{ab(a-b)-ac(c-a)}{b+c} \ge 0,$$

$$\sum \frac{ab(a-b)}{b+c} - \sum \frac{ba(a-b)}{c+a} \ge 0,$$

$$\sum \frac{ab(a-b)^2}{(b+c)(c+a)} \ge 0.$$

The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

Second Solution. Subtracting $a+b+c$ from the both sides, the desired inequality becomes as follows:

$$\frac{3(a^2+b^2+c^2)}{a+b+c}-(a+b+c) \ge \sum\left(\frac{a^2+b^2}{a+b}-\frac{a+b}{2}\right),$$

$$\sum \frac{(a-b)^2}{a+b+c} \ge \sum \frac{(a-b)^2}{2(a+b)},$$

$$\sum \frac{(a+b-c)(a-b)^2}{a+b} \ge 0.$$

Without loss of generality, assume that $a \ge b \ge c$. Since $a+b-c \ge 0$, it suffices to prove that

$$\frac{(a+c-b)(a-c)^2}{a+c} \ge \frac{(a-b-c)(b-c)^2}{b+c}.$$

This inequality is true because

$$a+c-b \ge a-b-c, \qquad a-c \ge b-c, \qquad \frac{a-c}{a+c} \ge \frac{b-c}{b+c}.$$

The last inequality reduces to $c(a-b) \ge 0$.

Third Solution. Write the inequality as follows:

$$\sum\left[\frac{3(a^2+b^2)}{2(a+b+c)}-\frac{a^2+b^2}{a+b}\right]\geq 0,$$

$$\sum\frac{(a^2+b^2)(a+b-2c)}{a+b}\geq 0,$$

$$\sum\frac{(a^2+b^2)(a-c)}{a+b}+\sum\frac{(a^2+b^2)(b-c)}{a+b}\geq 0,$$

$$\sum\frac{(a^2+b^2)(a-c)}{a+b}+\sum\frac{(b^2+c^2)(c-a)}{b+c}\geq 0,$$

$$\sum\frac{(a-c)^2(ab+bc+ca-b^2)}{(a+b)(b+c)}\geq 0.$$

It suffices to prove that

$$\sum\frac{(a-c)^2(ab+bc-ca-b^2)}{(a+b)(b+c)}\geq 0.$$

Since

$$ab+bc-ca-b^2=(a-b)(b-c),$$

this inequality is equivalent to

$$(a-b)(b-c)(c-a)\sum\frac{c-a}{(a+b)(b+c)}\geq 0,$$

which is true because

$$\sum\frac{c-a}{(a+b)(b+c)}=0.$$

□

P 1.16. *Let a,b,c be positive real numbers. Prove that*

$$\frac{1}{a^2+ab+b^2}+\frac{1}{b^2+bc+c^2}+\frac{1}{c^2+ca+a^2}\geq\frac{9}{(a+b+c)^2}.$$

(Vasile Cîrtoaje, 2000)

First Solution. Due to homogeneity, we may assume that

$$a+b+c=1.$$

Let $q=ab+bc+ca$. Since

$$b^2+bc+c^2=(a+b+c)^2-a(a+b+c)-(ab+bc+ca)=1-a-q,$$

we can write the inequality as

$$\sum \frac{1}{1-a-q} \geq 9,$$

$$9q^3 - 6q^2 - 3q + 1 + 9abc \geq 0.$$

From Schur's inequality

$$(a+b+c)^3 + 9abc \geq 4(a+b+c)(ab+bc+ca),$$

we get

$$1 + 9abc - 4q \geq 0.$$

Therefore,

$$9q^3 - 6q^2 - 3q + 1 + 9abc = (1 + 9abc - 4q) + q(3q-1)^2 \geq 0.$$

The equality holds for $a = b = c$.

Second Solution. Multiplying by $a^2 + b^2 + c^2 + ab + bc + ca$, the inequality can be written as

$$(a+b+c)\sum \frac{a}{b^2+bc+c^2} + \frac{9(ab+bc+ca)}{(a+b+c)^2} \geq 6.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a}{b^2+bc+c^2} \geq \frac{(a+b+c)^2}{\sum a(b^2+bc+c^2)} = \frac{a+b+c}{ab+bc+ca}.$$

Then, it suffices to show that

$$\frac{(a+b+c)^2}{ab+bc+ca} + \frac{9(ab+bc+ca)}{(a+b+c)^2} \geq 6.$$

This follows immediately from the AM-GM inequality.

□

P 1.17. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{a^2}{(2a+b)(2a+c)} + \frac{b^2}{(2b+c)(2b+a)} + \frac{c^2}{(2c+a)(2c+b)} \leq \frac{1}{3}.$$

(Tigran Sloyan, 2005)

First Solution. The inequality is equivalent to each of the inequalities

$$\sum \left[\frac{a^2}{(2a+b)(2a+c)} - \frac{a}{3(a+b+c)}\right] \leq 0,$$

$$\sum \frac{a(a-b)(a-c)}{(2a+b)(2a+c)} \geq 0.$$

Due to symmetry, we may consider

$$a \ge b \ge c.$$

Since $c(c-a)(c-b) \ge 0$, it suffices to prove that

$$\frac{a(a-b)(a-c)}{(2a+b)(2a+c)} + \frac{b(b-c)(b-a)}{(2b+c)(2b+a)} \ge 0.$$

This is equivalent to the obvious inequality

$$(a-b)^2[(a+b)(2ab-c^2) + c(a^2+b^2+5ab)] \ge 0.$$

The equality holds for $a = b = c$, and also for $a = b$ and $c = 0$ (or any cyclic permutation).

Second Solution (by *Vo Quoc Ba Can*). Apply the Cauchy-Schwarz inequality in the following manner

$$\frac{9a^2}{(2a+b)(2a+c)} = \frac{(2a+a)^2}{2a(a+b+c)+(2a^2+bc)} \le \frac{2a}{a+b+c} + \frac{a^2}{2a^2+bc}.$$

Then,

$$\sum \frac{9a^2}{(2a+b)(2a+c)} \le 2 + \sum \frac{a^2}{2a^2+bc} \le 3.$$

For the nontrivial case $a, b, c > 0$, the right inequality is equivalent to

$$\sum \frac{1}{2+bc/a^2} \le 1,$$

which follows immediately from P 1.2-(b).

Remark. From the inequality in P 1.17 and Hölder's inequality

$$\left[\sum \frac{a^2}{(2a+b)(2a+c)}\right]\left[\sum \sqrt{a(2a+b)(2a+c)}\right]^2 \ge (a+b+c)^3,$$

we get the following result:

- *If a, b, c are nonnegative real numbers such that $a+b+c = 3$, then*

$$\sqrt{a(2a+b)(2a+c)} + \sqrt{b(2b+c)(2b+a)} + \sqrt{c(2c+a)(2c+bc)} \ge 9,$$

with equality for $a = b = c = 1$, and for $(a,b,c) = \left(0, \frac{3}{2}, \frac{3}{2}\right)$ (or any cyclic permutation).

□

P 1.18. *Let a, b, c be positive real numbers. Prove that*

(a)
$$\sum \frac{a}{(2a+b)(2a+c)} \le \frac{1}{a+b+c};$$

(b)
$$\sum \frac{a^3}{(2a^2+b^2)(2a^2+c^2)} \le \frac{1}{a+b+c}.$$

(Vasile Cîrtoaje, 2005)

Solution. (a) Write the inequality as

$$\sum \left[\frac{1}{3} - \frac{a(a+b+c)}{(2a+b)(2a+c)}\right] \ge 0,$$

$$\sum \frac{(a-b)(a-c)}{(2a+b)(2a+c)} \ge 0.$$

Assume that

$$a \ge b \ge c.$$

Since $(a-b)(a-c) \ge 0$, it suffices to prove that

$$\frac{(b-c)(b-a)}{(2b+c)(2b+a)} + \frac{(a-c)(b-c)}{(2c+a)(2c+b)} \ge 0.$$

In addition, since $b-c \ge 0$ and $a-c \ge a-b \ge 0$, it is enough to show that

$$\frac{1}{(2c+a)(2c+b)} \ge \frac{1}{(2b+c)(2b+a)}.$$

This is equivalent to the obvious inequality

$$(b-c)(a+4b+4c) \ge 0.$$

The equality holds for $a = b = c$.

(b) We obtain the desired inequality by summing the inequalities

$$\frac{a^3}{(2a^2+b^2)(2a^2+c^2)} \le \frac{a}{(a+b+c)^2},$$

$$\frac{b^3}{(2b^2+c^2)(2b^2+a^2)} \le \frac{b}{(a+b+c)^2},$$

$$\frac{c^3}{(2c^2+a^2)(2c^2+b^2)} \le \frac{c}{(a+b+c)^2},$$

which are consequences of the Cauchy-Schwarz inequality. For example, from

$$(a^2+a^2+b^2)(c^2+a^2+a^2) \ge (ac+a^2+ba)^2,$$

the first inequality follows. The equality holds for $a = b = c$.

□

P 1.19. *If a, b, c are positive real numbers, then*

$$\sum \frac{1}{(a+2b)(a+2c)} \ge \frac{1}{(a+b+c)^2} + \frac{2}{3(ab+bc+ca)}.$$

Solution. Write the inequality as follows:

$$\sum \left[\frac{1}{(a+2b)(a+2c)} - \frac{1}{(a+b+c)^2}\right] \ge \frac{2}{3(ab+bc+ca)} - \frac{2}{(a+b+c)^2},$$

$$\sum \frac{(b-c)^2}{(a+2b)(a+2c)} \ge \sum \frac{(b-c)^2}{3(ab+bc+ca)},$$

$$(a-b)(b-c)(c-a)\sum \frac{b-c}{(a+2b)(a+2c)} \ge 0.$$

Since

$$\sum \frac{b-c}{(a+2b)(a+2c)} = \sum \left[\frac{b-c}{(a+2b)(a+2c)} - \frac{b-c}{3(ab+bc+ca)}\right]$$
$$= \frac{(a-b)(b-c)(c-a)}{3(ab+bc+ca)} \sum \frac{1}{(a+2b)(a+2c)},$$

the desired inequality is equivalent to the obvious inequality

$$(a-b)^2(b-c)^2(c-a)^2 \sum \frac{1}{(a+2b)(a+2c)} \ge 0.$$

The equality holds for $a=b$, or $b=c$, or $c=a$.

□

P 1.20. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

(a) $$\frac{1}{(a-b)^2} + \frac{1}{(b-c)^2} + \frac{1}{(c-a)^2} \ge \frac{4}{ab+bc+ca};$$

(b) $$\frac{1}{a^2-ab+b^2} + \frac{1}{b^2-bc+c^2} + \frac{1}{c^2-ca+a^2} \ge \frac{3}{ab+bc+ca};$$

(c) $$\frac{1}{a^2+b^2} + \frac{1}{b^2+c^2} + \frac{1}{c^2+a^2} \ge \frac{5}{2(ab+bc+ca)}.$$

Solution. Let

$$E_k(a,b,c) = \frac{ab+bc+ca}{a^2-kab+b^2} + \frac{ab+bc+ca}{b^2-kbc+c^2} + \frac{ab+bc+ca}{c^2-kca+a^2},$$

where $k \in [0,2]$. We will prove that

$$E_k(a,b,c) \ge \alpha_k,$$

where

$$\alpha_k = \begin{cases} \dfrac{5-2k}{2-k}, & 0 \le k \le 1 \\ 2+k, & 1 \le k \le 2 \end{cases}.$$

Assume that $a \le b \le c$ and show that

$$E_k(a,b,c) \ge E_k(0,b,c) \ge \alpha_k.$$

The left inequality is true because

$$\begin{aligned} &\frac{E_k(a,b,c) - E_k(0,b,c)}{a} = \\ &= \frac{b^2+(1+k)bc-ac}{b(a^2-kab+b^2)} + \frac{b+c}{b^2-kbc+c^2} + \frac{c^2+(1+k)bc-ab}{c(c^2-kca+a^2)} \\ &> \frac{bc-ac}{b(a^2-kab+b^2)} + \frac{b+c}{b^2-kbc+c^2} + \frac{bc-ab}{c(c^2-kca+a^2)} > 0. \end{aligned}$$

In order to prove the right inequality, $E_k(0,b,c) \ge \alpha_k$, where

$$E_k(0,b,c) = \frac{bc}{b^2-kbc+c^2} + \frac{b}{c} + \frac{c}{b},$$

we well use the AM-GM inequality. Thus, for $k \in [1,2]$, we have

$$E_k(0,b,c) = \frac{bc}{b^2-kbc+c^2} + \frac{b^2-kbc+c^2}{bc} + k \ge 2+k.$$

Also, for $k \in [0,1]$, we have

$$\begin{aligned} E_k(0,b,c) =& \frac{bc}{b^2-kbc+c^2} + \frac{b^2-kbc+c^2}{(2-k)^2bc} \\ &+ \left[1 - \frac{1}{(2-k)^2}\right]\left(\frac{b}{c} + \frac{c}{b}\right) + \frac{k}{(2-k)^2} \\ &\ge \frac{2}{2-k} + 2\left[1 - \frac{1}{(2-k)^2}\right] + \frac{k}{(2-k)^2} = \frac{5-2k}{2-k}. \end{aligned}$$

For $k \in [1,2]$, the equality holds when $a = 0$ and $\dfrac{b}{c} + \dfrac{c}{b} = 1+k$ (or any cyclic permutation). For $k \in [0,1]$, the equality holds when $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 1.21. *If a,b,c are positive real numbers, then*

$$\frac{(a^2+b^2)(a^2+c^2)}{(a+b)(a+c)} + \frac{(b^2+c^2)(b^2+a^2)}{(b+c)(b+a)} + \frac{(c^2+a^2)(c^2+b^2)}{(c+a)(c+b)} \ge a^2+b^2+c^2.$$

(Vasile Cîrtoaje, 2011)

Solution. Using the identity

$$(a^2+b^2)(a^2+c^2) = b^2c^2 + a^2(a^2+b^2+c^2),$$

we can write the inequality as follows:

$$\sum \frac{b^2c^2}{(a+b)(a+c)} \ge (a^2+b^2+c^2)\left[1 - \sum \frac{a^2}{(a+b)(a+c)}\right],$$

$$\sum b^2c^2(b+c) \ge 2abc(a^2+b^2+c^2),$$

$$\sum a^3(b^2+c^2) \ge 2\sum a^3bc,$$

$$\sum a^3(b-c)^2 \ge 0.$$

The equality holds for $a = b = c$.

□

P 1.22. *Let a, b, c be positive real numbers such that $a+b+c = 3$. Prove that*

$$\frac{1}{a^2+b+c} + \frac{1}{b^2+c+a} + \frac{1}{c^2+a+b} \le 1.$$

First Solution. By virtue of the Cauchy-Schwarz inequality, we have

$$(a^2+b+c)(1+b+c) \ge (a+b+c)^2.$$

Therefore,

$$\sum \frac{1}{a^2+b+c} \le \sum \frac{1+b+c}{(a+b+c)^2} = \frac{3+2(a+b+c)}{(a+b+c)^2} = 1.$$

The equality occurs for $a = b = c = 1$.

Second Solution. Rewrite the inequality as

$$\frac{1}{a^2-a+3} + \frac{1}{b^2-b+3} + \frac{1}{c^2-c+3} \le 1.$$

We see that the equality holds for $a = b = c = 1$. Thus, if there exists a real number k such that

$$\frac{1}{a^2-a+3} \le k + \left(\frac{1}{3} - k\right)a$$

for all $a \in [0,3]$, then

$$\sum \frac{1}{a^2-a+3} \le \sum \left[k + \left(\frac{1}{3} - k\right)a\right] = 3k + \left(\frac{1}{3} - k\right)\sum a = 1.$$

We have

$$k + \left(\frac{1}{3} - k\right) a - \frac{1}{a^2 - a + 3} = \frac{(a-1)f(a)}{3(a^2 - a + 3)},$$

where

$$f(a) = (1-3k)a^2 + 3ka + 3(1-3k).$$

From $f(1) = 0$, we get $k = 4/9$. Thus, setting $k = 4/9$, we get

$$k + \left(\frac{1}{3} - k\right) a - \frac{1}{a^2 - a + 3} = \frac{(a-1)^2(3-a)}{9(a^2 - a + 3)} \geq 0.$$

□

P 1.23. *Let a, b, c be real numbers such that $a + b + c = 3$. Prove that*

$$\frac{a^2 - bc}{a^2 + 3} + \frac{b^2 - ca}{b^2 + 3} + \frac{c^2 - ab}{c^2 + 3} \geq 0.$$

(Vasile Cîrtoaje, 2005)

Solution. Apply the SOS method. We have

$$\begin{aligned}
2\sum \frac{a^2 - bc}{a^2 + 3} &= \sum \frac{(a-b)(a+c) + (a-c)(a+b)}{a^2 + 3} \\
&= \sum \frac{(a-b)(a+c)}{a^2 + 3} + \sum \frac{(b-a)(b+c)}{b^2 + 3} \\
&= \sum (a-b)\left(\frac{a+c}{a^2+3} - \frac{b+c}{b^2+3}\right) \\
&= (3 - ab - bc - ca)\sum \frac{(a-b)^2}{(a^2+3)(b^2+3)} \geq 0.
\end{aligned}$$

Thus, it suffices to show that

$$3 - ab - bc - ca \geq 0.$$

This follows immediately from the known inequality

$$(a+b+c)^2 \geq 3(ab + bc + ca),$$

which is equivalent to

$$(a-b)^2 + (b-c)^2 + (c-a)^2 \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 1.24. *Let a,b,c be nonnegative real numbers such that $a+b+c=3$. Prove that*

$$\frac{1-bc}{5+2a}+\frac{1-ca}{5+2b}+\frac{1-ab}{5+2c}\ge 0.$$

Solution. We apply the SOS method. Since

$$9(1-bc)=(a+b+c)^2-9bc,$$

we can write the inequality as

$$\sum\frac{a^2+b^2+c^2+2a(b+c)-7bc}{5+2a}\ge 0.$$

From

$$(a-b)(a+kb+mc)+(a-c)(a+kc+mb)=$$
$$=2a^2-k(b^2+c^2)+(k+m-1)a(b+c)-2mbc,$$

choosing $k=-2$ and $m=7$, we get

$$(a-b)(a-2b+7c)+(a-c)(a-2c+7b)=2[a^2+b^2+c^2+2a(b+c)-7bc].$$

Therefore, the desired inequality becomes as follows:

$$\sum\frac{(a-b)(a-2b+7c)}{5+2a}+\sum\frac{(a-c)(a-2c+7b)}{5+2a}\ge 0,$$

$$\sum\frac{(a-b)(a-2b+7c)}{5+2a}+\sum\frac{(b-a)(b-2a+7c)}{5+2b}\ge 0,$$

$$\sum(a-b)(5+2c)[(5+2b)(a-2b+7c)-(5+2a)(b-2a+7c)]\ge 0,$$

$$\sum(a-b)^2(5+2c)(15+4a+4b-14c)\ge 0,$$

$$\sum(a-b)^2(5+2c)(a+b-c)\ge 0.$$

Without loss of generality, assume that $a\ge b\ge c$. Clearly, it suffices to show that

$$(a-c)^2(5+2b)(a+c-b)\ge(b-c)^2(5+2a)(a-b-c).$$

Since $a-c\ge b-c\ge 0$ and $a+c-b\ge a-b-c$, we only need to show that

$$(a-c)(5+2b)\ge(b-c)(5+2a).$$

Indeed,

$$(a-c)(5+2b)-(b-c)(5+2a)=(a-b)(5+2c)\ge 0.$$

The equality holds for $a=b=c=1$, and for $a=b=3/2$ and $c=0$ (or any cyclic permutation).

□

P 1.25. *Let a, b, c be nonnegative real numbers such that $a+b+c=3$. Prove that*

$$\frac{1}{a^2+b^2+2}+\frac{1}{b^2+c^2+2}+\frac{1}{c^2+a^2+2}\le\frac{3}{4}.$$

(Vasile Cîrtoaje, 2006)

Solution. Since

$$\frac{2}{a^2+b^2+2}=1-\frac{a^2+b^2}{a^2+b^2+2},$$

we may write the inequality as

$$\frac{a^2+b^2}{a^2+b^2+2}+\frac{b^2+c^2}{b^2+c^2+2}+\frac{c^2+a^2}{c^2+a^2+2}\ge\frac{3}{2}.$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\sum\frac{a^2+b^2}{a^2+b^2+2}&\ge\frac{\left(\sum\sqrt{a^2+b^2}\,\right)^2}{\sum(a^2+b^2+2)}\\&=\frac{2\sum a^2+2\sum\sqrt{(a^2+b^2)(a^2+c^2)}}{2\sum a^2+6}\\&\ge\frac{2\sum a^2+2\sum(a^2+bc)}{2\sum a^2+6}\\&=\frac{3\sum a^2+9}{2\sum a^2+6}=\frac{3}{2}.\end{aligned}$$

The equality holds for $a=b=c=1$.

□

P 1.26. *Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that*

$$\frac{1}{4a^2+b^2+c^2}+\frac{1}{4b^2+c^2+a^2}+\frac{1}{4c^2+a^2+b^2}\le\frac{1}{2}.$$

(Vasile Cîrtoaje, 2007)

Solution. According to the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\frac{9}{4a^2+b^2+c^2}&=\frac{(a+b+c)^2}{2a^2+(a^2+b^2)+(a^2+c^2)}\\&\le\frac{1}{2}+\frac{b^2}{a^2+b^2}+\frac{c^2}{a^2+c^2}.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum\frac{9}{4a^2+b^2+c^2}&\le\frac{3}{2}+\sum\left(\frac{b^2}{a^2+b^2}+\frac{c^2}{a^2+c^2}\right)\\&=\frac{3}{2}+\sum\left(\frac{b^2}{a^2+b^2}+\frac{a^2}{b^2+a^2}\right)=\frac{3}{2}+3=\frac{9}{2}.\end{aligned}$$

The equality holds for $a=b=c=1$.

Remark Similarly, we can prove the following generalization:

- *If $a_1,a_2,\dots,a_n$ are nonnegative real numbers such that $a_1+a_2+\cdots+a_n=n$, then*

$$\sum \frac{1}{(n+1)a_1^2+a_2^2+\cdots+a_n^2} \le \frac{1}{2}.$$

□

P 1.27. *Let a,b,c be nonnegative real numbers such that $a+b+c=2$. Prove that*

$$\frac{bc}{a^2+1}+\frac{ca}{b^2+1}+\frac{ab}{c^2+1} \le 1.$$

(Pham Kim Hung, 2005)

Solution. Let

$$p=a+b+c=2, \qquad q=ab+bc+ca, \qquad q \le p^2/3=4/3.$$

If $a=0$, then the inequality reduces to $4ab \le (a+b)^2$. Otherwise, for $a,b,c>0$, write the inequality as

$$\sum \frac{1}{a(a^2+1)} \le \frac{1}{abc},$$

$$\sum \left(\frac{1}{a}-\frac{a}{a^2+1}\right) \le \frac{1}{abc},$$

$$\sum \frac{a}{a^2+1} \ge \frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{1}{abc},$$

$$\sum \frac{a}{a^2+1} \ge \frac{q-1}{abc},$$

Using the inequality

$$\frac{2}{a^2+1} \ge 2-a,$$

which is equivalent to

$$a(a-1)^2 \ge 0,$$

we get

$$\sum \frac{a}{a^2+1} \ge \sum \frac{a(2-a)}{2} = \sum \frac{a(b+c)}{2} = q.$$

Therefore, it suffices to prove that

$$1+abcq \ge q.$$

By Schur's inequality of degree four, we have

$$abc \ge \frac{(p^2-q)(4q-p^2)}{6p} = \frac{(4-q)(q-1)}{3}.$$

Thus,

$$1+abcq-q \ge 1+\frac{q(4-q)(q-1)}{3}-q=\frac{(3-q)(q-1)^2}{3} \ge 0.$$

The equality holds if $a=0$ and $b=c=1$ (or any cyclic permutation).

□

P 1.28. *Let a,b,c be nonnegative real numbers such that $a+b+c=1$. Prove that*

$$\frac{bc}{a+1}+\frac{ca}{b+1}+\frac{ab}{c+1} \le \frac{1}{4}.$$

(Vasile Cîrtoaje, 2009)

First Solution. We have

$$\begin{aligned}\sum \frac{bc}{a+1} &= \sum \frac{bc}{(a+b)+(c+a)}\\ &\le \frac{1}{4}\sum bc\left(\frac{1}{a+b}+\frac{1}{c+a}\right)\\ &= \frac{1}{4}\sum \frac{bc}{a+b}+\frac{1}{4}\sum \frac{bc}{c+a}\\ &= \frac{1}{4}\sum \frac{bc}{a+b}+\frac{1}{4}\sum \frac{ca}{a+b}\\ &= \frac{1}{4}\sum \frac{bc+ca}{a+b}=\frac{1}{4}\sum c=\frac{1}{4}.\end{aligned}$$

The equality holds for $a=b=c=1/3$, and for $a=0$ and $b=c=1/2$ (or any cyclic permutation).

Second Solution. It is easy to check that the inequality is true if one of a,b,c is zero. Otherwise, write the inequality as

$$\frac{1}{a(a+1)}+\frac{1}{b(b+1)}+\frac{1}{c(c+1)} \le \frac{1}{4abc}.$$

Since

$$\frac{1}{a(a+1)}=\frac{1}{a}-\frac{1}{a+1},$$

we may write the required inequality as

$$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1} \ge \frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{1}{4abc}.$$

In virtue of the AM-HM inequality, we have

$$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1} \ge \frac{9}{(a+1)+(b+1)+(c+1)}=\frac{9}{4}.$$

Therefore, it suffices to prove that

$$\frac{9}{4} \ge \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{1}{4abc}.$$

This is equivalent to Schur's inequality

$$(a+b+c)^3 + 9abc \ge 4(a+b+c)(ab+bc+ca).$$

□

P 1.29. *Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that*

$$\frac{1}{a(2a^2+1)} + \frac{1}{b(2b^2+1)} + \frac{1}{c(2c^2+1)} \le \frac{3}{11abc}.$$

(Vasile Cîrtoaje, 2009)

Solution. Since

$$\frac{1}{a(2a^2+1)} = \frac{1}{a} - \frac{2a}{2a^2+1},$$

we can write the inequality as

$$\sum \frac{2a}{2a^2+1} \ge \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{3}{11abc}.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a}{2a^2+1} \ge \frac{(\sum a)^2}{\sum a(2a^2+1)} = \frac{1}{2(a^3+b^3+c^3)+1}.$$

Therefore, it suffices to show that

$$\frac{2}{2(a^3+b^3+c^3)+1} \ge \frac{11q-3}{11abc},$$

where

$$q = ab+bc+ca, \qquad q \le \frac{1}{3}(a+b+c)^2 = \frac{1}{3}.$$

Since

$$a^3+b^3+c^3 = 3abc + (a+b+c)^3 - 3(a+b+c)(ab+bc+ca) = 3abc + 1 - 3q,$$

we need to prove that

$$22abc \ge (11q-3)(6abc+3-6q),$$

or, equivalently,

$$2(20-33q)abc \ge 3(11q-3)(1-2q).$$

From Schur's inequality

$$(a+b+c)^3 + 9abc \geq 4(a+b+c)(ab+bc+ca),$$

we get

$$9abc \geq 4q - 1.$$

Thus,

$$2(20-33q)abc - 3(11q-3)(1-2q) \geq$$

$$\geq \frac{2(20-33q)(4q-1)}{9} - 3(11q-3)(1-2q)$$

$$= \frac{330q^2 - 233q + 41}{9} = \frac{(1-3q)(41-110q)}{9} \geq 0.$$

This completes the proof. The equality holds for $a = b = c = 1/3$.

□

P 1.30. *Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that*

$$\frac{1}{a^3+b+c} + \frac{1}{b^3+c+a} + \frac{1}{c^3+a+b} \leq 1.$$

(Vasile Cîrtoaje, 2009)

Solution. Write the inequality in the form

$$\frac{1}{a^3-a+3} + \frac{1}{b^3-b+3} + \frac{1}{c^3-c+3} \leq 1.$$

Assume that $a \geq b \geq c$. There are two cases to consider.

Case 1: $2 \geq a \geq b \geq c$. The desired inequality follows by adding the inequalities

$$\frac{1}{a^3-a+3} \leq \frac{5-2a}{9}, \quad \frac{1}{b^3-b+3} \leq \frac{5-2b}{9}, \quad \frac{1}{c^3-c+3} \leq \frac{5-2c}{9}.$$

These inequalities are true since

$$\frac{1}{a^3-a+3} - \frac{5-2a}{9} = \frac{(a-1)^2(a-2)(2a+3)}{9(a^3-a+3)} \leq 0.$$

Case 2: $a > 2$. From $a + b + c = 3$, we get $b + c < 1$. Since

$$\sum \frac{1}{a^3-a+3} < \frac{1}{a^3-a+3} + \frac{1}{3-b} + \frac{1}{3-c} < \frac{1}{9} + \frac{1}{3-b} + \frac{1}{3-c},$$

it suffices to prove that

$$\frac{1}{3-b} + \frac{1}{3-c} \leq \frac{8}{9}.$$

We have

$$\frac{1}{3-b}+\frac{1}{3-c}-\frac{8}{9}=\frac{-3-15(1-b-c)-8bc}{9(3-b)(3-c)}<0.$$

The equality holds for $a=b=c=1$.

□

P 1.31. *Let a,b,c be positive real numbers such that $a+b+c=3$. Prove that*

$$\frac{a^2}{1+b^3+c^3}+\frac{b^2}{1+c^3+a^3}+\frac{c^2}{1+a^3+b^3}\geq 1.$$

Solution. Using the Cauchy-Schwarz inequality, we have

$$\sum\frac{a^2}{1+b^3+c^3}\geq\frac{\left(\sum a^2\right)^2}{\sum a^2(1+b^3+c^3)},$$

and it remains to show that

$$(a^2+b^2+c^2)^2\geq(a^2+b^2+c^2)+\sum a^2b^2(a+b).$$

Let

$$p=a+b+c,\qquad q=ab+bc+ca,\qquad q\leq 3.$$

Since $a^2+b^2+c^2=9-2q$ and

$$\sum a^2b^2(a+b)=\sum a^2b^2(3-c)=3\sum a^2b^2-qabc=3q^2-(q+18)abc,$$

the desired inequality can be written as

$$(9-2q)^2\geq(9-2q)+3q^2-(q+18)abc,$$

$$q^2-34q+72+(q+18)abc\geq 0.$$

This inequality is clearly true for $q\leq 2$. Consider further that $2<q\leq 3$. By Schur's inequality of degree four, we get

$$abc\geq\frac{(p^2-q)(4q-p^2)}{6p}=\frac{(9-q)(4q-9)}{18}.$$

Therefore

$$q^2-34q+72+(q+18)abc\geq q^2-34q+72+\frac{(q+18)(9-q)(4q-9)}{18}$$

$$=\frac{(3-q)(4q^2+21q-54)}{18}\geq 0.$$

The equality holds for $a=b=c=1$.

□

P 1.32. *Let a, b, c be nonnegative real numbers such that $a+b+c=3$. Prove that*

$$\frac{1}{6-ab}+\frac{1}{6-bc}+\frac{1}{6-ca}\le\frac{3}{5}.$$

Solution. Rewrite the inequality as

$$108-48(ab+bc+ca)+13abc(a+b+c)-3a^2b^2c^2\ge 0,$$

$$4[9-4(ab+bc+ca)+3abc]+abc(1-abc)\ge 0.$$

By the AM-GM inequality,

$$1=\left(\frac{a+b+c}{3}\right)^3\ge abc.$$

Consequently, it suffices to show that

$$9-4(ab+bc+ca)+3abc\ge 0.$$

We see that the homogeneous form of this inequality is just Schur's inequality of third degree

$$(a+b+c)^3+9abc\ge 4(a+b+c)(ab+bc+ca).$$

The equality holds for $a=b=c=1$, as well as for $a=0$ and $b=c=3/2$ (or any cyclic permutation).

□

P 1.33. *Let a, b, c be nonnegative real numbers such that $a+b+c=3$. Prove that*

$$\frac{1}{2a^2+7}+\frac{1}{2b^2+7}+\frac{1}{2c^2+7}\le\frac{1}{3}.$$

(Vasile Cîrtoaje, 2005)

Solution. Use the mixing variables method. Assume that $a=\max\{a,b,c\}$ and prove that

$$E(a,b,c)\le E(a,s,s)\le\frac{1}{3},$$

where

$$s=\frac{b+c}{2},\quad 0\le s\le 1,$$

$$E(a,b,c)=\frac{1}{2a^2+7}+\frac{1}{2b^2+7}+\frac{1}{2c^2+7}.$$

We have

$$E(a,s,s)-E(a,b,c)=\left(\frac{1}{2s^2+7}-\frac{1}{2b^2+7}\right)+\left(\frac{1}{2s^2+7}-\frac{1}{2c^2+7}\right)$$
$$=\frac{1}{2s^2+7}\left[\frac{(b-c)(b+s)}{2b^2+7}+\frac{(c-b)(c+s)}{2c^2+7}\right]$$
$$=\frac{(b-c)^2(7-4s^2-2bc)}{(2s^2+7)(2b^2+7)(2c^2+7)}.$$

Since $bc\le s^2\le 1$, it follows that

$$7-4s^2-2bc=1+4(1-s^2)+2(1-bc)>0,$$

hence $E(a,s,s)\ge E(a,b,c)$. Also,

$$\frac{1}{3}-E(a,s,s)=\frac{1}{3}-E(3-2s,s,s)=\frac{4(s-1)^2(2s-1)^2}{3(2a^2+7)(2s^2+7)}\ge 0.$$

The equality holds for $a=b=c=1$, as well as for $a=2$ and $b=c=1/2$ (or any cyclic permutation).

□

P 1.34. *Let a,b,c be nonnegative real numbers such that $a+b+c=3$. Prove that*

$$\frac{1}{2a^2+3}+\frac{1}{2b^2+3}+\frac{1}{2c^2+3}\ge\frac{3}{5}.$$

(Vasile Cîrtoaje, 2005)

First Solution *(by Nguyen Van Quy).* Write the inequality as

$$\sum\left(\frac{1}{3}-\frac{1}{2a^2+3}\right)\le\frac{2}{5},$$

$$\sum\frac{a^2}{2a^2+5}\le\frac{3}{5}.$$

Using the Cauchy-Schwarz inequality gives

$$\frac{25}{3(2a^2+3)}=\frac{25}{6a^2+(a+b+c)^2}$$
$$=\frac{(2+2+1)^2}{2(2a^2+bc)+2a(a+b+c)+a^2+b^2+c^2}$$
$$\le\frac{2^2}{2(2a^2+bc)}+\frac{2^2}{2a(a+b+c)}+\frac{1}{a^2+b^2+c^2},$$

hence

$$\sum \frac{25a^2}{3(2a^2+3)} \le \sum \frac{2a^2}{2a^2+bc} + \sum \frac{2a}{a+b+c} + \sum \frac{a^2}{a^2+b^2+c^2}$$
$$= \sum \frac{2a^2}{2a^2+bc} + 3.$$

Therefore, it suffices to show that

$$\sum \frac{a^2}{2a^2+bc} \le 1.$$

For the nontrivial case $a, b, c > 0$, this is equivalent to

$$\sum \frac{1}{2+bc/a^2} \le 1,$$

which follows immediately from P 1.2-(b). The equality holds for $a = b = c = 1$, as well as for $a = 0$ and $b = c = 3/2$ (or any cyclic permutation).

Second Solution. First, we can check that the desired inequality becomes an equality for $a = b = c = 1$, and for $a = 0$ and $b = c = 3/2$. Consider then the inequality $f(x) \ge 0$, where

$$f(x) = \frac{1}{2x^2+3} - A - Bx, \qquad f'(x) = \frac{-4x}{(2x^2+3)^2} - B.$$

The conditions $f(1) = 0$ and $f'(1) = 0$ involve $A = 9/25$ and $B = -4/25$. Also, the conditions $f(3/2) = 0$ and $f'(3/2) = 0$ involve $A = 22/75$ and $B = -8/75$. Using these values of A and B, we obtain the identities

$$\frac{1}{2x^2+3} - \frac{9-4x}{25} = \frac{2(x-1)^2(4x-1)}{25(2x^2+3)},$$

$$\frac{1}{2x^2+3} - \frac{22-8x}{75} = \frac{(2x-3)^2(4x+1)}{75(2x^2+3)},$$

and the inequalities

$$\frac{1}{2x^2+3} \ge \frac{9-4x}{25}, \qquad x \ge \frac{1}{4},$$
$$\frac{1}{2x^2+3} \ge \frac{22-8x}{75}, \qquad x \ge 0.$$

Without loss of generality, assume that $a \ge b \ge c$.

Case 1: $a \ge b \ge c \ge \dfrac{1}{4}$. By summing the inequalities

$$\frac{1}{2a^2+3} \ge \frac{9-4a}{25}, \quad \frac{1}{2b^2+3} \ge \frac{9-4b}{25}, \quad \frac{1}{2c^2+3} \ge \frac{9-4c}{25},$$

we get

$$\frac{1}{2a^2+3}+\frac{1}{2b^2+3}+\frac{1}{2c^2+3} \ge \frac{27-4(a+b+c)}{25} = \frac{3}{5}.$$

Case 2: $a \ge b \ge \frac{1}{4} \ge c$. We have

$$\begin{aligned}\sum \frac{1}{2a^2+3} &\ge \frac{22-8a}{75}+\frac{22-8b}{75}+\frac{1}{2c^2+3}\\ &= \frac{44-8(a+b)}{75}+\frac{1}{2c^2+3} = \frac{20+8c}{75}+\frac{1}{2c^2+3}.\end{aligned}$$

Therefore, it suffices to show that

$$\frac{20+8c}{75}+\frac{1}{2c^2+3} \ge \frac{3}{5},$$

which is equivalent to the obvious inequality

$$c(8c^2-25c+12) \ge 0.$$

Case 3: $a \ge \frac{1}{4} \ge b \ge c$. We have

$$\sum \frac{1}{2a^2+3} > \frac{1}{2b^2+3}+\frac{1}{2c^2+3} \ge \frac{2}{1/8+3} > \frac{3}{5}.$$

□

P 1.35. *Let a, b, c be nonnegative real numbers such that $ab+bc+ca=3$. Prove that*

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \ge \frac{a+b+c}{6}+\frac{3}{a+b+c}.$$

(Vasile Cîrtoaje, 2007)

First Solution. Denoting

$$x = a+b+c, \qquad x \ge 3,$$

we have

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} = \frac{(a+b+c)^2+ab+bc+ca}{(a+b+c)(ab+bc+ca)-abc} = \frac{x^2+3}{3x-abc}.$$

Then, the inequality becomes

$$\frac{x^2+3}{3x-abc} \ge \frac{x}{6}+\frac{3}{x},$$

$$3(x^3+9abc-12x)+abc(x^2-9) \ge 0.$$

This inequality is true since

$$x^2 - 9 \ge 0, \qquad x^3 + 9abc - 12x \ge 0.$$

The last inequality is just Schur's inequality of degree three

$$(a+b+c)^3 + 9abc \ge 4(a+b+c)(ab+bc+ca).$$

The equality holds for $a = b = c = 1$, and for $a = 0$ and $b = c = \sqrt{3}$ (or any cyclic permutation).

Second Solution. We apply the SOS method. Write the inequality as follows:

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \ge \frac{a+b+c}{2(ab+bc+ca)} + \frac{3}{a+b+c},$$

$$2(a+b+c)\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) \ge \frac{(a+b+c)^2}{ab+bc+ca} + 6,$$

$$[(a+b)+(b+c)+(c+a)]\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) - 9 \ge \frac{(a+b+c)^2}{ab+bc+ca} - 3,$$

$$\sum \frac{(b-c)^2}{(a+b)(c+a)} \ge \frac{1}{2(ab+bc+ca)} \sum (b-c)^2,$$

$$\sum \frac{ab+bc+ca-a^2}{(a+b)(c+a)}(b-c)^2 \ge 0,$$

$$\sum \frac{3-a^2}{3+a^2}(b-c)^2 \ge 0,$$

Without loss of generality, assume that $a \ge b \ge c$. Since $3 - c^2 \ge 0$, it suffices to show that

$$\frac{3-a^2}{3+a^2}(b-c)^2 + \frac{3-b^2}{3+b^2}(c-a)^2 \ge 0.$$

Having in view that

$$3 - b^2 = ab+bc+ca-b^2 \ge b(a-b) \ge 0, \qquad (c-a)^2 \ge (b-c)^2,$$

it is enough to prove that

$$\frac{3-a^2}{3+a^2} + \frac{3-b^2}{3+b^2} \ge 0.$$

This is true since

$$\frac{3-a^2}{3+a^2} + \frac{3-b^2}{3+b^2} = \frac{2(9-a^2b^2)}{(3+a^2)(3+b^2)} = \frac{2c(a+b)(3+ab)}{(3+a^2)(3+b^2)} \ge 0.$$

□

P 1.36. *If a,b,c are nonnegative real numbers such that $ab+bc+ca=3$, then*

(a)
$$\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}\ge\frac{3}{2};$$

(b)
$$\frac{1}{a^2+2}+\frac{1}{b^2+2}+\frac{1}{c^2+2}\ge\frac{9}{10};$$

(c)
$$\frac{a(b+c)}{a^2+1}+\frac{b(c+a)}{b^2+1}+\frac{c(a+b)}{c^2+1}\le 3.$$

(Vasile Cîrtoaje, 2005)

Solution. (a) ***First Solution.*** After expanding, the inequality can be restated as

$$a^2+b^2+c^2+3\ge 3a^2b^2c^2+a^2b^2+b^2c^2+c^2a^2.$$

From
$$(a+b+c)(ab+bc+ca)-9abc=a(b-c)^2+b(c-a)^2+c(a-b)^2\ge 0,$$
we get
$$a+b+c\ge 3abc.$$

So, it suffices to show that

$$a^2+b^2+c^2+3\ge abc(a+b+c)+a^2b^2+b^2c^2+c^2a^2.$$

This is equivalent to the homogeneous inequalities

$$(ab+bc+ca)(a^2+b^2+c^2)+(ab+bc+ca)^2\ge 3abc(a+b+c)+3(a^2b^2+b^2c^2+c^2a^2),$$

$$ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2)\ge 2(a^2b^2+b^2c^2+c^2a^2),$$
$$ab(a-b)^2+bc(b-c)^2+ca(c-a)^2\ge 0.$$

The equality holds for $a=b=c=1$, and for $a=0$ and $b=c=\sqrt{3}$ (or any cyclic permutation).

Second Solution. Without loss of generality, assume that

$$a=\min\{a,b,c\},\qquad bc\ge 1.$$

From
$$(a+b+c)(ab+bc+ca)-9abc=a(b-c)^2+b(c-a)^2+c(a-b)^2\ge 0,$$
we get
$$a+b+c\ge 3abc.$$

The desired inequality follows by summing the inequalities

$$\frac{1}{b^2+1}+\frac{1}{c^2+1}\ge\frac{2}{bc+1},$$

$$\frac{1}{a^2+1}+\frac{2}{bc+1}\ge\frac{3}{2}.$$

We have

$$\frac{1}{b^2+1}+\frac{1}{c^2+1}-\frac{2}{bc+1}=\frac{b(c-b)}{(b^2+1)(bc+1)}+\frac{c(b-c)}{(c^2+1)(bc+1)}$$
$$=\frac{(b-c)^2(bc-1)}{(b^2+1)(c^2+1)(bc+1)}\ge 0$$

and

$$\frac{1}{a^2+1}+\frac{2}{bc+1}-\frac{3}{2}=\frac{a^2-bc+3-3a^2bc}{2(a^2+1)(bc+1)}=\frac{a(a+b+c-3abc)}{2(a^2+1)(bc+1)}\ge 0.$$

Third Solution. Since

$$\frac{1}{a^2+1}=1-\frac{a^2}{a^2+1},\quad \frac{1}{b^2+1}=1-\frac{b^2}{b^2+1},\quad \frac{1}{c^2+1}=1-\frac{c^2}{c^2+1},$$

we can rewrite the inequality as

$$\frac{a^2}{a^2+1}+\frac{b^2}{b^2+1}+\frac{c^2}{c^2+1}\le\frac{3}{2},$$

or, in the homogeneous form,

$$\sum\frac{a^2}{3a^2+ab+bc+ca}\le\frac{1}{2}.$$

According to the Cauchy-Schwarz inequality, we have

$$\frac{4a^2}{3a^2+ab+bc+ca}=\frac{(a+a)^2}{a(a+b+c)+(2a^2+bc)}\le\frac{a}{a+b+c}+\frac{a^2}{2a^2+bc},$$

hence

$$\sum\frac{4a^2}{3a^2+ab+bc+ca}\le 1+\sum\frac{a^2}{2a^2+bc}.$$

It suffices to show that

$$\sum\frac{a^2}{2a^2+bc}\le 1.$$

For the nontrivial case $a,b,c>0$, this is equivalent to

$$\sum\frac{1}{2+bc/a^2}\le 1,$$

which follows immediately from P 1.2-(b).

(b) After expanding, the inequality becomes

$$4(a^2+b^2+c^2)+48\ge 9a^2b^2c^2+8(a^2b^2+b^2c^2+c^2a^2).$$

From
$$(a+b+c)(ab+bc+ca)-9abc = a(b-c)^2+b(c-a)^2+c(a-b)^2 \ge 0,$$
we get
$$a+b+c \ge 3abc.$$
So, it suffices to show that
$$4(a^2+b^2+c^2)+48 \ge 3abc(a+b+c)+8(a^2b^2+b^2c^2+c^2a^2).$$
This is true if
$$4(a^2+b^2+c^2)+48 \ge 12abc(a+b+c)+8(a^2b^2+b^2c^2+c^2a^2),$$
which is equivalent to the homogeneous inequality
$$(ab+bc+ca)(a^2+b^2+c^2)+4(ab+bc+ca)^2 \ge 9abc(a+b+c)+6(a^2b^2+b^2c^2+c^2a^2),$$
i.e.
$$ab(a-b)^2+bc(b-c)^2+ca(c-a)^2 \ge 0.$$
The equality holds for $a=0$ and $b=c=\sqrt{3}$ (or any cyclic permutation).

(c) Denoting
$$p=a+b+c, \qquad q=ab+bc+ca, \qquad r=abc,$$
we may write the inequality as follows:
$$\sum\left(1-\frac{ab+ac}{a^2+1}\right) \ge 0.$$
$$\sum\frac{2a^2+1-ap}{a^2+1} \ge 0,$$
$$\sum(2a^2+1)(b^2+1)(c^2+1)-p\sum a(b^2+1)(c^2+1) \ge 0.$$
Since
$$\sum(2a^2+1)(b^2+1)(c^2+1) = 6a^2b^2c^2+5\sum a^2b^2+4\sum a^2+3$$
$$=6r^2+5(9-2pr)+4(p^2-6)+3 = 6r^2-10pr+4p^2+24$$
and
$$\sum a(b^2+1)(c^2+1) = abc\sum bc+\sum a(b^2+c^2)+\sum a$$
$$=\left(\sum a\right)\left(\sum ab\right)+\sum a = 4p,$$
the inequality becomes
$$6r^2-10pr+4p^2+24-4p^2 \ge 0,$$
$$3r^2-5pr+12 \ge 0,$$

which is equivalent to the homogeneous inequality

$$27r^2 - 15pqr + 4q^3 \geq 0.$$

We have

$$27r^2 - 15pqr + 4q^3 = 4q(q^2 - 3pr) - 3r(pq - 9r) = 2q\sum a^2(b-c)^2 - 3r\sum a(b-c)^2$$

$$= \sum a^2(2q - 3bc)(b-c)^2 = \sum(2ab - bc + 2ca)(ab - ca)^2.$$

Assuming that $a \leq b \leq c$, hence $ab \geq ca \geq bc$, we get

$$\sum(2ab - bc + 2ca)(ab - ca)^2 \geq (2bc - ca + 2ab)(bc - ab)^2 + (2ca - ab + 2bc)(ca - bc)^2$$

$$\geq (2bc - ca + 2ab)(ca - bc)^2 + (2ca - ab + 2bc)(ca - bc)^2 = (ab + 4bc + ca)(ca - bc)^2 \geq 0.$$

The equality occurs for $a = b = c = 1$.

Remark 1. We can write the inequality (a) in the homogeneous form

$$\frac{1}{1 + \dfrac{3a^2}{ab + bc + ca}} + \frac{1}{1 + \dfrac{3b^2}{ab + bc + ca}} + \frac{1}{1 + \dfrac{3c^2}{ab + bc + ca}} \geq \frac{3}{2}.$$

Using the substitutions

$$bc = x, \quad ca = y, \quad ab = z, \qquad a^2 = \frac{yz}{x}, \quad b^2 = \frac{zx}{y}, \quad c^2 = \frac{xy}{z},$$

the hypothesis and the inequality become respectively $x + y + z = 3$ and

$$\frac{1}{1 + \dfrac{3yz}{x(x + y + z)}} + \frac{1}{1 + \dfrac{3zx}{y(x + y + z)}} + \frac{1}{1 + \dfrac{3xy}{z(x + y + z}} \geq \frac{3}{2}.$$

So, we find the following result:

- *If x, y, z are nonnegative real numbers (no two of which are zero) such that $x+y+z = 3$, then*

$$\frac{x}{x + yz} + \frac{y}{y + zx} + \frac{z}{z + xy} \geq \frac{3}{2}.$$

Remark 2. Similarly, we can reformulate the statement (c) as follows:

- *If x, y, z are nonnegative real numbers (no two of which are zero) such that $x+y+z = 3$, then*

$$\frac{x(y + z)}{x + yz} + \frac{y(z + x)}{y + zx} + \frac{z(x + y)}{z + xy} \leq 3,$$

with equality when $x = y = z = 1$, and also when one of x, y, z is zero.

□

P 1.37. *Let a, b, c be positive real numbers such that $ab + bc + ca = 3$. Prove that*

$$\frac{a^2}{a^2+b+c}+\frac{b^2}{b^2+c+a}+\frac{c^2}{c^2+a+b} \ge 1.$$

(Vasile Cîrtoaje, 2005)

Solution. We apply the Cauchy-Schwarz inequality in the following way

$$\sum \frac{a^2}{a^2+b+c} \ge \frac{\left(a^{3/2}+b^{3/2}+c^{3/2}\right)^2}{\sum a(a^2+b+c)} = \frac{\sum a^3 + 2\sum (ab)^{3/2}}{\sum a^3 + 6}.$$

Then, we still have to show that

$$(ab)^{3/2} + (bc)^{3/2} + (ca)^{3/2} \ge 3.$$

By the AM-GM inequality, we have

$$(ab)^{3/2} = \frac{(ab)^{3/2}+(ab)^{3/2}+1}{2} - \frac{1}{2} \ge \frac{3ab}{2} - \frac{1}{2},$$

hence

$$(ab)^{3/2} + (bc)^{3/2} + (ca)^{3/2} \ge \frac{3}{2}(ab+bc+ca) - \frac{3}{2} = 3.$$

The equality holds for $a = b = c = 1$.

□

P 1.38. *Let a, b, c be positive real numbers such that $ab + bc + ca = 3$. Prove that*

$$\frac{bc+4}{a^2+4}+\frac{ca+4}{b^2+4}+\frac{ab+4}{c^2+4} \le 3 \le \frac{bc+2}{a^2+2}+\frac{ca+2}{b^2+2}+\frac{ab+2}{c^2+2}.$$

(Vasile Cîrtoaje, 2007)

Solution. More general, using the SOS method, we will show that

$$(k-3)\left(\frac{bc+k}{a^2+k}+\frac{ca+k}{b^2+k}+\frac{ab+k}{c^2+k}-3\right) \le 0$$

for $k > 0$. This inequality is equivalent to

$$(k-3)\sum \frac{a^2-bc}{a^2+k} \ge 0.$$

Since

$$\begin{aligned}
2\sum \frac{a^2-bc}{a^2+k} &= \sum \frac{(a-b)(a+c)+(a-c)(a+b)}{a^2+k} \\
&= \sum \frac{(a-b)(a+c)}{a^2+k} + \sum \frac{(b-a)(b+c)}{b^2+k} \\
&= (k-ab-bc-ca)\sum \frac{(a-b)^2}{(a^2+k)(b^2+k)} \\
&= (k-3)\sum \frac{(a-b)^2}{(a^2+k)(b^2+k)},
\end{aligned}$$

we have

$$2(k-3)\sum \frac{a^2-bc}{a^2+k} = (k-3)^2 \sum \frac{(a-b)^2}{(a^2+k)(b^2+k)} \ge 0.$$

The equality in both inequalities holds for $a = b = c = 1$.

□

P 1.39. *Let a, b, c be nonnegative real numbers such that $ab + bc + ca = 3$. If*

$$k \ge 2 + \sqrt{3},$$

then

$$\frac{1}{a+k} + \frac{1}{b+k} + \frac{1}{c+k} \le \frac{3}{1+k}.$$

(Vasile Cîrtoaje, 2007)

Solution. Let us denote

$$p = a + b + c, \qquad p \ge 3.$$

By expanding, the inequality becomes

$$k(k-2)p + 3abc \ge 3(k-1)^2.$$

Since this inequality is true for $p \ge 3(k-1)^2/(k^2-2k)$, consider further that

$$p \le \frac{3(k-1)^2}{k(k-2)}.$$

From Schur's inequality

$$(a+b+c)^3 + 9abc \ge 4(ab+bc+ca)(a+b+c),$$

we get

$$9abc \ge 12p - p^3.$$

Therefore, it suffices to prove that

$$3k(k-2)p + 12p - p^3 \ge 9(k-1)^2,$$

or, equivalently,

$$(p-3)[(3(k-1)^2 - p^2 - 3p] \ge 0.$$

Thus, it remains to prove that

$$3(k-1)^2 - p^2 - 3p \ge 0.$$

Since $p \le 3(k-1)^2/(k^2-2k)$ and $k \ge 2+\sqrt{3}$, we have

$$3(k-1)^2 - p^2 - 3p \ge 3(k-1)^2 - \frac{9(k-1)^4}{k^2(k-2)^2} - \frac{9(k-1)^2}{k(k-2)}$$
$$= \frac{3(k-1)^2(k^2-3)(k^2-4k+1)}{k^2(k-2)^2} \ge 0.$$

The equality holds for $a=b=c=1$. In the case $k=2+\sqrt{3}$, the equality holds also for $a=0$ and $b=c=\sqrt{3}$ (or any cyclic permutation).

□

P 1.40. *Let a,b,c be nonnegative real numbers such that $a^2+b^2+c^2=3$. Prove that*

$$\frac{a(b+c)}{1+bc} + \frac{b(c+a)}{1+ca} + \frac{c(a+b)}{1+ab} \le 3.$$

(Vasile Cîrtoaje, 2010)

Solution. Write the inequality in the homogeneous forms

$$\sum \frac{a(b+c)}{a^2+b^2+c^2+3bc} \le 1,$$

$$\sum \left[\frac{a(b+c)}{a^2+b^2+c^2+3bc} - \frac{a}{a+b+c}\right] \le 0,$$

$$\sum \frac{a(a-b)(a-c)}{a^2+b^2+c^2+3bc} \ge 0.$$

Without loss of generality, assume that $a \ge b \ge c$. Then, it suffices to prove that

$$\frac{a(a-b)(a-c)}{a^2+b^2+c^2+3bc} + \frac{b(b-c)(b-a)}{a^2+b^2+c^2+3ca} \ge 0,$$

which is true if

$$\frac{a(a-c)}{a^2+b^2+c^2+3bc} \ge \frac{b(b-c)}{a^2+b^2+c^2+3ca}.$$

Since

$$a(a-c) \ge b(b-c)$$

and

$$\frac{1}{a^2+b^2+c^2+3bc} \ge \frac{1}{a^2+b^2+c^2+3ca},$$

the conclusion follows. The equality holds for $a=b=c=1$, and for $a=b=\sqrt{3/2}$ and $c=0$ (or any cyclic permutation).

□

P 1.41. *Let a, b, c be positive real numbers such that $a^2+b^2+c^2=3$. Prove that*

$$\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}\geq 3.$$

(Cezar Lupu, 2005)

First Solution. We apply the SOS method. Write the inequality in the homogeneous forms

$$\sum\left(\frac{b^2+c^2}{b+c}-\frac{b+c}{2}\right)\geq\sqrt{3(a^2+b^2+c^2)}-a-b-c,$$

$$\sum\frac{(b-c)^2}{2(b+c)}\geq\frac{\sum(b-c)^2}{\sqrt{3(a^2+b^2+c^2)}+a+b+c}.$$

Since

$$\sqrt{3(a^2+b^2+c^2)}+a+b+c\geq 2(a+b+c)>2(b+c),$$

the conclusion follows. The equality holds for $a=b=c=1$.

Second Solution. By virtue of the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\sum\frac{a^2+b^2}{a+b}&\geq\frac{\left(\sum\sqrt{a^2+b^2}\right)^2}{\sum(a+b)}=\frac{2\sum a^2+2\sum\sqrt{(a^2+b^2)(a^2+c^2)}}{2\sum a}\\&\geq\frac{2\sum a^2+2\sum(a^2+bc)}{2\sum a}=\frac{3\sum a^2+\left(\sum a\right)^2}{2\sum a}\\&=\frac{9+\left(\sum a\right)^2}{2\sum a}=3+\frac{\left(\sum a-3\right)^2}{2\sum a}\geq 3.\end{aligned}$$

□

P 1.42. *Let a, b, c be positive real numbers such that $a^2+b^2+c^2=3$. Prove that*

$$\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a}+2\leq\frac{7}{6}(a+b+c).$$

(Vasile Cîrtoaje, 2011)

Solution. We apply the SOS method. Write the inequality as

$$3\sum\left(b+c-\frac{4bc}{b+c}\right)\geq 8(3-a-b-c).$$

Since

$$b+c-\frac{4bc}{b+c}=\frac{(b-c)^2}{b+c}$$

and

$$3-a-b-c=\frac{9-(a+b+c)^2}{3+a+b+c}=\frac{3(a^2+b^2+c^2)-(a+b+c)^2}{3+a+b+c}$$
$$=\frac{1}{3+a+b+c}\sum(b-c)^2,$$

we can write the inequality as

$$S_a(b-c)^2+S_b(c-a)^2+S_c(a-b)^2\ge 0,$$

where

$$S_a=\frac{3}{b+c}-\frac{8}{3+a+b+c}.$$

First Solution. Without loss of generality, assume that $a\ge b\ge c$, which involves $S_a\ge S_b\ge S_c$. If

$$S_b+S_c\ge 0,$$

then

$$S_a\ge S_b\ge 0,$$

hence

$$S_a(b-c)^2+S_b(c-a)^2+S_c(a-b)^2\ge S_b(c-a)^2+S_c(a-b)^2$$
$$\ge (S_b+S_c)(a-b)^2\ge 0.$$

By the AM-HM inequality, we have

$$S_b+S_c=3\left(\frac{1}{a+c}+\frac{1}{a+b}\right)-\frac{16}{3+a+b+c}$$
$$\ge\frac{12}{(a+c)+(a+b)}-\frac{16}{3+a+b+c}$$
$$=\frac{4(9-5a-b-c)}{(2a+b+c)(3+a+b+c)}.$$

Therefore,we only need to show that

$$9\ge 5a+b+c.$$

This follows immediately from the Cauchy-Schwarz inequality

$$(25+1+1)(a^2+b^2+c^2)\ge(5a+b+c)^2.$$

Thus, the proof is completed. The equality holds for $a=b=c=1$, and also for $a=5/3$ and $b=c=1/3$ (or any cyclic permutation).

Second Solution (by *Le Khanhsy*). Write the inequality as

$$3\sum\frac{(b-c)^2}{b+c}\ge\frac{8\sum(b-c)^2}{3+a+b+c}.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{(b-c)^2}{b+c} \ge \frac{\left[\sum(b-c)^2\right]^2}{\sum(b+c)(b-c)^2}.$$

Thus, it suffices to show that

$$\frac{3\left[\sum(b-c)^2\right]^2}{\sum(b+c)(b-c)^2} \ge \frac{8\sum(b-c)^2}{3+a+b+c},$$

which is true if

$$\frac{3\sum(b-c)^2}{\sum(b+c)(b-c)^2} \ge \frac{8}{3+a+b+c},$$

i.e.

$$\sum(9-A)(b-c)^2 \ge 0, \qquad A = 5b+5c-3a.$$

Since

$$9-A \ge \frac{81-A^2}{18},$$

it suffices to show that

$$\sum(81-A^2)(b-c)^2 \ge 0,$$

which is equivalent to

$$\sum \left[27(a^2+b^2+c^2)-(5b+5c-3a)^2\right](b-c)^2 \ge 0,$$

$$\sum \left[27(a^2+b^2+c^2)-(5b+5c-3a)^2-54(c-a)(a-b)\right](b-c)^2 \ge 0,$$

$$\sum(6a-b-c)^2(b-c)^2 \ge 0.$$

□

P 1.43. *Let a,b,c be positive real numbers such that $a^2+b^2+c^2=3$. Prove that*

(a)
$$\frac{1}{3-ab}+\frac{1}{3-bc}+\frac{1}{3-ca} \le \frac{3}{2};$$

(b)
$$\frac{1}{5-2ab}+\frac{1}{5-2bc}+\frac{1}{5-2ca} \le 1;$$

(c)
$$\frac{1}{\sqrt{6}-ab}+\frac{1}{\sqrt{6}-bc}+\frac{1}{\sqrt{6}-ca} \le \frac{3}{\sqrt{6}-1}.$$

(Vasile Cîrtoaje, 2005)

Solution. (a) Since

$$\frac{3}{3-ab}=1+\frac{ab}{3-ab}=1+\frac{2ab}{a^2+b^2+2c^2+(a-b)^2}$$
$$\le 1+\frac{2ab}{a^2+b^2+2c^2}\le 1+\frac{(a+b)^2}{2(a^2+b^2+2c^2)},$$

it suffices to prove that

$$\frac{(a+b)^2}{a^2+b^2+2c^2}+\frac{(b+c)^2}{b^2+c^2+2a^2}+\frac{(c+a)^2}{c^2+a^2+2b^2}\le 3.$$

By the Cauchy-Schwarz inequality, we have

$$\frac{(a+b)^2}{a^2+b^2+2c^2}=\frac{(a+b)^2}{(a^2+c^2)+(b^2+c^2)}\le\frac{a^2}{a^2+c^2}+\frac{b^2}{b^2+c^2}.$$

Thus,

$$\sum\frac{(a+b)^2}{a^2+b^2+2c^2}\le\sum\frac{a^2}{a^2+c^2}+\sum\frac{b^2}{b^2+c^2}=\sum\frac{a^2}{a^2+c^2}+\sum\frac{c^2}{c^2+a^2}=3.$$

The equality holds for $a=b=c=1$.

(b) Write the inequality in the homogeneous form

$$\sum\frac{a^2+b^2+c^2}{5(a^2+b^2+c^2)-6bc}\le 1.$$

Since

$$\frac{2(a^2+b^2+c^2)}{5(a^2+b^2+c^2)-6bc}=1-\frac{3a^2+3(b-c)^2}{5(a^2+b^2+c^2)-6bc},$$

the inequality is equivalent to

$$\sum\frac{a^2+(b-c)^2}{5(a^2+b^2+c^2)-6bc}\ge\frac{1}{3}.$$

Assume that

$$a\ge b\ge c.$$

By the Cauchy-Schwarz inequality, we have

$$\sum\frac{a^2}{5(a^2+b^2+c^2)-6bc}\ge\frac{\left(\sum a\right)^2}{\sum[5(a^2+b^2+c^2)-6bc]}=\frac{\sum a^2+2\sum ab}{15\sum a^2-6\sum ab}.$$

$$\sum\frac{(b-c)^2}{5(a^2+b^2+c^2)-6bc}\ge\frac{[(b-c)+(a-c)+(a-b)]^2}{\sum[5(a^2+b^2+c^2)-6bc]}=\frac{4(a-c)^2}{15\sum a^2-6\sum ab}.$$

Therefore, it suffices to show that

$$\frac{\sum a^2 + 2\sum ab + 4(a-c)^2}{15\sum a^2 - 6\sum ab} \ge \frac{1}{3},$$

which is equivalent to

$$\sum ab + (a-c)^2 \ge \sum a^2,$$

$$(a-b)(b-c) \ge 0.$$

(c) According to P 1.32, the following inequality holds

$$\frac{1}{6-a^2b^2} + \frac{1}{6-b^2c^2} + \frac{1}{6-c^2a^2} \le \frac{3}{5}.$$

Since

$$\frac{2\sqrt{6}}{6-a^2b^2} = \frac{1}{\sqrt{6}-ab} + \frac{1}{\sqrt{6}+ab},$$

this inequality becomes

$$\sum \frac{1}{\sqrt{6}-ab} + \sum \frac{1}{\sqrt{6}+ab} \le \frac{6\sqrt{6}}{5}.$$

Thus, it suffices to show that

$$\sum \frac{1}{\sqrt{6}+ab} \ge \frac{3}{\sqrt{6}+1}.$$

Since $ab + bc + ca \le a^2 + b^2 + c^2 = 3$, by the AM-HM inequality, we have

$$\sum \frac{1}{\sqrt{6}+ab} \ge \frac{9}{\sum(\sqrt{6}+ab)} = \frac{9}{3\sqrt{6}+ab+bc+ca} \ge \frac{3}{\sqrt{6}+1}.$$

The equality holds for $a = b = c = 1$.

□

P 1.44. *Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that*

$$\frac{1}{1+a^5} + \frac{1}{1+b^5} + \frac{1}{1+c^5} \ge \frac{3}{2}.$$

(Vasile Cîrtoaje, 2007)

Solution. Let $a = \min\{a, b, c\}$. There are two cases to consider.

Case 1: $a \ge \frac{1}{2}$. The desired inequality follows by summing the inequalities

$$\frac{8}{1+a^5} \ge 9 - 5a^2, \quad \frac{8}{1+b^5} \ge 9 - 5b^2, \quad \frac{8}{1+c^5} \ge 9 - 5c^2.$$

To obtain these inequalities, we consider the inequality

$$\frac{8}{1+x^5} \ge p + qx^2,$$

where the real coefficients p and q will be determined such that $(x-1)^2$ is a factor of the polynomial

$$P(x) = 8 - (1+x^5)(p+qx^2).$$

It is easy to check that $P(1) = 0$ involves $p + q = 4$, hence

$$P(x) = 4(2 - x^2 - x^7) - p(1 - x^2 + x^5 - x^7) = (1-x)Q(x),$$

where

$$Q(x) = 4(2 + 2x + x^2 + x^3 + x^4 + x^5 + x^6) - p(1 + x + x^5 + x^6).$$

In addition, $Q(1) = 0$ involves $p = 9$, hence

$$\begin{aligned} P(x) &= (1-x)^2(5x^5 + 10x^4 + 6x^3 + 2x^2 - 2x - 1) \\ &= (1-x)^2[x^5 + (2x-1)(2x^4 + 6x^3 + 6x^2 + 4x + 1)]. \end{aligned}$$

Clearly, we have $P(x) \ge 0$ for $x \ge \dfrac{1}{2}$.

Case 2: $a \le \dfrac{1}{2}$. Write the desired inequality as

$$\frac{1}{1+a^5} - \frac{1}{2} \ge \frac{b^5c^5 - 1}{(1+b^5)(1+c^5)}.$$

Since

$$\frac{1}{1+a^5} - \frac{1}{2} \ge \frac{32}{33} - \frac{1}{2} = \frac{31}{66}$$

and

$$(1+b^5)(1+c^5) \ge (1 + \sqrt{b^5c^5}\ )^2,$$

it suffices to show that

$$31(1 + \sqrt{b^5c^5}\ )^2 \ge 66(b^5c^5 - 1).$$

For the nontrivial case $bc > 1$, this inequality is equivalent to

$$31(1 + \sqrt{b^5c^5}\ ) \ge 66(\sqrt{b^5c^5} - 1),$$

$$bc \le (97/35)^{2/5}.$$

Indeed, from

$$3 = a^2 + b^2 + c^2 > b^2 + c^2 \ge 2bc,$$

we get

$$bc < 3/2 < (97/35)^{2/5}.$$

This completes the proof. The equality holds for $a = b = c = 1$.

□

P 1.45. *Let a, b, c be positive real numbers such that $abc = 1$. Prove that*

$$\frac{1}{a^2+a+1}+\frac{1}{b^2+b+1}+\frac{1}{c^2+c+1} \ge 1.$$

First Solution. Using the substitution

$$a = \frac{yz}{x^2}, \qquad b = \frac{zx}{y^2}, \qquad c = \frac{xy}{z^2},$$

where x, y, z are positive real numbers, the inequality becomes

$$\sum \frac{x^4}{x^4+x^2yz+y^2z^2} \ge 1.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{x^4}{x^4+x^2yz+y^2z^2} \ge \frac{\left(\sum x^2\right)^2}{\sum(x^4+x^2yz+y^2z^2)} = \frac{\sum x^4 + 2\sum y^2z^2}{\sum x^4 + xyz\sum x + \sum y^2z^2}.$$

Therefore, it suffices to show that

$$\sum y^2z^2 \ge xyz \sum x,$$

which is equivalent to $\sum x^2(y-z)^2 \ge 0$. The equality holds for $a = b = c = 1$.

Second Solution. Using the substitution

$$a = \frac{y}{x}, \qquad b = \frac{z}{y}, \qquad c = \frac{x}{z},$$

where $x, y, z > 0$, we need to prove that

$$\frac{x^2}{x^2+xy+y^2}+\frac{y^2}{y^2+yz+z^2}+\frac{z^2}{z^2+zx+z^2} \ge 1.$$

Since

$$\frac{x^2(x^2+y^2+z^2+xy+yz+zx)}{x^2+xy+y^2} = x^2 + \frac{x^2z(x+y+z)}{x^2+xy+y^2},$$

multiplying by $x^2+y^2+z^2+xy+yz+zx$, the inequality can be written as

$$\sum \frac{x^2z}{x^2+xy+y^2} \ge \frac{xy+yz+zx}{x+y+z}.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{x^2z}{x^2+xy+y^2} \ge \frac{\left(\sum xz\right)^2}{\sum z(x^2+xy+y^2)} = \frac{xy+yz+zx}{x+y+z}.$$

Remark. The inequality in P 1.45 is a particular case of the following more general inequality (*Vasile Cîrtoaje*, 2009).

- *Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be positive real numbers such that $a_1a_2\cdots a_n = 1$. If $p, q \ge 0$ such that $p + q = n - 1$, then*

$$\sum_{i=1}^{i=n} \frac{1}{1 + pa_i + qa_i^2} \ge 1.$$

□

P 1.46. *Let a, b, c be positive real numbers such that $abc = 1$. Prove that*

$$\frac{1}{a^2 - a + 1} + \frac{1}{b^2 - b + 1} + \frac{1}{c^2 - c + 1} \le 3.$$

First Solution. Since

$$\frac{1}{a^2 - a + 1} + \frac{1}{a^2 + a + 1} = \frac{2(a^2 + 1)}{a^4 + a^2 + 1} = 2 - \frac{2a^4}{a^4 + a^2 + 1},$$

we can rewrite the inequality as

$$\sum \frac{1}{a^2 + a + 1} + 2\sum \frac{a^4}{a^4 + a^2 + 1} \ge 3.$$

Thus, it suffices to show that

$$\sum \frac{1}{a^2 + a + 1} \ge 1$$

and

$$\sum \frac{a^4}{a^4 + a^2 + 1} \ge 1.$$

The first inequality is just the inequality in P 1.45, while the second follows from the first by substituting a, b, c with a^{-2}, b^{-2}, c^{-2}, respectively. The equality holds for $a = b = c = 1$.

Second Solution. Write the inequality as

$$\sum \left(\frac{4}{3} - \frac{1}{a^2 - a + 1} \right) \ge 1,$$

$$\sum \frac{(2a - 1)^2}{a^2 - a + 1} \ge 3.$$

Let $p = a + b + c$ and $q = ab + bc + ca$. By the Cauchy-Schwarz inequality, we have

$$\sum \frac{(2a - 1)^2}{a^2 - a + 1} \ge \frac{\left(2\sum a - 3\right)^2}{\sum(a^2 - a + 1)} = \frac{(2p - 3)^2}{p^2 - 2q - p + 3}.$$

Thus, it suffices to show that

$$(2p-3)^2 \geq 3(p^2-2q-p+3),$$

which is equivalent to

$$p^2+6q-9p \geq 0.$$

From the known inequality

$$(ab+bc+ca)^2 \geq 3abc(a+b+c),$$

we get $q^2 \geq 3p$. Using this inequality and the AM-GM inequality, we find

$$p^2+6q = p^2+3q+3q \geq 3\sqrt[3]{9p^2q^2} \geq 3\sqrt[3]{9p^2(3p)} = 9p.$$

□

P 1.47. *Let a,b,c be positive real numbers such that $abc=1$. Prove that*

$$\frac{3+a}{(1+a)^2}+\frac{3+b}{(1+b)^2}+\frac{3+c}{(1+c)^2} \geq 3.$$

Solution. Using the inequality in P 1.1, we have

$$\begin{aligned}\sum \frac{3+a}{(1+a)^2} &= \sum \frac{2}{(1+a)^2}+\sum \frac{1}{1+a} \\ &= \sum \left[\frac{1}{(1+a)^2}+\frac{1}{(1+b)^2}\right]+\sum \frac{1}{1+c} \\ &\geq \sum \frac{1}{1+ab}+\sum \frac{ab}{1+ab} = 3.\end{aligned}$$

The equality holds for $a=b=c=1$.

□

P 1.48. *Let a,b,c be positive real numbers such that $abc=1$. Prove that*

$$\frac{7-6a}{2+a^2}+\frac{7-6b}{2+b^2}+\frac{7-6c}{2+c^2} \geq 1.$$

(Vasile Cîrtoaje, 2008)

Solution. Write the inequality as

$$\left(\frac{7-6a}{2+a^2}+1\right)+\left(\frac{7-6b}{2+b^2}+1\right)+\left(\frac{7-6c}{2+c^2}+1\right)\ge 4,$$

$$\frac{(3-a)^2}{2+a^2}+\frac{(3-b)^2}{2+b^2}+\frac{(3-c)^2}{2+c^2}\ge 4.$$

Substituting a,b,c by $1/a,1/b,1/c$, respectively, we need to prove that $abc=1$ involves

$$\frac{(3a-1)^2}{2a^2+1}+\frac{(3b-1)^2}{2b^2+1}+\frac{(3c-1)^2}{2c^2+1}\ge 4.$$

By the Cauchy-Schwarz inequality, we have

$$\sum\frac{(3a-1)^2}{2a^2+1}\ge\frac{\left(3\sum a-3\right)^2}{\sum(2a^2+1)}=\frac{9\sum a^2+18\sum ab-18\sum a+9}{2\sum a^2+3}.$$

Thus, it suffices to prove that

$$9\sum a^2+18\sum ab-18\sum a+9\ge 4\left(2\sum a^2+3\right),$$

which is equivalent to

$$f(a)+f(b)+f(c)\ge 3,$$

where

$$f(x)=x^2+18\left(\frac{1}{x}-x\right).$$

We use the mixing variables technique. Without loss of generality, assume that

$$a=\max\{a,b,c\},\qquad a\ge 1,\quad bc\le 1.$$

Since

$$f(b)+f(c)-2f(\sqrt{bc})=(b-c)^2+18(\sqrt{b}-\sqrt{c})^2\left(\frac{1}{bc}-1\right)\ge 0,$$

it suffices to show that

$$f(a)+2f(\sqrt{bc})\ge 3,$$

which is equivalent to

$$f(x^2)+2f\left(\frac{1}{x}\right)\ge 3,\qquad x=\sqrt{a},$$

$$x^6-18x^4+36x^3-3x^2-36x+20\ge 0,$$

$$(x-1)^2(x-2)^2(x+1)(x+5)\ge 0.$$

The equality holds for $a=b=c=1$, and also for $a=1/4$ and $b=c=2$ (or any cyclic permutation).

□

P 1.49. *Let a, b, c be positive real numbers such that $abc = 1$. Prove that*

$$\frac{a^6}{1+2a^5} + \frac{b^6}{1+2b^5} + \frac{c^6}{1+2c^5} \geq 1.$$

(Vasile Cîrtoaje, 2008)

Solution. Using the substitutions

$$a = \sqrt[3]{\frac{x^2}{yz}}, \quad b = \sqrt[3]{\frac{y^2}{zx}}, \quad c = \sqrt[3]{\frac{z^2}{xy}},$$

the inequality becomes

$$\sum \frac{x^4}{y^2z^2 + 2x^3\sqrt[3]{xyz}} \geq 1.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{x^4}{y^2z^2 + 2x^3\sqrt[3]{xyz}} \geq \frac{(\sum x^2)^2}{\sum(y^2z^2 + 2x^3\sqrt[3]{xyz})} = \frac{(\sum x^2)^2}{\sum x^2y^2 + 2\sqrt[3]{xyz}\sum x^3}.$$

Therefore, we only need to show that

$$\left(\sum x^2\right)^2 \geq \sum x^2y^2 + 2\sqrt[3]{xyz}\sum x^3.$$

Since, by the AM-GM inequality,

$$x + y + z \geq 3\sqrt[3]{xyz},$$

it suffices to prove that

$$3(\sum x^2)^2 \geq 3\sum x^2y^2 + 2(\sum x)(\sum x^3);$$

that is,

$$\sum x^4 + 3\sum x^2y^2 \geq 2\sum xy(x^2+y^2),$$

$$\sum (x-y)^4 \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 1.50. *Let a, b, c be positive real numbers such that $abc = 1$. Prove that*

$$\frac{a}{a^2+5} + \frac{b}{b^2+5} + \frac{c}{c^2+5} \leq \frac{1}{2}.$$

(Vasile Cîrtoaje, 2008)

Solution. Let

$$F(a,b,c) = \frac{a}{a^2+5} + \frac{b}{b^2+5} + \frac{c}{c^2+5}.$$

Without loss of generality, assume that $a = \min\{a,b,c\}$.

Case 1: $a \le 1/5$. We have

$$F(a,b,c) < \frac{a}{5} + \frac{b}{2\sqrt{5b^2}} + \frac{c}{2\sqrt{5c^2}} \le \frac{1}{25} + \frac{1}{\sqrt{5}} < \frac{1}{2}.$$

Case 2: $a > 1/5$. Use the mixing variables method. We will show that

$$F(a,b,c) \le F(a,x,x) \le \frac{1}{2},$$

where

$$x = \sqrt{bc}, \qquad a = 1/x^2, \quad x < \sqrt{5}.$$

The left inequality, $F(a,b,c) \le F(a,x,x)$, is equivalent to

$$(\sqrt{b}-\sqrt{c})^2[10x(b+c)+10x^2-25-x^4] \ge 0.$$

This is true since

$$10x(b+c)+10x^2-25-x^4 \ge 20x^2+10x^2-25x^2-x^4 = x^2(5-x^2) > 0.$$

The right inequality, $F(a,x,x) \le \frac{1}{2}$, is equivalent to

$$(x-1)^2(5x^4-10x^3-2x^2+6x+5) \ge 0.$$

It is also true since

$$5x^4-10x^3-2x^2+6x+5 = 5(x-1)^4+2x(5x^2-16x+13)$$

and

$$5x^2+13 \ge 2\sqrt{65x^2} > 16x.$$

The equality holds for $a=b=c=1$.

□

P 1.51. *Let a,b,c be positive real numbers such that $abc=1$. Prove that*

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{2}{(1+a)(1+b)(1+c)} \ge 1.$$

(Pham Van Thuan, 2006)

First Solution. There are two of a, b, c either larger than or equal to 1, or less than or equal to 1. Let b and c be these numbers; that is, $(1-b)(1-c) \geq 0$. Since

$$\frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} \geq \frac{1}{1+bc}$$

(see P 1.1), it suffices to show that

$$\frac{1}{(1+a)^2} + \frac{1}{1+bc} + \frac{2}{(1+a)(1+b)(1+c)} \geq 1.$$

This inequality is equivalent to

$$\frac{b^2c^2}{(1+bc)^2} + \frac{1}{1+bc} + \frac{2bc}{(1+bc)(1+b)(1+c)} \geq 1,$$

which can be written in the obvious form

$$\frac{bc(1-b)(1-c)}{(1+bc)(1+b)(1+c)} \geq 0.$$

The equality holds for $a = b = c = 1$.

Second Solution. Setting

$$a = yz/x^2, \quad b = zx/y^2, \quad c = xy/z^2,$$

where $x, y, z > 0$, the inequality becomes

$$\sum \frac{x^4}{(x^2+yz)^2} + \frac{2x^2y^2z^2}{(x^2+yz)(y^2+zx)(z^2+xy)} \geq 1.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{x^4}{(x^2+yz)^2} \geq \sum \frac{x^4}{(x^2+y^2)(x^2+z^2)} = 1 - \frac{2x^2y^2z^2}{(x^2+y^2)(y^2+z^2)(z^2+x^2)}.$$

Then, it suffices to show that

$$(x^2+y^2)(y^2+z^2)(z^2+x^2) \geq (x^2+yz)(y^2+zx)(z^2+xy).$$

This inequality follows by multiplying the inequalities

$$(x^2+y^2)(x^2+z^2) \geq (x^2+yz)^2,$$

$$(y^2+z^2)(y^2+x^2) \geq (y^2+zx)^2,$$

$$(z^2+x^2)(z^2+y^2) \geq (z^2+xy)^2.$$

Third Solution. We make the substitution

$$\frac{1}{1+a} = \frac{1+x}{2}, \quad \frac{1}{1+b} = \frac{1+y}{2}, \quad \frac{1}{1+c} = \frac{1+z}{2},$$

which is equivalent to

$$a = \frac{1-x}{1+x}, \quad b = \frac{1-y}{1+y}, \quad c = \frac{1-z}{1+z},$$

where

$$-1 < x, y, z < 1, \qquad x + y + z + xyz = 0.$$

The desired inequality becomes

$$(1+x)^2 + (1+y)^2 + (1+z)^2 + (1+x)(1+y)(1+z) \geq 4,$$

$$x^2 + y^2 + z^2 + (x+y+z)^2 + 4(x+y+z) \geq 0.$$

By virtue of the AM-GM inequality, we have

$$x^2 + y^2 + z^2 + (x+y+z)^2 + 4(x+y+z) = x^2 + y^2 + z^2 + x^2y^2z^2 - 4xyz$$

$$\geq 4\sqrt[4]{x^4y^4z^4} - 4xyz = 4|xyz| - 4xyz \geq 0.$$

□

P 1.52. *Let a, b, c be nonnegative real numbers such that*

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} = \frac{3}{2}.$$

Prove that

$$\frac{3}{a+b+c} \geq \frac{2}{ab+bc+ca} + \frac{1}{a^2+b^2+c^2}.$$

Solution. Write the inequality in the homogeneous form

$$\frac{2}{a+b+c}\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) \geq \frac{2}{ab+bc+ca} + \frac{1}{a^2+b^2+c^2}.$$

Due to homogeneity, we may assume that

$$a + b + c = 1, \qquad 0 \leq a, b, c < 1.$$

Denote $q = ab + bc + ca$. From the known inequality $(a+b+c)^2 \geq 3(ab+bc+ca)$, we get

$$1 - 3q \geq 0.$$

Rewrite the desired inequality as follows:

$$2\left(\frac{1}{1-c} + \frac{1}{1-a} + \frac{1}{1-b}\right) \geq \frac{2}{q} + \frac{1}{1-2q},$$

$$\frac{2(q+1)}{q-abc} \geq \frac{2-3q}{q(1-2q)},$$

$$q^2(1-4q)+(2-3q)abc \ge 0.$$

By Schur's inequality, we have

$$(a+b+c)^3+9abc \ge 4(a+b+c)(ab+bc+ca),$$

$$1-4q \ge -9abc.$$

Then,

$$\begin{aligned} q^2(1-4q)+(2-3q)abc &\ge -9q^2abc+(2-3q)abc \\ &= (1-3q)(2+3q)abc \ge 0. \end{aligned}$$

The equality holds for $a=b=c=1$, and for $a=0$ and $b=c=\frac{5}{3}$ (or any cyclic permutation).

□

P 1.53. *Let a,b,c be nonnegative real numbers such that*

$$7(a^2+b^2+c^2)=11(ab+bc+ca).$$

Prove that

$$\frac{51}{28} \le \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \le 2.$$

Solution. Due to homogeneity, we may assume that $b+c=2$. Let us denote

$$x=bc, \quad 0 \le x \le 1.$$

By the hypothesis $7(a^2+b^2+c^2)=11(ab+bc+ca)$, we get

$$x=\frac{7a^2-22a+28}{25}.$$

Notice that the condition $x \le 1$ involves

$$\frac{1}{7} \le a \le 3.$$

Since

$$\begin{aligned} \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} &= \frac{a}{b+c}+\frac{a(b+c)+(b+c)^2-2bc}{a^2+(b+c)a+bc} \\ &= \frac{a}{2}+\frac{2(a+2-x)}{a^2+2a+x}=\frac{4a^3+27a+11}{8a^2+7a+7}, \end{aligned}$$

the required inequalities become

$$\frac{51}{28} \le \frac{4a^3+27a+11}{8a^2+7a+7} \le 2.$$

We have

$$\frac{4a^3+27a+11}{8a^2+7a+7} - \frac{51}{28} = \frac{(7a-1)(4a-7)^2}{28(8a^2+7a+7)} \ge 0$$

and

$$2 - \frac{4a^3+27a+11}{8a^2+7a+7} = \frac{(3-a)(2a-1)^2}{8a^2+7a+7} \ge 0.$$

This completes the proof. The left inequality becomes an equality for $7a = b = c$ (or any cyclic permutation), while the right inequality is an equality for $\frac{a}{3} = b = c$ (or any cyclic permutation).

□

P 1.54. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{1}{a^2+b^2} + \frac{1}{b^2+c^2} + \frac{1}{c^2+a^2} \ge \frac{10}{(a+b+c)^2}.$$

Solution. Assume that $a = \min\{a, b, c\}$, and denote

$$x = b + \frac{a}{2}, \quad y = c + \frac{a}{2}.$$

Since

$$a^2+b^2 \le x^2, \quad b^2+c^2 \le x^2+y^2, \quad c^2+a^2 \le y^2,$$
$$(a+b+c)^2 = (x+y)^2 \ge 4xy,$$

it suffices to show that

$$\frac{1}{x^2} + \frac{1}{x^2+y^2} + \frac{1}{y^2} \ge \frac{5}{2xy}.$$

We have

$$\frac{1}{x^2} + \frac{1}{x^2+y^2} + \frac{1}{y^2} - \frac{5}{2xy} = \left(\frac{1}{x^2} + \frac{1}{y^2} - \frac{2}{xy}\right) + \left(\frac{1}{x^2+y^2} - \frac{1}{2xy}\right)$$
$$= \frac{(x-y)^2}{x^2y^2} - \frac{(x-y)^2}{2xy(x^2+y^2)}$$
$$= \frac{(x-y)^2(2x^2-xy+2y^2)}{2x^2y^2(x^2+y^2)} \ge 0.$$

The equality holds for $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 1.55. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{1}{a^2-ab+b^2}+\frac{1}{b^2-bc+c^2}+\frac{1}{c^2-ca+a^2}\ge\frac{3}{\max\{ab,bc,ca\}}.$$

Solution. Assume that

$$a=\min\{a,b,c\},\qquad bc=\max\{ab,bc,ca\}.$$

Since

$$\frac{1}{a^2-ab+b^2}+\frac{1}{b^2-bc+c^2}+\frac{1}{c^2-ca+a^2}\ge\frac{1}{b^2}+\frac{1}{b^2-bc+c^2}+\frac{1}{c^2},$$

it suffices to show that

$$\frac{1}{b^2}+\frac{1}{b^2-bc+c^2}+\frac{1}{c^2}\ge\frac{3}{bc}.$$

We have

$$\frac{1}{b^2}+\frac{1}{b^2-bc+c^2}+\frac{1}{c^2}-\frac{3}{bc}=\frac{(b-c)^4}{b^2c^2(b^2-bc+c^2)}\ge 0.$$

The equality holds for $a=b=c$, and also $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.56. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{a(2a+b+c)}{b^2+c^2}+\frac{b(2b+c+a)}{c^2+a^2}+\frac{c(2c+a+b)}{a^2+b^2}\ge 6.$$

Solution. By the Cauchy-Schwarz inequality, we have

$$\sum\frac{a(2a+b+c)}{b^2+c^2}\ge\frac{\left[\sum a(2a+b+c)\right]^2}{\sum a(2a+b+c)(b^2+c^2)}.$$

Thus, we still need to show that

$$2\left(\sum a^2+\sum ab\right)^2\ge 3\sum a(2a+b+c)(b^2+c^2),$$

which is equivalent to

$$2\sum a^4+2abc\sum a+\sum ab(a^2+b^2)\ge 6\sum a^2b^2.$$

We can obtain this inequality by adding Schur's inequality of degree four

$$\sum a^4+abc\sum a\ge\sum ab(a^2+b^2)$$

and

$$\sum ab(a^2+b^2)\ge 2\sum a^2b^2,$$

multiplied by 2 and 3, respectively. The equality occurs for $a=b=c$, and for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.57. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{a^2(b+c)^2}{b^2+c^2}+\frac{b^2(c+a)^2}{c^2+a^2}+\frac{c^2(a+b)^2}{a^2+b^2}\ge 2(ab+bc+ca).$$

Solution. We apply the SOS method. Since

$$\frac{a^2(b+c)^2)}{b^2+c^2}=a^2+\frac{2a^2bc}{b^2+c^2},$$

we can write the inequality as

$$2\left(\sum a^2-\sum ab\right)-\sum a^2\left(1-\frac{2bc}{b^2+c^2}\right)\ge 0,$$

$$\sum (b-c)^2-\sum\frac{a^2(b-c)^2}{b^2+c^2}\ge 0,$$

$$\sum\left(1-\frac{a^2}{b^2+c^2}\right)(b-c)^2\ge 0.$$

Without loss of generality, assume that $a\ge b\ge c$. Since $1-\dfrac{c^2}{a^2+b^2}>0$, it suffices to prove that

$$\left(1-\frac{a^2}{b^2+c^2}\right)(b-c)^2+\left(1-\frac{b^2}{c^2+a^2}\right)(a-c)^2\ge 0,$$

which is equivalent to

$$\frac{(a^2-b^2+c^2)(a-c)^2}{a^2+c^2}\ge\frac{(a^2-b^2-c^2)(b-c)^2}{b^2+c^2}.$$

This inequality follows by multiplying the inequalities

$$a^2-b^2+c^2\ge a^2-b^2-c^2,\qquad \frac{(a-c)^2}{a^2+c^2}\ge\frac{(b-c)^2}{b^2+c^2}.$$

The latter inequality is true since

$$\frac{(a-c)^2}{a^2+c^2}-\frac{(b-c)^2}{b^2+c^2}=\frac{2bc}{b^2+c^2}-\frac{2ac}{a^2+c^2}=\frac{2c(a-b)(ab-c^2)}{(b^2+c^2)(a^2+c^2)}\ge 0.$$

The equality occurs for $a=b=c$, and for $a=b$ and $c=0$ (or any cyclic permutation).

□

P 1.58. *If a, b, c are positive real numbers, then*

$$3\sum\frac{a}{b^2-bc+c^2}+5\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\right)\ge 8\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

(Vasile Cîrtoaje, 2011)

Solution. In order to apply the SOS method, we multiply the inequality by abc and write it as follows:

$$8\left(\sum a^2 - \sum bc\right) - 3\sum a^2\left(1 - \frac{bc}{b^2 - bc + c^2}\right) \geq 0,$$

$$4\sum (b-c)^2 - 3\sum \frac{a^2(b-c)^2}{b^2 - bc + c^2} \geq 0,$$

$$\sum \frac{(b-c)^2(4b^2 - 4bc + 4c^2 - 3a^2)}{b^2 - bc + c^2} \geq 0.$$

Without loss of generality, assume that $a \geq b \geq c$. Since

$$4a^2 - 4ab + 4b^2 - 3c^2 = (2a-b)^2 + 3(b^2 - c^2) \geq 0,$$

it suffices to prove that

$$\frac{(a-c)^2(4a^2 - 4ac + 4c^2 - 3b^2)}{a^2 - ac + c^2} \geq \frac{(b-c)^2(3a^2 - 4b^2 + 4bc - 4c^2)}{b^2 - bc + c^2}.$$

Notice that

$$4a^2 - 4ac + 4c^2 - 3b^2 = (a - 2c)^2 + 3(a^2 - b^2) \geq 0.$$

Thus, the desired inequality follows by multiplying the inequalities

$$4a^2 - 4ac + 4c^2 - 3b^2 \geq 3a^2 - 4b^2 + 4bc - 4c^2$$

and

$$\frac{(a-c)^2}{a^2 - ac + c^2} \geq \frac{(b-c)^2}{b^2 - bc + c^2}.$$

The first inequality is equivalent to

$$(a - 2c)^2 + (b - 2c)^2 \geq 0.$$

Also, we have

$$\frac{(a-c)^2}{a^2 - ac + c^2} - \frac{(b-c)^2}{b^2 - bc + c^2} = \frac{bc}{b^2 - bc + c^2} - \frac{ac}{a^2 - ac + c^2}$$

$$= \frac{c(a-b)(ab - c^2)}{(b^2 - bc + c^2)(a^2 - ac + c^2)} \geq 0.$$

The equality occurs for $a = b = c$, and for $2a = b = c$ (or any cyclic permutation).

□

P 1.59. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

(a) $$2abc\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) + a^2 + b^2 + c^2 \geq 2(ab + bc + ca);$$

(b) $$\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+a} \leq \frac{3(a^2 + b^2 + c^2)}{2(a+b+c)}.$$

Solution. (a) **First Solution.** We have

$$\begin{aligned} 2abc\sum\frac{1}{b+c}+\sum a^2 &= \sum\frac{a(2bc+ab+ac)}{b+c} \\ &= \sum\frac{ab(a+c)}{b+c}+\sum\frac{ac(a+b)}{b+c} \\ &= \sum\frac{ab(a+c)}{b+c}+\sum\frac{ba(b+c)}{c+a} \\ &= \sum ab\left(\frac{a+c}{b+c}+\frac{b+c}{a+c}\right)\ge 2\sum ab. \end{aligned}$$

The equality occurs for $a=b=c$, and for $a=0$ and $b=c$ (or any cyclic permutation).

Second Solution. Write the inequality as

$$\sum\left(\frac{2abc}{b+c}+a^2-ab-ac\right)\ge 0.$$

We have

$$\begin{aligned} \sum\left(\frac{2abc}{b+c}+a^2-ab-ac\right) &= \sum\frac{ab(a-b)+ac(a-c)}{b+c} \\ &= \sum\frac{ab(a-b)}{b+c}+\sum\frac{ba(b-a)}{c+a} \\ &= \sum\frac{ab(a-b)^2}{(b+c)(c+a)}\ge 0. \end{aligned}$$

(b) Since

$$\sum\frac{a^2}{a+b}=\sum\left(a-\frac{ab}{a+b}\right)=a+b+c-\sum\frac{ab}{a+b},$$

we can write the desired inequality as

$$\sum\frac{ab}{a+b}+\frac{3(a^2+b^2+c^2)}{2(a+b+c)}\ge a+b+c.$$

Multiplying by $2(a+b+c)$, the inequality can be written as

$$2\sum\left(1+\frac{a}{b+c}\right)bc+3(a^2+b^2+c^2)\ge 2(a+b+c)^2,$$

or

$$2abc\sum\frac{1}{b+c}+a^2+b^2+c^2\ge 2(ab+bc+ca),$$

which is just the inequality in (a).

□

P 1.60. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

(a) $$\frac{a^2-bc}{b^2+c^2}+\frac{b^2-ca}{c^2+a^2}+\frac{c^2-ab}{a^2+b^2}+\frac{3(ab+bc+ca)}{a^2+b^2+c^2}\ge 3;$$

(b) $$\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2}+\frac{ab+bc+ca}{a^2+b^2+c^2}\ge \frac{5}{2};$$

(c) $$\frac{a^2+bc}{b^2+c^2}+\frac{b^2+ca}{c^2+a^2}+\frac{c^2+ab}{a^2+b^2}\ge \frac{ab+bc+ca}{a^2+b^2+c^2}+2.$$

(Vasile Cîrtoaje, 2014)

Solution. (a) Use the SOS method. Write the inequality as follows:

$$\sum\left(\frac{2a^2}{b^2+c^2}-1\right)+\sum\left(1-\frac{2bc}{b^2+c^2}\right)-6\left(1-\frac{ab+bc+ca}{a^2+b^2+c^2}\right)\ge 0,$$

$$\sum\frac{2a^2-b^2-c^2}{b^2+c^2}+\sum\frac{(b-c)^2}{b^2+c^2}-3\sum\frac{(b-c)^2}{a^2+b^2+c^2}\ge 0.$$

Since

$$\sum\frac{2a^2-b^2-c^2}{b^2+c^2}=\sum\frac{a^2-b^2}{b^2+c^2}+\sum\frac{a^2-c^2}{b^2+c^2}=\sum\frac{a^2-b^2}{b^2+c^2}+\sum\frac{b^2-a^2}{c^2+a^2}$$
$$=\sum\frac{(a^2-b^2)^2}{(b^2+c^2)(c^2+a^2)}=\sum\frac{(b^2-c^2)^2}{(a^2+b^2)(a^2+c^2)},$$

we can write the inequality as

$$\sum(b-c)^2S_a\ge 0,$$

where

$$S_a=\frac{(b+c)^2}{(a^2+b^2)(a^2+c^2)}+\frac{1}{b^2+c^2}-\frac{3}{a^2+b^2+c^2}.$$

It suffices to show that $S_a, S_b, S_c\ge 0$ for all nonnegative real numbers a, b, c, no two of which are zero. Denoting $x^2=b^2+c^2$, we have

$$S_a=\frac{x^2+2bc}{a^4+a^2x^2+b^2c^2}+\frac{1}{x^2}-\frac{3}{a^2+x^2},$$

and the inequality $S_a\ge 0$ becomes

$$(a^2-2x^2)b^2c^2+2x^2(a^2+x^2)bc+(a^2+x^2)(a^2-x^2)^2\ge 0.$$

Clearly, this is true if

$$-2x^2b^2c^2+2x^4bc\ge 0.$$

Indeed,

$$-2x^2b^2c^2+2x^4bc=2x^2bc(x^2-bc)=2bc(b^2+c^2)(b^2+c^2-bc)\ge 0.$$

The equality occurs for $a=b=c$, and for $a=0$ and $b=c$ (or any cyclic permutation).

(b) ***First Solution.*** We get the desired inequality by summing the inequality in (a) and the inequality

$$\frac{bc}{b^2+c^2}+\frac{ca}{c^2+a^2}+\frac{ab}{a^2+b^2}+\frac{1}{2}\geq\frac{2(ab+bc+ca)}{a^2+b^2+c^2}.$$

This inequality is equivalent to

$$\sum\left(\frac{2bc}{b^2+c^2}+1\right)\geq\frac{4(ab+bc+ca)}{a^2+b^2+c^2}+2,$$

$$\sum\frac{(b+c)^2}{b^2+c^2}\geq\frac{2(a+b+c)^2}{a^2+b^2+c^2}.$$

By the Cauchy-Schwarz inequality, we have

$$\sum\frac{(b+c)^2}{b^2+c^2}\geq\frac{\left[\sum(b+c)\right]^2}{\sum(b^2+c^2)}=\frac{2(a+b+c)^2}{a^2+b^2+c^2}.$$

The equality occurs for $a=b=c$, and for $a=0$ and $b=c$ (or any cyclic permutation).

Second Solution. Let

$$p=a+b+c,\qquad q=ab+bc+ca,\qquad r=abc.$$

By the Cauchy-Schwarz inequality, we have

$$\sum\frac{a^2}{b^2+c^2}\geq\frac{\left(\sum a^2\right)^2}{\sum a^2(b^2+c^2)}=\frac{(p^2-2q)^2}{2(q^2-2pr)}.$$

Therefore, it suffices to show that

$$\frac{(p^2-2q)^2}{q^2-2pr}+\frac{2q}{p^2-2q}\geq 5. \tag{*}$$

Consider the following cases: $p^2\geq 4q$ and $3q\leq p^2<4q$.

Case 1: $p^2\geq 4q$. The inequality (*) is true if

$$\frac{(p^2-2q)^2}{q^2}+\frac{2q}{p^2-2q}\geq 5,$$

which is equivalent to the obvious inequality

$$(p^2-4q)\left[(p^2-q)^2-2q^2\right]\geq 0.$$

Case 2: $3q\leq p^2<4q$. Using Schur's inequality of degree four

$$6pr\geq(p^2-q)(4q-p^2),$$

the inequality (*) is true if

$$\frac{3(p^2-2q)^2}{3q^2-(p^2-q)(4q-p^2)}+\frac{2q}{p^2-2q}\geq 5,$$

which is equivalent to the obvious inequality

$$(p^2-3q)(p^2-4q)(2p^2-5q)\leq 0.$$

Third Solution (by *Nguyen Van Quy*). Write the inequality (*) from the previous solution as follows:

$$\frac{(a^2+b^2+c^2)^2}{a^2b^2+b^2c^2+c^2a^2}+\frac{2(ab+bc+ca)}{a^2+b^2+c^2}\geq 5,$$

$$\frac{(a^2+b^2+c^2)^2}{a^2b^2+b^2c^2+c^2a^2}-3\geq 2-\frac{2(ab+bc+ca)}{a^2+b^2+c^2},$$

$$\frac{a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2}{a^2b^2+b^2c^2+c^2a^2}\geq\frac{2(a^2+b^2+c^2-ab-bc-ca)}{a^2+b^2+c^2}.$$

Since

$$2(a^2b^2+b^2c^2+c^2a^2)\leq\sum ab(a^2+b^2)\leq(ab+bc+ca)(a^2+b^2+c^2),$$

it suffices to prove that

$$\frac{a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2}{ab+bc+ca}\geq a^2+b^2+c^2-ab-bc-ca,$$

which is just Schur's inequality of degree four

$$a^4+b^4+c^4+abc(a+b+c)\geq ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2).$$

(c) We get the desired inequality by summing the inequality in (a) and the inequality

$$\frac{2bc}{b^2+c^2}+\frac{2ca}{c^2+a^2}+\frac{2ab}{a^2+b^2}+1\geq\frac{4(ab+bc+ca)}{a^2+b^2+c^2},$$

which was proved at the first solution of (b). The equality occurs for $a=b=c$, and for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.61. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2}\geq\frac{(a+b+c)^2}{2(ab+bc+ca)}.$$

Solution. Applying the Cauchy-Schwarz inequality, we get

$$\sum \frac{a^2}{b^2+c^2} \ge \frac{\left(\sum a^2\right)^2}{\sum a^2(b^2+c^2)} = \frac{(a^2+b^2+c^2)^2}{2(a^2b^2+b^2c^2+c^2a^2)}.$$

Therefore, it suffices to show that

$$\frac{(a^2+b^2+c^2)^2}{2(a^2b^2+b^2c^2+c^2a^2)} \ge \frac{(a+b+c)^2}{2(ab+bc+ca)},$$

which is equivalent to

$$\frac{(a^2+b^2+c^2)^2}{a^2b^2+b^2c^2+c^2a^2} - 3 \ge \frac{(a+b+c)^2}{ab+bc+ca} - 3,$$

$$\frac{a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2}{a^2b^2+b^2c^2+c^2a^2} \ge \frac{a^2+b^2+c^2-ab-bc-ca}{ab+bc+ca}.$$

Since $a^2b^2+b^2c^2+c^2a^2 \le (ab+bc+ca)^2$, it suffices to show that

$$a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2 \ge (a^2+b^2+c^2-ab-bc-ca)(ab+bc+ca),$$

which is just Schur's inequality of degree four

$$a^4+b^4+c^4+abc(a+b+c) \ge ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2).$$

The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.62. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{2ab}{(a+b)^2} + \frac{2bc}{(b+c)^2} + \frac{2ca}{(c+a)^2} + \frac{a^2+b^2+c^2}{ab+bc+ca} \ge \frac{5}{2}.$$

(Vasile Cîrtoaje, 2006)

First Solution. We use the SOS method. Write the inequality as follows:

$$\frac{a^2+b^2+c^2}{ab+bc+ca} - 1 \ge \sum \left[\frac{1}{2} - \frac{2bc}{(b+c)^2}\right],$$

$$\sum \frac{(b-c)^2}{ab+bc+ca} \ge \sum \frac{(b-c)^2}{(b+c)^2},$$

$$(b-c)^2S_a + (c-a)^2S_b + (a-b)^2S_c \ge 0,$$

where

$$S_a = 1 - \frac{ab+bc+ca}{(b+c)^2}, \quad S_b = 1 - \frac{ab+bc+ca}{(c+a)^2}, \quad S_c = 1 - \frac{ab+bc+ca}{(a+b)^2}.$$

Without loss of generality, assume that $a \ge b \ge c$. We have $S_c > 0$ and

$$S_b \ge 1 - \frac{(c+a)(c+b)}{(c+a)^2} = \frac{a-b}{c+a} \ge 0.$$

If $b^2S_a + a^2S_b \ge 0$, then

$$\sum (b-c)^2S_a \ge (b-c)^2S_a + (c-a)^2S_b \ge (b-c)^2S_a + \frac{a^2}{b^2}(b-c)^2S_b$$
$$= \frac{(b-c)^2(b^2S_a + a^2S_b)}{b^2} \ge 0.$$

We have

$$b^2S_a + a^2S_b = a^2 + b^2 - (ab+bc+ca)\left[\left(\frac{b}{b+c}\right)^2 + \left(\frac{a}{c+a}\right)^2\right]$$
$$\ge a^2 + b^2 - (b+c)(c+a))\left[\left(\frac{b}{b+c}\right)^2 + \left(\frac{a}{c+a}\right)^2\right]$$
$$= a^2\left(1 - \frac{b+c}{c+a}\right) + b^2\left(1 - \frac{c+a}{b+c}\right)$$
$$= \frac{(a-b)^2(ab+bc+ca)}{(b+c)(c+a)} \ge 0.$$

The equality occurs for $a = b = c$, and for $a = b$ and $c = 0$ (or any cyclic permutation).

Second Solution. Multiplying by $ab+bc+ca$, the inequality becomes

$$\sum \frac{2a^2b^2}{(a+b)^2} + 2abc\sum \frac{1}{a+b} + a^2 + b^2 + c^2 \ge \frac{5}{2}(ab+bc+ca),$$

$$2abc\sum \frac{1}{a+b} + a^2 + b^2 + c^2 - 2(ab+bc+ca) - \sum \frac{1}{2}ab\left[1 - \sum \frac{4ab}{(a+b)^2}\right] \ge 0.$$

According to the second solution of P 1.59-(a), we can write the inequality as follows:

$$\sum \frac{ab(a-b)^2}{(b+c)(c+a)} - \sum \frac{ab(a-b)^2}{2(a+b)^2} \ge 0,$$

$$(b-c)^2S_a + (c-a)^2S_b + (a-b)^2S_c \ge 0,$$

where

$$S_a = \frac{bc}{b+c}[2(b+c)^2 - (a+b)(a+c)].$$

Without loss of generality, assume that $a \ge b \ge c$. We have $S_c > 0$ and

$$S_b = \frac{ac}{a+c}[2(a+c)^2 - (a+b)(b+c)] \ge \frac{ac}{a+c}[2(a+c)^2 - (2a)(a+c)]$$
$$= \frac{2ac^2(a+c)}{a+c} \ge 0.$$

If $S_a + S_b \geq 0$, then

$$\sum (b-c)^2 S_a \geq (b-c)^2 S_a + (a-c)^2 S_b \geq (b-c)^2 (S_a + S_b) \geq 0.$$

The inequality $S_a + S_b \geq 0$ is equivalent to

$$\frac{ac}{a+c}[2(a+c)^2 - (a+b)(b+c)] \geq \frac{bc}{b+c}[(a+b)(a+c) - 2(b+c)^2].$$

Since

$$\frac{ac}{a+c} \geq \frac{bc}{b+c},$$

it suffices to show that

$$2(a+c)^2 - (a+b)(b+c) \geq (a+b)(a+c) - 2(b+c)^2.$$

This is true since is equivalent to

$$(a-b)^2 + 2c(a+b) + 4c^2 \geq 0.$$

□

P 1.63. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{ab}{(a+b)^2} + \frac{bc}{(b+c)^2} + \frac{ca}{(c+a)^2} + \frac{1}{4} \geq \frac{ab+bc+ca}{a^2+b^2+c^2}.$$

(Vasile Cîrtoaje, 2011)

First Solution. We use the SOS method. Write the inequality as follows:

$$1 - \frac{ab+bc+ca}{a^2+b^2+c^2} \geq \sum \left[\frac{1}{4} - \frac{bc}{(b+c)^2}\right],$$

$$2\sum \frac{(b-c)^2}{a^2+b^2+c^2} \geq \sum \frac{(b-c)^2}{(b+c)^2},$$

$$\sum (b-c)^2 \left[2 - \frac{a^2+b^2+c^2}{(b+c)^2}\right] \geq 0.$$

Since

$$2 - \frac{a^2+b^2+c^2}{(b+c)^2} = 1 + \frac{2bc-a^2}{(b+c)^2} \geq 1 - \left(\frac{a}{b+c}\right)^2,$$

it suffices to show that

$$(b-c)^2 S_a + (c-a)^2 S_b + (a-b)^2 S_c \geq 0,$$

where

$$S_a = 1 - \left(\frac{a}{b+c}\right)^2, \quad S_b = 1 - \left(\frac{b}{c+a}\right)^2, \quad S_c = 1 - \left(\frac{c}{a+b}\right)^2.$$

Without loss of generality, assume that $a \ge b \ge c$. Since $S_b \ge 0$ and $S_c > 0$, if $b^2S_a + a^2S_b \ge 0$, then

$$\begin{aligned}\sum(b-c)^2S_a &\ge (b-c)^2S_a + (c-a)^2S_b \ge (b-c)^2S_a + \frac{a^2}{b^2}(b-c)^2S_b\\ &= \frac{(b-c)^2(b^2S_a + a^2S_b)}{b^2} \ge 0.\end{aligned}$$

We have

$$\begin{aligned}b^2S_a + a^2S_b &= a^2 + b^2 - \left(\frac{ab}{b+c}\right)^2 - \left(\frac{ab}{c+a}\right)^2\\ &= a^2\left[1 - \left(\frac{b}{b+c}\right)^2\right] + b^2\left[1 - \left(\frac{a}{c+a}\right)^2\right] \ge 0.\end{aligned}$$

The equality occurs for $a = b = c$, and for $a = b$ and $c = 0$ (or any cyclic permutation).

Second Solution. Since $(a+b)^2 \le 2(a^2+b^2)$, it suffices to prove that

$$\sum \frac{ab}{2(a^2+b^2)} + \frac{1}{4} \ge \frac{ab+bc+ca}{a^2+b^2+c^2},$$

which is equivalent to

$$\sum \frac{2ab}{a^2+b^2} + 1 \ge \frac{4(ab+bc+ca)}{a^2+b^2+c^2},$$

$$\sum \frac{(a+b)^2}{a^2+b^2} \ge 2 + \frac{4(ab+bc+ca)}{a^2+b^2+c^2},$$

$$\sum \frac{(a+b)^2}{a^2+b^2} \ge \frac{2(a+b+c)^2}{a^2+b^2+c^2}.$$

The last inequality follows immediately by the Cauchy-Schwarz inequality

$$\sum \frac{(a+b)^2}{a^2+b^2} \ge \frac{[\sum(a+b)]^2}{\sum(a^2+b^2)}.$$

Remark. The following generalization of the inequalities in P 1.62 and P 1.63 holds:

- *Let a, b, c be nonnegative real numbers, no two of which are zero. If $0 \le k \le 2$, then*

$$\sum \frac{4ab}{(a+b)^2} + k\frac{a^2+b^2+c^2}{ab+bc+ca} \ge 3k - 1 + 2(2-k)\frac{ab+bc+ca}{a^2+b^2+c^2}.$$

with equality for $a = b = c$, and for $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 1.64. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{3ab}{(a+b)^2}+\frac{3bc}{(b+c)^2}+\frac{3ca}{(c+a)^2}\le\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{5}{4}.$$

(Vasile Cîrtoaje, 2011)

Solution. We use the SOS method. Write the inequality as follows:

$$3\sum\left[\frac{1}{4}-\frac{bc}{(b+c)^2}\right]\ge 1-\frac{ab+bc+ca}{a^2+b^2+c^2},$$

$$3\sum\frac{(b-c)^2}{(b+c)^2}\ge 2\sum\frac{(b-c)^2}{a^2+b^2+c^2},$$

$$(b-c)^2S_a+(c-a)^2S_b+(a-b)^2S_c\ge 0,$$

where

$$S_a=\frac{3(a^2+b^2+c^2)}{(b+c)^2}-2,\quad S_b=\frac{3(a^2+b^2+c^2)}{(c+a)^2}-2,\quad S_c=\frac{3(a^2+b^2+c^2)}{(a+b)^2}-2.$$

Without loss of generality, assume that $a\ge b\ge c$. Since $S_a>0$ and

$$S_b=\frac{a^2+3b^2+c^2-4ac}{(c+a)^2}=\frac{(a-2c)^2+3(b^2-c^2)}{(c+a)^2}\ge 0,$$

if $S_b+S_c\ge 0$, then

$$\sum(b-c)^2S_a\ge(c-a)^2S_b+(a-b)^2S_c\ge(a-b)^2(S_b+S_c)\ge 0.$$

Using the AM-HM Inequality, we have

$$S_b+S_c=3(a^2+b^2+c^2)\left[\frac{1}{(c+a)^2}+\frac{1}{(a+b)^2}\right]-4$$

$$\ge\frac{12(a^2+b^2+c^2)}{(c+a)^2+(a+b)^2}-4=\frac{4(a-b-c)^2+4(b-c)^2}{(c+a)^2+(a+b)^2}\ge 0.$$

The equality occurs for $a=b=c$, and for $\frac{a}{2}=b=c$ (or any cyclic permutation).

□

P 1.65. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

(a) $$\frac{a^3+abc}{b+c}+\frac{b^3+abc}{c+a}+\frac{c^3+abc}{a+b}\ge a^2+b^2+c^2;$$

(b) $$\frac{a^3+2abc}{b+c}+\frac{b^3+2abc}{c+a}+\frac{c^3+2abc}{a+b}\ge\frac{1}{2}(a+b+c)^2;$$

(c) $$\frac{a^3+3abc}{b+c}+\frac{b^3+3abc}{c+a}+\frac{c^3+3abc}{a+b}\ge 2(ab+bc+ca);$$

(d) $$\frac{a^2+2bc}{b+c}+\frac{b^2+2ca}{c+a}+\frac{c^2+2ab}{a+b}\ge\frac{3}{2}(ab+bc+ca).$$

Solution. (a) **First Solution.** Write the inequality as

$$\sum \left(\frac{a^3 + abc}{b + c} - a^2 \right) \geq 0,$$

$$\sum \frac{a(a-b)(a-c)}{b+c} \geq 0.$$

Assume that $a \geq b \geq c$. Since $(c-a)(c-b) \geq 0$ and

$$\frac{a(a-b)(a-c)}{b+c} + \frac{b(b-c)(b-a)}{b+c} = \frac{(a-b)^2(a^2+b^2+c^2+ab)}{(b+c)(c+a)} \geq 0,$$

the conclusion follows. The equality occurs for $a = b = c$, and for $a = b$ and $c = 0$ (or any cyclic permutation).

(b) Taking into account the inequality in (a), it suffices to show that

$$\frac{abc}{b+c} + \frac{abc}{c+a} + \frac{abc}{a+b} + a^2 + b^2 + c^2 \geq \frac{1}{2}(a+b+c)^2,$$

which is just the inequality (a) from P 1.59. The equality occurs for $a = b = c$, and for $a = b$ and $c = 0$ (or any cyclic permutation).

(c) The desired inequality follows by adding the inequality in (a) and the inequality (a) from P 1.59. The equality occurs for $a = b = c$, and for $a = b$ and $c = 0$ (or any cyclic permutation).

(d) Multiplying by $a+b+c$, we get the inequality in (b). The equality occurs for $a = b = c$, and for $a = b$ and $c = 0$ (or any cyclic permutation).

□

P 1.66. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{a^3 + 3abc}{(b+c)^2} + \frac{b^3 + 3abc}{(c+a)^2} + \frac{c^3 + 3abc}{(a+b)^2} \geq a + b + c.$$

(Vasile Cîrtoaje, 2005)

Solution. We use the SOS method. We have

$$\sum \frac{a^3 + 3abc}{(b+c)^2} - \sum a = \sum \left[\frac{a^3 + 3abc}{(b+c)^2} - a \right] = \sum \frac{a^3 - a(b^2 - bc + c^2)}{(b+c)^2}$$

$$= \sum \frac{a^3(b+c) - a(b^3 + c^3)}{(b+c)^3} = \sum \frac{ab(a^2 - b^2) + ac(a^2 - c^2)}{(b+c)^3}$$

$$= \sum \frac{ab(a^2 - b^2)}{(b+c)^3} + \sum \frac{ba(b^2 - a^2)}{(c+a)^3} = \sum \frac{ab(a^2 - b^2)[(c+a)^3 - (b+c)^3]}{(b+c)^3(c+a)^3}$$

$$= \sum \frac{ab(a+b)(a-b)^2[(c+a)^2 + (c+a)(b+c) + (b+c)^2]}{(b+c)^3(c+a)^3} \geq 0.$$

The equality occurs for $a = b = c$, and for $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 1.67. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

(a)
$$\frac{a^3+3abc}{(b+c)^3}+\frac{b^3+3abc}{(c+a)^3}+\frac{c^3+3abc}{(a+b)^3}\ge\frac{3}{2};$$

(b)
$$\frac{3a^3+13abc}{(b+c)^3}+\frac{3b^3+13abc}{(c+a)^3}+\frac{3c^3+13abc}{(a+b)^3}\ge 6.$$

(Vasile Cîrtoaje and Ji Chen, 2005)

Solution. (a) **First Solution.** Use the SOS method. We have

$$\begin{aligned}\sum\frac{a^3+3abc}{(b+c)^3}&=\sum\frac{a(b+c)^2+a(a^2+bc-b^2-c^2)}{(b+c)^3}\\&=\sum\frac{a}{b+c}+\sum\frac{a^3-a(b^2-bc+c^2)}{(b+c)^3}\\&\ge\frac{3}{2}+\sum\frac{a^3(b+c)-a(b^3+c^3)}{(b+c)^4}\\&=\frac{3}{2}+\sum\frac{ab(a^2-b^2)+ac(a^2-c^2)}{(b+c)^4}\\&=\frac{3}{2}+\sum\frac{ab(a^2-b^2)}{(b+c)^4}+\sum\frac{ba(b^2-a^2)}{(c+a)^4}\\&=\frac{3}{2}+\sum\frac{ab(a+b)(a-b)[(c+a)^4-(b+c)^4]}{(b+c)^4(c+a)^4}\ge 0.\end{aligned}$$

The equality occurs for $a=b=c$.

Second Solution. Assume that $a\ge b\ge c$. Since

$$\frac{a^3+3abc}{b+c}\ge\frac{b^3+3abc}{c+a}\ge\frac{c^3+3abc}{a+b}$$

and

$$\frac{1}{(b+c)^2}\ge\frac{1}{(c+a)^2}\ge\frac{1}{(a+b)^2},$$

by Chebyshev's inequality, we get

$$\sum\frac{a^3+3abc}{(b+c)^3}\ge\frac{1}{3}\left(\sum\frac{a^3+3abc}{b+c}\right)\sum\frac{1}{(b+c)^2}.$$

Thus, it suffices to show that

$$\left(\sum\frac{a^3+3abc}{b+c}\right)\sum\frac{1}{(b+c)^2}\ge\frac{9}{2}.$$

We can obtain this inequality by multiplying the known inequality (Iran-1996)

$$\sum\frac{1}{(b+c)^2}\ge\frac{9}{4(ab+bc+ca)}$$

and the inequality (c) from P 1.65.

(b) We have

$$\sum \frac{3a^3+13abc}{(b+c)^3} = \sum \frac{3a(b+c)^2+4abc+3a(a^2+bc-b^2-c^2)}{(b+c)^3}$$

$$= \sum \frac{3a}{b+c} + 4abc \sum \frac{1}{(b+c)^3} + 3\sum \frac{a^3-a(b^2-bc+c^2)}{(b+c)^3}.$$

Since

$$\sum \frac{1}{(b+c)^3} \geq \frac{3}{(a+b)(b+c)(c+a)}$$

(by the AM-GM inequality) and

$$\sum \frac{a^3-a(b^2-bc+c^2)}{(b+c)^3} = \sum \frac{a^3(b+c)-a(b^3+c^3)}{(b+c)^4}$$

$$= \sum \frac{ab(a^2-b^2)+ac(a^2-c^2)}{(b+c)^4} = \sum \frac{ab(a^2-b^2)}{(b+c)^4} + \sum \frac{ba(b^2-a^2)}{(c+a)^4}$$

$$= \sum \frac{ab(a+b)(a-b)[(c+a)^4-(b+c)^4]}{(b+c)^4(c+a)^4} \geq 0,$$

it suffices to prove that

$$\sum \frac{3a}{b+c} + \frac{12abc}{(a+b)(b+c)(c+a)} \geq 6.$$

This inequality is equivalent to the third degree Schur's inequality

$$a^3+b^3+c^3+3abc \geq \sum ab(a+b).$$

The equality occurs for $a=b=c$, and for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.68. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

(a) $$\frac{a^3}{b+c}+\frac{b^3}{c+a}+\frac{c^3}{a+b}+ab+bc+ca \geq \frac{3}{2}(a^2+b^2+c^2);$$

(b) $$\frac{2a^2+bc}{b+c}+\frac{2b^2+ca}{c+a}+\frac{2c^2+ab}{a+b} \geq \frac{9(a^2+b^2+c^2)}{2(a+b+c)}.$$

(Vasile Cîrtoaje, 2006)

Solution. (a) We apply the SOS method. Write the inequality as

$$\sum \left(\frac{2a^3}{b+c} - a^2 \right) \ge \sum (a-b)^2.$$

Since

$$\sum \left(\frac{2a^3}{b+c} - a^2 \right) = \sum \frac{a^2(a-b)+a^2(a-c)}{b+c}$$

$$= \sum \frac{a^2(a-b)}{b+c} + \sum \frac{b^2(b-a)}{c+a} = \sum \frac{(a-b)^2(a^2+b^2+ab+bc+ca)}{(b+c)(c+a)},$$

we can write the inequality as

$$(b-c)^2 S_a + (c-a)^2 S_b + (a-b)^2 S_c \ge 0,$$

where

$$S_a = (b+c)(b^2+c^2-a^2), \quad S_b = (c+a)(c^2+a^2-b^2), \quad S_c = (a+b)(a^2+b^2-c^2).$$

Without loss of generality, assume that $a \ge b \ge c$. Since $S_b \ge 0$, $S_c \ge 0$ and

$$S_a + S_b = (a+b)(a-b)^2 + c^2(a+b+2c) \ge 0,$$

we have

$$\sum (b-c)^2 S_a \ge (b-c)^2 S_a + (a-c)^2 S_b \ge (b-c)^2(S_a+S_b) \ge 0.$$

The equality occurs for $a=b=c$, and for $a=b$ and $c=0$ (or any cyclic permutation).

(b) Multiplying by $a+b+c$, the inequality can be written as

$$\sum \left(1 + \frac{a}{b+c}\right)(2a^2+bc) \ge \frac{9}{2}(a^2+b^2+c^2),$$

$$\sum \frac{2a^3+abc}{b+c} + ab+bc+ca \ge \frac{5}{2}(a^2+b^2+c^2).$$

This inequality follows using the inequality in (a) and the first inequality from P 1.59. The equality occurs for $a=b=c$, and for $a=b$ and $c=0$ (or any cyclic permutation).

□

P 1.69. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{a(b+c)}{b^2+bc+c^2} + \frac{b(c+a)}{c^2+ca+a^2} + \frac{c(a+b)}{a^2+ab+b^2} \ge 2.$$

First Solution. Apply the SOS method. We have

$$(a+b+c)\left[\sum \frac{a(b+c)}{b^2+bc+c^2}-2\right]=\sum\left[\frac{a(b+c)(a+b+c)}{b^2+bc+c^2}-2a\right]$$

$$=\sum \frac{a(ab+ac-b^2-c^2)}{b^2+bc+c^2}=\sum \frac{ab(a-b)-ca(c-a)}{b^2+bc+c^2}$$

$$=\sum \frac{ab(a-b)}{b^2+bc+c^2}-\sum \frac{ab(a-b)}{c^2+ca+a^2}$$

$$=(a+b+c)\sum \frac{ab(a-b)^2}{(b^2+bc+c^2)(c^2+ca+a^2)}\ge 0.$$

The equality occurs for $a=b=c$, and for $a=0$ and $b=c$ (or any cyclic permutation).

Second Solution (by *Choy Fai Lam*). By the AM-HM inequality, we have

$$\frac{1}{b^2+bc+c^2}+\frac{1}{ab+bc+ca}\ge \frac{4}{(b^2+bc+c^2)+(ab+bc+ca)}=\frac{4}{(b+c)(a+b+c)},$$

hence

$$\frac{a(b+c)}{b^2+bc+c^2}\ge \frac{4a}{a+b+c}-\frac{a(b+c)}{ab+bc+ca}.$$

So,

$$\sum \frac{a(b+c)}{b^2+bc+c^2}\ge 4\sum \frac{a}{a+b+c}-\sum \frac{a(b+c)}{ab+bc+ca}=4-2=2.$$

Fourth Solution. By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a(b+c)}{b^2+bc+c^2}\ge \frac{(a+b+c)^2}{\sum \frac{a(b^2+bc+c^2)}{b+c}}.$$

Thus, it is enough to show that

$$(a+b+c)^2\ge 2\sum \frac{a(b^2+bc+c^2)}{b+c}.$$

Since

$$\frac{a(b^2+bc+c^2)}{b+c}=a\left(b+c-\frac{bc}{b+c}\right)=ab+ca-\frac{abc}{b+c},$$

$$\sum \frac{a(b^2+bc+c^2)}{b+c}=2(ab+bc+ca)-abc\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right),$$

this inequality is equivalent to

$$2abc\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)+a^2+b^2+c^2\ge 2(ab+bc+ca),$$

which is just the inequality (a) from P 1.59.

Fifth Solution. By direct calculation, we can write the inequality as

$$\sum ab(a^4+b^4) \ge \sum a^2b^2(a^2+b^2),$$

which is equivalent to the obvious inequality

$$\sum ab(a-b)(a^3-b^3) \ge 0.$$

□

P 1.70. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{a(b+c)}{b^2+bc+c^2}+\frac{b(c+a)}{c^2+ca+a^2}+\frac{c(a+b)}{a^2+ab+b^2} \ge 2+4\prod\left(\frac{a-b}{a+b}\right)^2.$$

(Vasile Cîrtoaje, 2011)

Solution. For $b=c=1$, the inequality reduces to $a(a-1)^2 \ge 0$. Assume further that

$$a > b > c.$$

As we have shown in the first solution of the previous P 1.69,

$$\sum \frac{a(b+c)}{b^2+bc+c^2} - 2 = \sum \frac{bc(b-c)^2}{(a^2+ab+b^2)(a^2+ac+c^2)}.$$

Therefore, it remains to show that

$$\sum \frac{bc(b-c)^2}{(a^2+ab+b^2)(a^2+ac+c^2)} \ge 4\prod\left(\frac{a-b}{a+b}\right)^2.$$

Since

$$(a^2+ab+b^2)(a^2+ac+c^2) \le (a+b)^2(a+c)^2,$$

it suffices to show that

$$\sum \frac{bc(b-c)^2}{(a+b)^2(a+c)^2} \ge 4\prod\left(\frac{a-b}{a+b}\right)^2,$$

which is equivalent to

$$\sum \frac{bc(b+c)^2}{(a-b)^2(a-c)^2} \ge 4.$$

We have

$$\sum \frac{bc(b+c)^2}{(a-b)^2(a-c)^2} \ge \frac{ab(a+b)^2}{(a-c)^2(b-c)^2}$$
$$\ge \frac{ab(a+b)^2}{a^2b^2} = \frac{(a+b)^2}{ab} \ge 4.$$

The equality occurs for $a=b=c$, and for $a=b$ and $c=0$ (or any cyclic permutation). □

P 1.71. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{ab-bc+ca}{b^2+c^2}+\frac{bc-ca+ab}{c^2+a^2}+\frac{ca-ab+bc}{a^2+b^2}\ge\frac{3}{2}.$$

Solution. Use the SOS method. We have

$$\sum\left(\frac{ab-bc+ca}{b^2+c^2}-\frac{1}{2}\right)=\sum\frac{(b+c)(2a-b-c)}{2(b^2+c^2)}$$

$$=\sum\frac{(b+c)(a-b)}{2(b^2+c^2)}+\sum\frac{(b+c)(a-c)}{2(b^2+c^2)}$$

$$=\sum\frac{(b+c)(a-b)}{2(b^2+c^2)}+\sum\frac{(c+a)(b-a)}{2(c^2+a^2)}$$

$$=\sum\frac{(a-b)^2(ab+bc+ca-c^2)}{2(b^2+c^2)(c^2+a^2)}.$$

Since

$$ab+bc+ca-c^2=(b-c)(c-a)+2ab\ge(b-c)(c-a),$$

it suffices to show that

$$\sum(a^2+b^2)(a-b)^2(b-c)(c-a)\ge 0.$$

This inequality is equivalent to

$$(a-b)(b-c)(c-a)\sum(a-b)(a^2+b^2)\ge 0,$$

$$(a-b)^2(b-c)^2(c-a)^2\ge 0.$$

The equality occurs for $a=b=c$, and for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.72. *Let a, b, c be nonnegative real numbers, no two of which are zero. If $k>-2$, then*

$$\sum\frac{ab+(k-1)bc+ca}{b^2+kbc+c^2}\ge\frac{3(k+1)}{k+2}.$$

(Vasile Cîrtoaje, 2005)

First Solution. Apply the SOS method. Write the inequality as

$$\sum\left[\frac{ab+(k-1)bc+ca}{b^2+kbc+c^2}-\frac{k+1}{k+2}\right]\ge 0,$$

$$\sum\frac{A}{b^2+kbc+c^2}\ge 0,$$

where

$$A = (b+c)(2a-b-c) + k(ab+ac-b^2-c^2).$$

Since

$$\begin{aligned} A =&(b+c)[(a-b)+(a-c)] + k[b(a-b)+c(a-c)] \\ &= (a-b)[(k+1)b+c] + (a-c)[(k+1)c+b], \end{aligned}$$

the inequality is equivalent to

$$\sum \frac{(a-b)[(k+1)b+c]}{b^2+kbc+c^2} + \sum \frac{(a-c)[(k+1)c+b]}{b^2+kbc+c^2} \geq 0,$$

$$\sum \frac{(a-b)[(k+1)b+c]}{b^2+kbc+c^2} + \sum \frac{(b-a)[(k+1)a+c]}{c^2+kca+a^2} \geq 0,$$

$$\sum (b-c)^2 R_a S_a \geq 0,$$

where

$$R_a = b^2+kbc+c^2, \quad S_a = a(b+c-a)+(k+1)bc.$$

Without loss of generality, assume that

$$a \geq b \geq c.$$

Case 1: $k \geq -1$. Since $S_a \geq a(b+c-a)$, it suffices to show that

$$\sum a(b+c-a)(b-c)^2 R_a \geq 0.$$

We have

$$\begin{aligned} \sum a(b+c-a)(b-c)^2 R_a &\geq a(b+c-a)(b-c)^2 R_a + b(c+a-b)(c-a)^2 R_b \\ &\geq (b-c)^2[a(b+c-a)R_a + b(c+a-b)R_b]. \end{aligned}$$

Thus, it is enough to prove that

$$a(b+c-a)R_a + b(c+a-b)R_b \geq 0.$$

Since $b+c-a \geq -(c+a-b)$, we have

$$\begin{aligned} a(b+c-a)R_a + b(c+a-b)R_b &\geq (c+a-b)(bR_b - aR_a) \\ &= (c+a-b)(a-b)(ab-c^2) \geq 0. \end{aligned}$$

Case 2: $-2 < k \leq 1$. Since

$$S_a = (a-b)(c-a) + (k+2)bc \geq (a-b)(c-a),$$

we have

$$\sum (b-c)^2 R_a S_a \geq (a-b)(b-c)(c-a) \sum (b-c) R_a.$$

From

$$\sum(b-c)R_a = \sum(b-c)[b^2+bc+c^2-(1-k)bc]$$
$$= \sum(b^3-c^3)-(1-k)\sum bc(b-c)$$
$$= (1-k)(a-b)(b-c)(c-a),$$

we get

$$(a-b)(b-c)(c-a)\sum(b-c)R_a = (1-k)(a-b)^2(b-c)^2(c-a)^2 \ge 0.$$

This completes the proof. The equality occurs for $a=b=c$, and also for $a=b$ and $c=0$ (or any cyclic permutation).

Second Solution. Use the *highest coefficient method* (see P 3.76 in Volume 1). Let

$$p = a+b+c, \quad q = ab+bc+ca.$$

Write the inequality in the form $f_6(a,b,c) \ge 0$, where

$$f_6(a,b,c) = (k+2)\sum[a(b+c)+(k-1)bc](a^2+kab+b^2)(a^2+kac+c^2)$$
$$- 3(k+1)\prod(b^2+kbc+c^2).$$

Since

$$a(b+c)+(k-1)bc = (k-2)bc+q,$$
$$(a^2+kab+b^2)(a^2+kac+c^2) = (p^2-2q+kab-c^2)(p^2-2q+kac-b^2),$$

$f_6(a,b,c)$ has the same highest coefficient A as

$$(k+2)(k-2)P_2(a,b,c)-3(k+1)P_4(a,b,c),$$

where

$$P_2(a,b,c) = \sum bc(kab-c^2)(kac-b^2), \qquad P_4(a,b,c) = \prod(b^2+kbc+c^2).$$

According to Remark 2 from the proof of P 2.75 in Volume 1,

$$A = (k+2)(k-2)P_2(1,1,1)-3(k+1)(k-1)^3 = -9(k-1)^2.$$

Since $A \le 0$, according to P 3.76-(a) in Volume 1, it suffices to prove the original inequality for $b=c=1$, and for $a=0$.

For $b=c=1$, the inequality becomes as follows:

$$\frac{2a+k-1}{k+2}+\frac{2(ka+1)}{a^2+ka+1} \ge \frac{3(k+1)}{k+2},$$

$$\frac{a-k-2}{k+2}+\frac{ka+1}{a^2+ka+1} \ge 0,$$

$$\frac{a(a-1)^2}{(k+2)(a^2+ka+1)} \geq 0.$$

For $a = 0$, the inequality becomes:

$$\frac{(k-1)bc}{b^2+c^2+kbc} + \frac{b}{c} + \frac{c}{b} \geq \frac{3(k+1)}{k+2},$$

$$\frac{k-1}{x+k} + x \geq \frac{3(k+1)}{k+2}, \qquad x = \frac{b}{c} + \frac{c}{b}, \quad x \geq 2,$$

$$\frac{(x-2)[(k+2)x+k^2+k+1]}{(k+2)(x+k)} \geq 0,$$

$$(b-c)^2[(k+2)(b^2+c^2)+(k^2+k+1)bc] \geq 0.$$

Remark. For $k = 1$ and $k = 0$, from P 1.72, we get the inequalities in P 1.69 and P 1.71, respectively. Besides, for $k = 2$, we get the well-known inequality (Iran 1996):

$$\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} \geq \frac{9}{4(ab+bc+ca)}.$$

□

P 1.73. *Let a, b, c be nonnegative real numbers, no two of which are zero. If $k > -2$, then*

$$\sum \frac{3bc - a(b+c)}{b^2+kbc+c^2} \leq \frac{3}{k+2}.$$

(Vasile Cîrtoaje, 2011)

Solution. Write the inequality in P 1.72 as

$$\sum \left[1 - \frac{ab+(k-1)bc+ca}{b^2+kbc+c^2}\right] \leq \frac{3}{k+2},$$

$$\sum \frac{b^2+c^2+bc-a(b+c)}{b^2+kbc+c^2} \leq \frac{3}{k+2}.$$

Since $b^2 + c^2 \geq 2bc$, we get

$$\sum \frac{3bc - a(b+c)}{b^2+kbc+c^2} \leq \frac{3}{k+2},$$

which is just the desired inequality. The equality occurs for $a = b = c$.

□

P 1.74. *Let a, b, c be nonnegative real numbers such that $ab + bc + ca = 3$. Prove that*

$$\frac{ab+1}{a^2+b^2} + \frac{bc+1}{b^2+c^2} + \frac{ca+1}{c^2+a^2} \geq \frac{4}{3}.$$

Solution. Write the inequality in the homogeneous form $E(a,b,c) \geq 4$, where

$$E(a,b,c) = \frac{4ab+bc+ca}{a^2+b^2} + \frac{4bc+ca+ab}{b^2+c^2} + \frac{4ca+ab+bc}{c^2+a^2}.$$

Without loss of generality, assume that $a = \min\{a,b,c\}$. We will show that

$$E(a,b,c) \geq E(0,b,c) \geq 4.$$

We have

$$\frac{E(a,b,c)-E(0,b,c)}{a} = \frac{4b^2+c(b-a)}{b(a^2+b^2)} + \frac{b+c}{b^2+c^2} + \frac{4c^2+b(c-a)}{c(c^2+a^2)} > 0,$$

$$E(0,b,c) - 4 = \frac{b}{c} + \frac{4bc}{b^2+c^2} + \frac{c}{b} - 4 = \frac{(b-c)^4}{bc(b^2+c^2)} \geq 0.$$

The equality holds for $a = 0$ and $b = c = \sqrt{3}$ (or any cyclic permutation).

□

P 1.75. *Let a,b,c be nonnegative real numbers such that $ab+bc+ca = 3$. Prove that*

$$\frac{5ab+1}{(a+b)^2} + \frac{5bc+1}{(b+c)^2} + \frac{5ca+1}{(c+a)^2} \geq 2.$$

Solution. Write the inequality as $E(a,b,c) \geq 6$, where

$$E(a,b,c) = \frac{16ab+bc+ca}{(a+b)^2} + \frac{16bc+ca+ab}{(b+c)^2} + \frac{16ca+ab+bc}{(c+a)^2}.$$

Without loss of generality, assume that

$$a \leq b \leq c.$$

Case 1: $16b^2 \geq c(a+b)$. We will show that

$$E(a,b,c) \geq E(0,b,c) \geq 6.$$

Indeed,

$$\frac{E(a,b,c)-E(0,b,c)}{a} = \frac{16b^2-c(a+b)}{b(a+b)^2} + \frac{1}{b+c} + \frac{16c^2-b(a+c)}{c(c+a)^2} > 0,$$

$$E(0,b,c) - 6 = \frac{b}{c} + \frac{16bc}{(b+c)^2} + \frac{c}{b} - 6 = \frac{(b-c)^4}{bc(b+c)^2} \geq 0.$$

Case 2: $16b^2 < c(a+b)$. We have

$$E(a,b,c) - 6 > \frac{16ab+bc+ca}{(a+b)^2} - 6 > \frac{16ab+16b^2}{(a+b)^2} - 6 = \frac{2(5b-3a)}{a+b} > 0.$$

The equality holds for $a = 0$ and $b = c = \sqrt{3}$ (or any cyclic permutation).

□

P 1.76. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{a^2-bc}{2b^2-3bc+2c^2}+\frac{b^2-ca}{2c^2-3ca+2a^2}+\frac{c^2-ab}{2a^2-3ab+2b^2}\ge 0.$$

(Vasile Cîrtoaje, 2005)

Solution. The hint is applying the Cauchy-Schwarz inequality after we made the numerators of the fractions to be nonnegative and as small as possible. Thus, we write the inequality as

$$\sum\left(\frac{a^2-bc}{2b^2-3bc+2c^2}+1\right)\ge 3,$$

$$\sum\frac{a^2+2(b-c)^2}{2b^2-3bc+2c^2}\ge 3.$$

Without loss of generality, assume that

$$a\ge b\ge c.$$

Using the Cauchy-Schwarz inequality gives

$$\sum\frac{a^2}{2b^2-3bc+2c^2}\ge\frac{\left(\sum a^2\right)^2}{\sum a^2(2b^2-3bc+2c^2)}=\frac{\sum a^4+2\sum a^2b^2}{4\sum a^2b^2-3abc\sum a}$$

and

$$\sum\frac{(b-c)^2}{2b^2-3bc+2c^2}\ge\frac{[a(b-c)+b(a-c)+c(a-b)]^2}{\sum a^2(2b^2-3bc+2c^2)}=\frac{4b^2(a-c)^2}{4\sum a^2b^2-3abc\sum a}.$$

Therefore, it suffices to show that

$$\frac{\sum a^4+2\sum a^2b^2+8b^2(a-c)^2}{4\sum a^2b^2-3abc\sum a}\ge 3.$$

By Schur's inequality of degree four, we have

$$\sum a^4+abc\sum a\ge\sum ab(a^2+b^2)\ge 2\sum a^2b^2.$$

Thus,it is enough to prove that

$$\frac{4\sum a^2b^2-abc\sum a+8b^2(a-c)^2}{4\sum a^2b^2-3abc\sum a}\ge 3,$$

which is equivalent to

$$abc\sum a+b^2(a-c)^2\ge\sum a^2b^2,$$

$$ac(a-b)(b-c)\ge 0.$$

The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.77. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{2a^2-bc}{b^2-bc+c^2}+\frac{2b^2-ca}{c^2-ca+a^2}+\frac{2c^2-ab}{a^2-ab+b^2}\geq 3.$$

(Vasile Cîrtoaje, 2005)

Solution. Write the inequality such that the numerators of the fractions are nonnegative and as small as possible:

$$\sum\left(\frac{2a^2-bc}{b^2-bc+c^2}+1\right)\geq 6,$$

$$\sum\frac{2a^2+(b-c)^2}{b^2-bc+c^2}\geq 6.$$

Applying the Cauchy-Schwarz inequality, we get

$$\sum\frac{2a^2+(b-c)^2}{b^2-bc+c^2}\geq\frac{4\left(2\sum a^2-\sum ab\right)^2}{\sum[2a^2+(b-c)^2](b^2-bc+c^2)}.$$

Thus, we still have to prove that

$$2\left(2\sum a^2-\sum ab\right)^2\geq 3\sum[2a^2+(b-c)^2](b^2-bc+c^2).$$

This inequality is equivalent to

$$2\sum a^4+2abc\sum a+\sum ab(a^2+b^2)\geq 6\sum a^2b^2.$$

We can obtain it by summing up Schur's inequality of degree four

$$\sum a^4+abc\sum a\geq\sum ab(a^2+b^2)$$

and

$$\sum ab(a^2+b^2)\geq 2\sum a^2b^2,$$

multiplied by 2 and 3, respectively. The equality holds for $a=b=c$, and for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.78. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{a^2}{2b^2-bc+2c^2}+\frac{b^2}{2c^2-ca+2a^2}+\frac{c^2}{2a^2-ab+2b^2}\geq 1.$$

(Vasile Cîrtoaje, 2005)

Solution. By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a^2}{2b^2-bc+2c^2} \ge \frac{\left(\sum a^2\right)^2}{\sum a^2(2b^2-bc+2c^2)}.$$

Therefore, it suffices to show that

$$\left(\sum a^2\right)^2 \ge \sum a^2(2b^2-bc+2c^2),$$

which is equivalent to

$$\sum a^4 + abc\sum a \ge 2\sum a^2b^2.$$

This inequality follows by adding Schur's inequality of degree four

$$\sum a^4 + abc\sum a \ge \sum ab(a^2+b^2)$$

and

$$\sum ab(a^2+b^2) \ge 2\sum a^2b^2.$$

The equality holds for $a=b=c$, and for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.79. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{1}{4b^2-bc+4c^2} + \frac{1}{4c^2-ca+4a^2} + \frac{1}{4a^2-ab+4b^2} \ge \frac{9}{7(a^2+b^2+c^2)}.$$

(Vasile Cîrtoaje, 2005)

Solution. Use the SOS method. Without loss of generality, assume that

$$a \ge b \ge c.$$

Write the inequality as

$$\sum \left[\frac{7(a^2+b^2+c^2)}{4b^2-bc+4c^2} - 3\right] \ge 0,$$

$$\sum \frac{7a^2-5b^2-5c^2+3bc}{4b^2-bc+4c^2} \ge 0,$$

$$\sum \frac{5(2a^2-b^2-c^2)-3(a^2-bc)}{4b^2-bc+4c^2} \ge 0.$$

Since

$$2a^2-b^2-c^2 = (a-b)(a+b)+(a-c)(a+c),$$

and

$$2(a^2-bc) = (a-b)(a+c)+(a-c)(a+b)$$

we have

$$\begin{aligned}
&10(2a^2 - b^2 - c^2) - 6(a^2 - bc) = \\
&= (a - b)[10(a + b) - 3(a + c)] + (a - c)[10(a + c) - 3(a + b)] \\
&= (a - b)(7a + 10b - 3c) + (a - c)(7a + 10c - 3b).
\end{aligned}$$

Thus, we can write the desired inequality as follows:

$$\sum \frac{(a-b)(7a+10b-3c)}{4b^2 - bc + 4c^2} + \sum \frac{(a-c)(7a+10c-3b)}{4b^2 - bc + 4c^2} \ge 0,$$

$$\sum \frac{(a-b)(7a+10b-3c)}{4b^2 - bc + 4c^2} + \sum \frac{(b-a)(7b+10a-3c)}{4c^2 - ca + 4a^2} \ge 0,$$

$$\sum \frac{(a-b)^2(28a^2 + 28b^2 - 9c^2 + 68ab - 19bc - 19ca)}{(4b^2 - bc + 4c^2)(4c^2 - ca + 4a^2)},$$

$$\sum \frac{(a-b)^2[(b-c)(28b+9c) + a(28a+68b-19c)]}{(4b^2 - bc + 4c^2)(4c^2 - ca + 4a^2)},$$

$$\sum (a-b)^2 R_c S_c \ge 0,$$

where

$$R_a = 4b^2 - bc + 4c^2, \quad R_b = 4c^2 - ca + 4a^2, \quad R_c = 4a^2 - ab + 4b^2,$$

$$S_a = (c - a)(28c + 9a) + b(28b + 68c - 19a),$$

$$S_b = (a - b)(28a + 9b) + c(28c + 68a - 19b),$$

$$S_c = (b - c)(28b + 9c) + a(28a + 68b - 19c).$$

Since $S_b \ge 0$, $S_c > 0$ and $R_c \ge R_b \ge R_a > 0$, we have

$$\begin{aligned}
\sum (b-c)^2 R_a S_a &\ge (b-c)^2 R_a S_a + (a-c)^2 R_b S_b \\
&\ge (b-c)^2 R_a S_a + (b-c)^2 R_a S_b \\
&= (b-c)^2 R_a (S_a + S_b).
\end{aligned}$$

Thus, we only need to show that $S_a + S_b \ge 0$. Indeed,

$$S_a + S_b = 19(a-b)^2 + 49(a-b)c + 56c^2 \ge 0.$$

The equality holds for $a = b = c$, and for $a = b$ and $c = 0$ (or any cyclic permutation).

□

P 1.80. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{2a^2 + bc}{b^2 + c^2} + \frac{2b^2 + ca}{c^2 + a^2} + \frac{2c^2 + ab}{a^2 + b^2} \ge \frac{9}{2}.$$

(Vasile Cîrtoaje, 2005)

First Solution. We apply the SOS method. Since

$$\sum\left[\frac{2(2a^2+bc)}{b^2+c^2}-3\right]=2\sum\frac{2a^2-b^2-c^2}{b^2+c^2}-\sum\frac{(b-c)^2}{b^2+c^2}$$

and

$$\sum\frac{2a^2-b^2-c^2}{b^2+c^2}=\sum\frac{a^2-b^2}{b^2+c^2}+\sum\frac{a^2-c^2}{b^2+c^2}=\sum\frac{a^2-b^2}{b^2+c^2}+\sum\frac{b^2-a^2}{c^2+a^2}$$
$$=\sum(a^2-b^2)\left(\frac{1}{b^2+c^2}-\frac{1}{c^2+a^2}\right)=\sum\frac{(a^2-b^2)^2}{(b^2+c^2)(c^2+a^2)}$$
$$\geq\sum\frac{(a-b)^2(a^2+b^2)}{(b^2+c^2)(c^2+a^2)},$$

we can write the inequality as

$$2\sum\frac{(b-c)^2(b^2+c^2)}{(c^2+a^2)(a^2+b^2)}\geq\sum\frac{(b-c)^2}{b^2+c^2},$$

or

$$(b-c)^2S_a+(c-a)^2S_b+(a-b)^2S_c\geq 0,$$

where

$$S_a=2(b^2+c^2)^2-(c^2+a^2)(a^2+b^2).$$

Without loss of generality, assume that $a\geq b\geq c$, which involves $S_a\leq S_b\leq S_c$. If

$$S_a+S_b\geq 0,$$

then

$$S_c\geq S_b\geq 0,$$

hence

$$(b-c)^2S_a+(c-a)^2S_b+(a-b)^2S_c\geq(b-c)^2S_a+(a-c)^2S_b$$
$$\geq(b-c)^2(S_a+S_b)\geq 0.$$

We have

$$S_a+S_b=(a^2-b^2)^2+2c^2(a^2+b^2+2c^2)\geq 0.$$

The equality holds for $a=b=c$, and for $a=b$ and $c=0$ (or any cyclic permutation).

Second Solution. Since

$$bc\geq\frac{2b^2c^2}{b^2+c^2},$$

we have

$$\sum\frac{2a^2+bc}{b^2+c^2}\geq\sum\frac{2a^2+\frac{2b^2c^2}{b^2+c^2}}{b^2+c^2}=2(a^2b^2+b^2c^2+c^2a^2)\sum\frac{1}{(b^2+c^2)^2}.$$

Therefore, it suffices to show that

$$\sum \frac{1}{(b^2+c^2)^2} \ge \frac{9}{4(a^2b^2+b^2c^2+c^2a^2)},$$

which is just the known Iran-1996 inequality (see Remark from P 1.72).

Third Solution. We get the desired inequality by summing the inequality in P 1.60-(a), namely

$$\frac{2a^2-2bc}{b^2+c^2}+\frac{2b^2-2ca}{c^2+a^2}+\frac{2c^2-2ab}{a^2+b^2}+\frac{6(ab+bc+ca)}{a^2+b^2+c^2} \ge 6,$$

and the inequality

$$\frac{3bc}{b^2+c^2}+\frac{3ca}{c^2+a^2}+\frac{3ab}{a^2+b^2}+\frac{3}{2} \ge \frac{6(ab+bc+ca)}{a^2+b^2+c^2}.$$

This inequality is equivalent to

$$\sum \left(\frac{2bc}{b^2+c^2}+1\right) \ge \frac{4(ab+bc+ca)}{a^2+b^2+c^2}+2,$$

$$\sum \frac{(b+c)^2}{b^2+c^2} \ge \frac{2(a+b+c)^2}{a^2+b^2+c^2}.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{(b+c)^2}{b^2+c^2} \ge \frac{\left[\sum(b+c)\right]^2}{\sum(b^2+c^2)} = \frac{2(a+b+c)^2}{a^2+b^2+c^2}.$$

□

P 1.81. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{2a^2+3bc}{b^2+bc+c^2}+\frac{2b^2+3ca}{c^2+ca+a^2}+\frac{2c^2+3ab}{a^2+ab+b^2} \ge 5.$$

(Vasile Cîrtoaje, 2005)

Solution. We apply the SOS method. Write the inequality as

$$\sum \left[\frac{3(2a^2+3bc)}{b^2+bc+c^2}-5\right] \ge 0,$$

or

$$\sum \frac{6a^2+4bc-5b^2-5c^2}{b^2+bc+c^2} \ge 0.$$

Since

$$2a^2-b^2-c^2=(a-b)(a+b)+(a-c)(a+c)$$

and

$$2(a^2-bc)=(a-b)(a+c)+(a-c)(a+b),$$

we have

$$6a^2+4bc-5b^2-5c^2=5(2a^2-b^2-c^2)-4(a^2-bc)$$
$$=(a-b)[5(a+b)-2(a+c)]+(a-c)[5(a+c)-2(a+b)]$$
$$=(a-b)(3a+5b-2c)+(a-c)(3a+5c-2b).$$

Thus, we can write the desired inequality as follows:

$$\sum\frac{(a-b)(3a+5b-2c)}{b^2+bc+c^2}+\sum\frac{(a-c)(3a+5c-2b)}{b^2+bc+c^2}\ge 0,$$

$$\sum\frac{(a-b)(3a+5b-2c)}{b^2+bc+c^2}+\sum\frac{(b-a)(3b+5a-2c)}{c^2+ca+a^2}\ge 0,$$

$$\sum\frac{(a-b)^2(3a^2+3b^2-4c^2+8ab+bc+ca)}{(b^2+bc+c^2)(c^2+ca+a^2)}\ge 0,$$

$$(b-c)^2S_a+(c-a)^2S_b+(a-b)^2S_c\ge 0,$$

where

$$S_a=(b^2+bc+c^2)(-4a^2+3b^2+3c^2+ab+8bc+ca),$$
$$S_b=(c^2+ca+a^2)(-4b^2+3c^2+3a^2+bc+8ca+ab),$$
$$S_c=(a^2+ab+b^2)(-4c^2+3a^2+3b^2+ca+8ab+bc).$$

Assume that $a\ge b\ge c$. Since $S_c>0$,

$$S_b=(c^2+ca+a^2)[(a-b)(3a+4b)+c(8a+b+3c)]\ge 0,$$

$$S_a+S_b\ge(b^2+bc+c^2)(b-a)(3b+4a)+(c^2+ca+a^2)(a-b)(3a+4b)$$
$$=(a-b)^2[3(a+b)(a+b+c)+ab-c^2]\ge 0,$$

we have

$$(b-c)^2S_a+(c-a)^2S_b+(a-b)^2S_c\ge(b-c)^2S_a+(a-c)^2S_b$$
$$\ge(b-c)^2(S_a+S_b)\ge 0.$$

The equality holds for $a=b=c$, and for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.82. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{2a^2+5bc}{(b+c)^2}+\frac{2b^2+5ca}{(c+a)^2}+\frac{2c^2+5ab}{(a+b)^2}\ge\frac{21}{4}.$$

(Vasile Cîrtoaje, 2005)

Solution. Use the SOS method.Write the inequality as follows:

$$\sum\left[\frac{2a^2+5bc}{(b+c)^2}-\frac{7}{4}\right]\ge 0,$$

$$\sum\frac{4(a^2-b^2)+4(a^2-c^2)-3(b-c)^2}{(b+c)^2}\ge 0,$$

$$4\sum\frac{b^2-c^2}{(c+a)^2}+4\sum\frac{c^2-b^2}{(a+b)^2}-3\sum\frac{(b-c)^2}{(b+c)^2}\ge 0,$$

$$4\sum\frac{(b-c)^2(b+c)(2a+b+c)}{(c+a)^2(a+b)^2}-3\sum\frac{(b-c)^2}{(b+c)^2}\ge 0.$$

Substituting $b+c=x$, $c+a=y$ and $a+b=z$, we can rewrite the inequality in the form

$$(y-z)^2S_x+(z-x)^2S_y+(x-y)^2S_z\ge 0,$$

where

$$S_x=4x^3(y+z)-3y^2z^2,\quad S_y=4y^3(z+x)-3z^2x^2,\quad S_z=4z^3(x+y)-3x^2y^2.$$

Without loss of generality, assume that

$$0<x\le y\le z,\qquad z\le x+y,$$

which involves $S_x\le S_y\le S_z$. If

$$S_x+S_y\ge 0,$$

then

$$S_z\ge S_y\ge 0,$$

hence

$$\begin{aligned}(y-z)^2S_x+(z-x)^2S_y+(x-y)^2S_z&\ge (y-z)^2S_x+(z-x)^2S_y\\&\ge (y-z)^2(S_x+S_y)\ge 0.\end{aligned}$$

We have

$$\begin{aligned}S_x+S_y&=4xy(x^2+y^2)+4(x^3+y^3)z-3(x^2+y^2)z^2\\&\ge 4xy(x^2+y^2)+4(x^3+y^3)z-3(x^2+y^2)(x+y)z\\&=4xy(x^2+y^2)+(x^2-4xy+y^2)(x+y)z.\end{aligned}$$

For the nontrivial case $x^2-4xy+y^2<0$, we get

$$\begin{aligned}S_x+S_y&\ge 4xy(x^2+y^2)+(x^2-4xy+y^2)(x+y)^2\\&\ge 2xy(x+y)^2+(x^2-4xy+y^2)(x+y)^2\\&=(x-y)^2(x+y)^2.\end{aligned}$$

The equality holds for $a=b=c$, and for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.83. *Let a,b,c be nonnegative real numbers, no two of which are zero. If $k > -2$, then*

$$\sum \frac{2a^2+(2k+1)bc}{b^2+kbc+c^2} \ge \frac{3(2k+3)}{k+2}.$$

(Vasile Cîrtoaje, 2005)

First Solution. There are two cases to consider.

Case 1: $-2 < k \le -1/2$. Write the inequality as

$$\sum \left[\frac{2a^2+(2k+1)bc}{b^2+kbc+c^2} - \frac{2k+1}{k+2}\right] \ge \frac{6}{k+2},$$

$$\sum \frac{2(k+2)a^2-(2k+1)(b-c)^2}{b^2+kbc+c^2} \ge 6.$$

Since $2(k+2)a^2-(2k+1)(b-c)^2 \ge 0$ for $-2 < k \le -1/2$, we can apply the Cauchy-Schwarz inequality. Thus, it suffices to show that

$$\frac{\left[2(k+2)\sum a^2-(2k+1)\sum(b-c)^2\right]^2}{\sum[2(k+2)a^2-(2k+1)(b-c)^2](b^2+kbc+c^2)} \ge 6,$$

which is equivalent to each of the following inequalities

$$\frac{2[(1-k)\sum a^2+(2k+1)\sum ab]^2}{\sum[2(k+2)a^2-(2k+1)(b-c)^2](b^2+kbc+c^2)} \ge 3,$$

$$2(k+2)\sum a^4+2(k+2)abc\sum a-(2k+1)\sum ab(a^2+b^2) \ge 6\sum a^2b^2,$$

$$2(k+2)\left[\sum a^4+abc\sum a-\sum ab(a^2+b^2)\right]+3\sum ab(a-b)^2 \ge 0.$$

The last inequality is true since, by Schur's inequality of degree four, we have

$$\sum a^4+abc\sum a-\sum ab(a^2+b^2) \ge 0.$$

Case 2: $k \ge -9/5$. Use the SOS method. Without loss of generality, assume that $a \ge b \ge c$. Write the inequality as

$$\sum \left[\frac{2a^2+(2k+1)bc}{b^2+kbc+c^2} - \frac{2k+3}{k+2}\right] \ge 0,$$

$$\sum \frac{2(k+2)a^2-(2k+3)(b^2+c^2)+2(k+1)bc}{b^2+kbc+c^2} \ge 0,$$

$$\sum \frac{(2k+3)(2a^2-b^2-c^2)-2(k+1)(a^2-bc)}{b^2+kbc+c^2} \ge 0.$$

Since

$$2a^2-b^2-c^2 = (a-b)(a+b)+(a-c)(a+c)$$

and

$$2(a^2-bc) = (a-b)(a+c) + (a-c)(a+b),$$

we have

$$(2k+3)(2a^2-b^2-c^2) - 2(k+1)(a^2-bc) =$$

$$= (a-b)[(2k+3)(a+b) - (k+1)(a+c)] + (a-c)[(2k+3)(a+c) - (k+1)(a+b)]$$

$$= (a-b)[(k+2)a + (2k+3)b - (k+1)c] + (a-c)[(k+2)a + (2k+3)c - (k+1)b)].$$

Thus, we can write the desired inequality as

$$\sum \frac{(a-b)[(k+2)a+(2k+3)b-(k+1)c]}{b^2+kbc+c^2} +$$

$$+\sum \frac{(a-c)[(k+2)a+(2k+3)c-(k+1)b]}{b^2+kbc+c^2} \geq 0,$$

or

$$\sum \frac{(a-b)[(k+2)a+(2k+3)b-(k+1)c]}{b^2+kbc+c^2} +$$

$$+\sum \frac{(b-a)[(k+2)b+(2k+3)a-(k+1)c]}{c^2+kca+a^2} \geq 0,$$

or

$$(b-c)^2 R_a S_a + (c-a)^2 R_b S_b + (a-b)^2 R_c S_c \geq 0,$$

where

$$R_a = b^2+kbc+c^2, \quad R_b = c^2+kca+a^2, \quad R_c = a^2+kab+b^2,$$

$$S_a = (k+2)(b^2+c^2) - (k+1)^2a^2 + (3k+5)bc + (k^2+k-1)a(b+c)$$

$$= -(a-b)[(k+1)^2a + (k+2)b] + c[(k^2+k-1)a + (3k+5)b + (k+2)c],$$

$$S_b = (k+2)(c^2+a^2) - (k+1)^2b^2 + (3k+5)ca + (k^2+k-1)b(c+a)$$

$$= (a-b)[(k+2)a + (k+1)^2b] + c[(3k+5)a + (k^2+k-1)b + (k+2)c],$$

$$S_c = (k+2)(a^2+b^2) - (k+1)^2c^2 + (3k+5)ab + (k^2+k-1)c(a+b)$$

$$= (k+2)(a^2+b^2) + (3k+5)ab + c[(k^2+k-1)(a+b) - (k+1)^2c]$$

$$\geq (5k+9)ab + c[(k^2+k-1)(a+b) - (k+1)^2c].$$

We have $S_b \geq 0$, since for the nontrivial case

$$(3k+5)a + (k^2+k-1)b + (k+2)c < 0,$$

we get

$$S_b \geq (a-b)[(k+2)a + (k+1)^2b] + b[(3k+5)a + (k^2+k-1)b + (k+2)c]$$

$$= (k+2)(a^2-b^2) + (k+2)^2ab + (k+2)bc > 0.$$

Also, we have $S_c \ge 0$ for $k \ge -9/5$, since

$$\begin{aligned}(5k+9)ab + c[(k^2+k-1)(a+b) - (k+1)^2c] \ge \\ \ge (5k+9)ac + c[(k^2+k-1)(a+b) - (k+1)^2c] \\ = (k+2)(k+4)ac + (k^2+k-1)bc - (k+1)^2c^2 \\ \ge (2k^2+7k+7)bc - (k+1)^2c^2 \\ \ge (k+2)(k+3)c^2 \ge 0.\end{aligned}$$

Therefore, it suffices to show that $R_aS_a + R_bS_b \ge 0$. From

$$bR_b - aR_a = (a-b)(ab-c^2) \ge 0,$$

we get

$$R_aS_a + R_bS_b \ge R_a\left(S_a + \frac{a}{b}S_b\right).$$

Thus, it suffices to show that

$$S_a + \frac{a}{b}S_b \ge 0.$$

We have

$$\begin{aligned}bS_a + aS_b &= (k+2)(a+b)(a-b)^2 + cf(a,b,c) \\ &\ge 2(k+2)b(a-b)^2 + cf(a,b,c),\end{aligned}$$

hence

$$S_a + \frac{a}{b}S_b \ge 2(k+2)(a-b)^2 + \frac{c}{b}f(a,b,c),$$

where

$$f(a,b,c) = b[(k^2+k-1)a + (3k+5)b] + a[(3k+5)a + (k^2+k-1)b]$$
$$+(k+2)c(a+b) = (3k+5)(a^2+b^2) + 2(k^2+k-1)ab + (k+2)c(a+b).$$

For the nontrivial case $f(a,b,c) < 0$, we have

$$S_a + \frac{a}{b}S_b \ge 2(k+2)(a-b)^2 + f(a,b,c)$$

$$\begin{aligned}&\ge 2(k+2)(a-b)^2 + (3k+5)(a^2+b^2) + 2(k^2+k-1)ab \\ &= (5k+9)(a^2+b^2) + 2(k^2-k-5)ab \ge 2(k+2)^2ab \ge 0.\end{aligned}$$

The proof is completed. The equality holds for $a = b = c$, and for $a = b$ and $c = 0$ (or any cyclic permutation).

Second Solution. We use the *highest coefficient method* (see P 3.76 in Volume 1). Let

$$p = a+b+c, \qquad q = ab+bc+ca.$$

Write the inequality as $f_6(a,b,c) \ge 0$, where

$$f_6(a,b,c) = (k+2)\sum[2a^2 + (2k+1)bc](a^2+kab+b^2)(a^2+kac+c^2)$$

$$-3(2k+3)\prod(b^2+kbc+c^2).$$

Since

$$(a^2+kab+b^2)(a^2+kac+c^2)=(p^2-2q+kab-c^2)(p^2-2q+kac-b^2),$$

$f_6(a,b,c)$ has the same highest coefficient A as

$$(k+2)P_2(a,b,c)-3(2k+3)P_4(a,b,c),$$

where

$$P_2(a,b,c)=\sum[2a^2+(2k+1)bc](kab-c^2)(kac-b^2),$$

$$P_4(a,b,c)=\prod(b^2+kbc+c^2).$$

According to Remark 2 from the proof of P 2.75 in Volume 1, we have

$$A=(k+2)P_2(1,1,1)-3(2k+3)(k-1)^3=9(2k+3)(k-1)^2.$$

On the other hand,

$$f_6(a,1,1)=2(k+2)a(a^2+ka+1)(a-1)^2(a+k+2)\ge 0,$$

$$\frac{f_6(0,b,c)}{(b-c)^2}=2(k+2)(b^2+c^2)^2+2(k+2)^2bc(b^2+c^2)+(4k^2+6k-1)b^2c^2.$$

For $-2<k\le -3/2$, we have $A\le 0$. According to P 3.76-(a) in Volume 1, it suffices to show that $f_6(a,1,1)\ge 0$ and $f_6(0,b,c)\ge 0$ for all $a,b,c\ge 0$. The first condition is clearly satisfied. The second condition is satisfied for all $k>-2$ since

$$\begin{aligned}2(k+2)(b^2+c^2)^2+(4k^2+6k-1)b^2c^2 &\ge [8(k+2)+4k^2+6k-1]b^2c^2\\ &=(4k^2+14k+15)b^2c^2\ge 0.\end{aligned}$$

For $k>-3/2$, when $A>0$, we will apply the *highest coefficient cancellation method.* Consider two cases: $p^2\le 4q$ and $p^2>4q$.

Case 1: $p^2\le 4q$. Since

$$f_6(1,1,1)=f_6(0,1,1)=0,$$

define the homogeneous function

$$P(a,b,c)=abc+B(a+b+c)^3+C(a+b+c)(ab+bc+ca)$$

such that $P(1,1,1)=P(0,1,1)=0$; that is,

$$P(a,b,c)=abc+\frac{1}{9}(a+b+c)^3-\frac{4}{9}(a+b+c)(ab+bc+ca).$$

We will prove the sharper inequality $g_6(a,b,c)\ge 0$, where

$$g_6(a,b,c)=f_6(a,b,c)-9(2k+3)(k-1)^2P^2(a,b,c).$$

Clearly, $g_6(a,b,c)$ has the highest coefficient $A=0$. Then, according to Remark 1 from the proof of P 3.76 in Volume 1, it suffices to prove that $g_6(a,1,1)\ge 0$ for $0\le a\le 4$. We have

$$P(a,1,1)=\frac{a(a-1)^2}{9},$$

hence

$$g_6(a,1,1)=f_6(a,1,1)-9(2k+3)(k-1)^2P^2(a,1,1)=\frac{a(a-1)^2g(a)}{9},$$

where

$$g(a)=18(k+2)(a^2+ka+1)(a+k+2)-(2k+3)(k-1)^2a(a-1)^2.$$

Since $a^2+ka+1\ge(k+2)a$, it suffices to show that

$$18(k+2)^2(a+k+2)\ge(2k+3)(k-1)^2(a-1)^2.$$

Also, since $(a-1)^2\le 2a+1$, it is enough to prove that $h(a)\ge 0$, where

$$h(a)=18(k+2)^2(a+k+2)-(2k+3)(k-1)^2(2a+1).$$

Since $h(a)$ is a linear function, the inequality $h(a)\ge 0$ is true if $h(0)\ge 0$ and $h(4)\ge 0$. Setting $x=2k+3$, $x>0$, we get

$$h(0)=18(k+2)^3-(2k+3)(k-1)^2=\frac{1}{4}(8x^3+37x^2+2x+9)>0.$$

Also,

$$\frac{1}{9}h(4)=2(k+2)^2(k+6)-(2k+3)(k-1)^2=3(7k^2+20k+15)>0.$$

Case 2: $p^2>4q$. We will prove the sharper inequality $g_6(a,b,c)\ge 0$, where

$$g_6(a,b,c)=f_6(a,b,c)-9(2k+3)(k-1)^2a^2b^2c^2.$$

We see that $g_6(a,b,c)$ has the highest coefficient $A=0$. According to Remark 1 from the proof of P 3.76 in Volume 1, it suffices to prove that $g_6(a,1,1)\ge 0$ for $a>4$ and $g_6(0,b,c)\ge 0$ for all $b,c\ge 0$. We have

$$g_6(a,1,1)=f_6(a,1,1)-9(2k+3)(k-1)^2a^2$$

$$=a[2(k+2)(a^2+ka+1)(a-1)^2(a+k+2)-9(2k+3)(k-1)^2a].$$

Since

$$a^2+ka+1>(k+2)a,\qquad (a-1)^2>9,$$

it suffices to show that

$$2(k+2)^2(a+k+2)\ge(2k+3)(k-1)^2.$$

Indeed,

$$2(k+2)^2(a+k+2)-(2k+3)(k-1)^2 > 2(k+2)^2(k+6)-(2k+3)(k-1)^2$$
$$= 3(7k^2+20k+15) > 0.$$

Also,

$$g_6(0,b,c) = f_6(0,b,c) \ge 0.$$

□

P 1.84. *Let a,b,c be nonnegative real numbers, no two of which are zero. If $k > -2$, then*

$$\sum \frac{3bc-2a^2}{b^2+kbc+c^2} \le \frac{3}{k+2}.$$

(Vasile Cîrtoaje, 2011)

First Solution. Write the inequality as

$$\sum \left[\frac{2a^2-3bc}{b^2+kbc+c^2}+\frac{3}{k+2}\right] \ge \frac{6}{k+2},$$

$$\sum \frac{2(k+2)a^2+3(b-c)^2}{b^2+kbc+c^2} \ge 6.$$

Applying the Cauchy-Schwarz inequality, it suffices to show that

$$\frac{\left[2(k+2)\sum a^2+3\sum(b-c)^2\right]^2}{\sum[2(k+2)a^2+3(b-c)^2](b^2+kbc+c^2)} \ge 6,$$

which is equivalent to each of the following inequalities

$$\frac{2\left[(k+5)\sum a^2-3\sum ab\right]^2}{\sum[2(k+2)a^2+3(b-c)^2](b^2+kbc+c^2)} \ge 3,$$

$$2(k+8)\sum a^4+2(2k+19)\sum a^2b^2 \ge 6(k+2)abc\sum a+21\sum ab(a^2+b^2),$$

$$2(k+2)f(a,b,c)+3g(a,b,c) \ge 0,$$

where

$$f(a,b,c) = \sum a^4+2\sum a^2b^2-3abc\sum a,$$

$$g(a,b,c) = 4\sum a^4+10\sum a^2b^2-7\sum ab(a^2+b^2).$$

We need to show that $f(a,b,c) \ge 0$ and $g(a,b,c) \ge 0$. Indeed,

$$f(a,b,c) = \left(\sum a^2\right)^2-3abc\sum a \ge \left(\sum ab\right)^2-3abc\sum a \ge 0$$

and

$$g(a,b,c) = \sum[2(a^4+b^4)+10a^2b^2-7ab(a^2+b^2)]$$
$$= \sum(a-b)^2(2a^2-3ab+2b^2) \ge 0.$$

The equality occurs for $a = b = c$.

Second Solution. Write the inequality in P 1.83 as

$$\sum\left[2-\frac{2a^2+(2k+1)bc}{b^2+kbc+c^2}\right] \le \frac{3}{k+2},$$

$$\sum\frac{2(b^2+c^2)-bc-2a^2}{b^2+kbc+c^2} \le \frac{3}{k+2}.$$

Since $b^2+c^2 \ge 2bc$, we get

$$\sum\frac{3bc-2a^2}{b^2+kbc+c^2} \le \frac{3}{k+2},$$

which is just the desired inequality.

□

P 1.85. *If a, b, c are nonnegative real numbers, no two of which are zero, then*

$$\frac{a^2+16bc}{b^2+c^2}+\frac{b^2+16ca}{c^2+a^2}+\frac{c^2+16ab}{a^2+b^2} \ge 10.$$

(Vasile Cîrtoaje, 2005)

Solution. Assume that $a \le b \le c$ and denote

$$E(a,b,c) = \frac{a^2+16bc}{b^2+c^2}+\frac{b^2+16ca}{c^2+a^2}+\frac{c^2+16ab}{a^2+b^2}.$$

Consider two cases.

Case 1: $16b^3 \ge ac^2$. We will show that

$$E(a,b,c) \ge E(0,b,c) \ge 10.$$

We have

$$E(a,b,c)-E(0,b,c) = \frac{a^2}{b^2+c^2}+\frac{a(16c^3-ab^2)}{c^2(c^2+a^2)}+\frac{a(16b^3-ac^2)}{b^2(a^2+b^2)} \ge 0.$$

Also,

$$E(0,b,c)-10 = \frac{16bc}{b^2+c^2}+\frac{b^2}{c^2}+\frac{c^2}{b^2}-10$$
$$= \frac{(b-c)^4(b^2+c^2+4bc)}{b^2c^2(b^2+c^2)} \ge 0.$$

Case 2: $16b^3 \le ac^2$. It suffices to show that

$$\frac{c^2+16ab}{a^2+b^2} \ge 10.$$

Indeed,

$$\frac{c^2+16ab}{a^2+b^2} - 10 \ge \frac{\dfrac{16b^3}{a}+16ab}{a^2+b^2} - 10$$
$$= \frac{16b}{a} - 10 \ge 16 - 10 > 0.$$

This completes the proof. The equality holds for $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 1.86. *If a, b, c are nonnegative real numbers, no two of which are zero, then*

$$\frac{a^2+128bc}{b^2+c^2} + \frac{b^2+128ca}{c^2+a^2} + \frac{c^2+128ab}{a^2+b^2} \ge 46.$$

(Vasile Cîrtoaje, 2005)

Solution. Let

$$a \le b \le c,$$
$$E(a,b,c) = \frac{a^2+128bc}{b^2+c^2} + \frac{b^2+128ca}{c^2+a^2} + \frac{c^2+128ab}{a^2+b^2}.$$

Consider two cases.

Case 1: $128b^3 \ge ac^2$. We will show that

$$E(a,b,c) \ge E(0,b,c) \ge 46.$$

We have

$$E(a,b,c) - E(0,b,c) = \frac{a^2}{b^2+c^2} + \frac{a(128c^3-ab^2)}{c^2(c^2+a^2)} + \frac{a(128b^3-ac^2)}{b^2(a^2+b^2)} \ge 0.$$

Also,

$$E(0,b,c) - 46 = \frac{128bc}{b^2+c^2} + \frac{b^2}{c^2} + \frac{c^2}{b^2} - 46$$
$$= \frac{(b^2+c^2-4bc)^2(b^2+c^2+8bc)}{b^2c^2(b^2+c^2)} \ge 0.$$

Case 2: $128b^3 \le ac^2$. It suffices to show that

$$\frac{c^2+128ab}{a^2+b^2} \ge 46.$$

Indeed,

$$\frac{c^2+128ab}{a^2+b^2} - 46 \ge \frac{\frac{128b^3}{a}+128ab}{a^2+b^2} - 46$$
$$= \frac{128b}{a} - 46 \ge 128 - 46 > 0.$$

This completes the proof. The equality holds for $a = 0$ and $\frac{b}{c} + \frac{c}{b} = 4$ (or any cyclic permutation).

□

P 1.87. *If a, b, c are nonnegative real numbers, no two of which are zero, then*

$$\frac{a^2+64bc}{(b+c)^2} + \frac{b^2+64ca}{(c+a)^2} + \frac{c^2+64ab}{(a+b)^2} \ge 18.$$

(Vasile Cîrtoaje, 2005)

Solution. Let

$$a \le b \le c,$$
$$E(a,b,c) = \frac{a^2+64bc}{(b+c)^2} + \frac{b^2+64ca}{(c+a)^2} + \frac{c^2+64ab}{(a+b)^2}.$$

Consider two cases.

Case 1: $64b^3 \ge c^2(a+2b)$. We will show that

$$E(a,b,c) \ge E(0,b,c) \ge 18.$$

We have

$$E(a,b,c) - E(0,b,c) = \frac{a^2}{(b+c)^2} + \frac{a[64c^3 - b^2(a+2c)]}{c^2(c+a)^2} + \frac{a[64b^3 - c^2(a+2b)]}{b^2(a+b)^2}$$
$$\ge 0.$$

Also,

$$E(0,b,c) - 18 = \frac{64bc}{(b+c)^2} + \frac{b^2}{c^2} + \frac{c^2}{b^2} - 18$$
$$= \frac{(b-c)^4(b^2+c^2+6bc)}{b^2c^2(b+c)^2} \ge 0.$$

Case 2: $64b^3 \le c^2(a+2b)$. It suffices to show that

$$\frac{c^2+64ab}{(a+b)^2} \ge 18.$$

Indeed,

$$\frac{c^2+64ab}{(a+b)^2} - 18 \ge \frac{\dfrac{64b^3}{a+2b}+64ab}{(a+b)^2} - 18$$
$$= \frac{64b}{a+2b} - 18 \ge \frac{64}{3} - 18 > 0.$$

This completes the proof. The equality holds for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.88. *Let a,b,c be nonnegative real numbers, no two of which are zero. If $k \ge -1$, then*

$$\sum \frac{a^2(b+c)+kabc}{b^2+kbc+c^2} \ge a+b+c.$$

Solution. We apply the SOS method. Write the inequality as follows:

$$\sum \left[\frac{a^2(b+c)+kabc}{b^2+kbc+c^2} - a\right] \ge 0,$$

$$\sum \frac{a(ab+ac-b^2-c^2)}{b^2+kbc+c^2} \ge 0,$$

$$\sum \frac{ab(a-b)}{b^2+kbc+c^2} + \sum \frac{ac(a-c)}{b^2+kbc+c^2} \ge 0,$$

$$\sum \frac{ab(a-b)}{b^2+kbc+c^2} + \sum \frac{ba(b-a)}{c^2+kca+a^2} \ge 0,$$

$$\sum ab(a^2+kab+b^2)(a+b+kc)(a-b)^2 \ge 0.$$

Without loss of generality, assume that

$$a \ge b \ge c.$$

Since $a+b+kc \ge a+b-c > 0$, it suffices to show that

$$b(b^2+kbc+c^2)(b+c+ka)(b-c)^2 + a(c^2+kca+a^2)(c+a+kb)(c-a)^2 \ge 0.$$

Since

$$c+a+kb \ge c+a-b \ge 0, \qquad c^2+kca+a^2 \ge b^2+kbc+c^2,$$

it is enough to prove that

$$b(b+c+ka)(b-c)^2+a(c+a+kb)(c-a)^2 \geq 0.$$

We have

$$b(b+c+ka)(b-c)^2+a(c+a+kb)(c-a)^2 \geq$$
$$\geq [b(b+c+ka)+a(c+a+kb)](b-c)^2$$
$$= [a^2+b^2+2kab+c(a+b)](b-c)^2$$
$$\geq [(a-b)^2+c(a+b)](b-c)^2 \geq 0.$$

The equality holds for $a=b=c$, and for $a=b$ and $c=0$ (or any cyclic permutation).

□

P 1.89. *Let a,b,c be nonnegative real numbers, no two of which are zero. If $k \geq \frac{-3}{2}$, then*

$$\sum \frac{a^3+(k+1)abc}{b^2+kbc+c^2} \geq a+b+c.$$

(Vasile Cîrtoaje, 2009)

Solution. Use the SOS method. Write the inequality as follows:

$$\sum \left[\frac{a^3+(k+1)abc}{b^2+kbc+c^2}-a\right] \geq 0, \quad \sum \frac{a^3-a(b^2-bc+c^2)}{b^2+kbc+c^2} \geq 0,$$

$$\sum \frac{(b+c)a^3-a(b^3+c^3)}{(b+c)(b^2+kbc+c^2)} \geq 0, \quad \sum \frac{ab(a^2-b^2)+ac(a^2-c^2)}{(b+c)(b^2+kbc+c^2)} \geq 0,$$

$$\sum \frac{ab(a^2-b^2)}{(b+c)(b^2+kbc+c^2)}+\sum \frac{ba(b^2-a^2)}{(c+a)(c^2+kca+a^2)} \geq 0,$$

$$\sum (a^2-b^2)^2ab(a^2+kab+b^2)[a^2+b^2+ab+(k+1)c(a+b+c)] \geq 0,$$

$$\sum (b^2-c^2)^2bc(b^2+kbc+c^2)S_a \geq 0,$$

where

$$S_a=b^2+c^2+bc+(k+1)a(a+b+c).$$

Without loss of generality, assume that

$$a \geq b \geq c.$$

Since $S_c>0$, it suffices to show that

$$(b^2-c^2)^2b(b^2+kbc+c^2)S_a+(c^2-a^2)^2a(c^2+kca+a^2)S_b \geq 0.$$

Since

$$(c^2-a^2)^2 \ge (b^2-c^2)^2, \qquad a \ge b,$$
$$c^2+kca+a^2-(b^2+kbc+c^2) = (a-b)(a+b+kc) \ge 0,$$

$$S_b = a^2+c^2+ac+(k+1)b(a+b+c) \ge a^2+c^2+ac-\frac{1}{2}b(a+b+c)$$
$$= \frac{(a-b)(2a+b)+c(2a+2c-b)}{2} \ge 0,$$

it is enough to show that $S_a+S_b \ge 0$. Indeed,

$$S_a+S_b = a^2+b^2+2c^2+c(a+b)+(k+1)(a+b)(a+b+c)$$
$$\ge a^2+b^2+2c^2+c(a+b)-\frac{1}{2}(a+b)(a+b+c)$$
$$= \frac{(a-b)^2+c(a+b+4c)}{2} \ge 0.$$

This completes the proof. The equality holds for $a=b=c$, and also for $a=b$ and $c=0$ (or any cyclic permutation).

□

P 1.90. *Prove that* $\frac{7}{8}$ *is the least positive value of* k *such that*

$$\sum \frac{1}{b^2+kbc+c^2} \ge \frac{3}{k+2}$$

for any nonnegative real numbers a,b,c, *no two of which are zero, with* $a+b+c=3$.

(Vasile Cîrtoaje, 2023)

Solution. For the assignment $a=0$ and $b=c=\frac{3}{2}$, the inequality becomes

$$\frac{1}{(k+2)c^2}+\frac{2}{c^2} \ge \frac{3}{k+2}, \qquad k \ge \frac{3c^2-5}{2} = \frac{7}{8}.$$

To show that $\frac{7}{8}$ is the least positive value of k, we need to prove the homogeneous inequality

$$\sum \frac{1}{b^2+kbc+c^2} \ge \frac{27}{(k+2)(a+b+c)^2}$$

for $k=\frac{7}{8}$. According to P 1.83, the following inequality holds for $k>-2$:

$$\sum \frac{2a^2+(2k+1)bc}{b^2+kbc+c^2} \ge \frac{3(2k+3)}{k+2}.$$

Since

$$\frac{2a^2+(2k+1)bc}{b^2+kbc+c^2}+2=\frac{2(a^2+b^2+c^2)+(4k+1)bc}{b^2+kbc+c^2},$$

we have

$$\sum \frac{2(a^2+b^2+c^2)+(4k+1)bc}{b^2+kbc+c^2} \ge \frac{3(4k+7)}{k+2}.$$

According to P 1.72, the following inequality holds for $k > -2$:

$$\sum \frac{4a(b+c)+4(k-1)bc}{b^2+kbc+c^2} \ge \frac{12(k+1)}{k+2}.$$

Adding these inequalities, we obtain

$$\sum \frac{2(a+b+c)^2+(8k-7)bc}{b^2+kbc+c^2} \ge \frac{3(8k+11)}{k+2}.$$

Choosing $k = \frac{7}{8}$, we get the desired inequality. For $k = \frac{7}{8}$, the equality occurs when $a = b = c = 1$, and also when $a = 0$ and $b = c = \frac{3}{2}$ (or any cyclic permutation).

□

P 1.91. *If a, b, c are the lengths of the sides of a triangle, then*

(a) $$\frac{b+c-a}{b^2-bc+c^2}+\frac{c+a-b}{c^2-ca+a^2}+\frac{a+b-c}{a^2-ab+b^2} \ge \frac{2(a+b+c)}{a^2+b^2+c^2};$$

(b) $$\frac{2bc-a^2}{b^2-bc+c^2}+\frac{2ca-b^2}{c^2-ca+a^2}+\frac{2ab-c^2}{a^2-ab+b^2} \ge 0.$$

(Vasile Cîrtoaje, 2009)

Solution. (a) By the Cauchy-Schwarz inequality, we get

$$\sum \frac{b+c-a}{b^2-bc+c^2} \ge \frac{\left[\sum(b+c-a)\right]^2}{\sum(b+c-a)(b^2-bc+c^2)}$$
$$= \frac{\left(\sum a\right)^2}{2\sum a^3-\sum a^2(b+c)+3abc}.$$

On the other hand, from

$$(b+c-a)(c+a-b)(a+b-c) \ge 0,$$

we get

$$2abc \le \sum a^2(b+c)-\sum a^3,$$

hence

$$2\sum a^3 - \sum a^2(b+c) + 3abc \le \frac{\sum a^3 + \sum a^2(b+c)}{2} = \frac{\left(\sum a\right)\left(\sum a^2\right)}{2}.$$

Therefore,

$$\sum \frac{b+c-a}{b^2-bc+c^2} \ge \frac{2\sum a}{\sum a^2}.$$

The equality holds for a degenerate triangle with $a = b+c$ (or any cyclic permutation).

(b) Since

$$\frac{2bc-a^2}{b^2-bc+c^2} = \frac{(b-c)^2+(b+c)^2-a^2}{b^2-bc+c^2} - 2,$$

we can write the inequality as

$$\sum \frac{(b-c)^2}{b^2-bc+c^2} + (a+b+c)\sum \frac{b+c-a}{b^2-bc+c^2} \ge 6.$$

Using the inequality in (a), it suffices to prove that

$$\sum \frac{(b-c)^2}{b^2-bc+c^2} + \frac{2(a+b+c)^2}{a^2+b^2+c^2} \ge 6.$$

Write this inequality as

$$\sum \frac{(b-c)^2}{b^2-bc+c^2} \ge \sum \frac{2(b-c)^2}{a^2+b^2+c^2},$$

$$\sum \frac{(b-c)^2(a-b+c)(a+b-c)}{b^2-bc+c^2} \ge 0.$$

Clearly, the last inequality is true. The equality holds for degenerate triangles with either $a/2 = b = c$ (or any cyclic permutation), or $a = 0$ and $b = c$ (or any cyclic permutation).

Remark. The following generalization of the inequality in (b) holds (*Vasile Cîrtoaje*, 2009):

- *Let a, b, c be the lengths of the sides of a triangle. If $k \ge -1$, then*

$$\sum \frac{2(k+2)bc-a^2}{b^2+kbc+c^2} \ge 0.$$

with equality for $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 1.92. *If a, b, c are nonnegative real numbers, then*

(a) $$\frac{a^2}{5a^2+(b+c)^2} + \frac{b^2}{5b^2+(c+a)^2} + \frac{c^2}{5c^2+(a+b)^2} \le \frac{1}{3};$$

(b) $$\frac{a^3}{13a^3+(b+c)^3} + \frac{b^3}{13b^3+(c+a)^3} + \frac{c^3}{13c^3+(a+b)^3} \le \frac{1}{7}.$$

(Vo Quoc Ba Can and Vasile Cîrtoaje, 2009)

Solution. (a) Apply the Cauchy-Schwarz inequality in the following manner

$$\frac{9}{5a^2+(b+c)^2}=\frac{(1+2)^2}{(a^2+b^2+c^2)+2(2a^2+bc)}\le\frac{1}{a^2+b^2+c^2}+\frac{2}{2a^2+bc}.$$

Then,

$$\sum\frac{9a^2}{5a^2+(b+c)^2}\le\sum\frac{a^2}{a^2+b^2+c^2}+\sum\frac{2a^2}{2a^2+bc}=1+2\sum\frac{a^2}{2a^2+bc},$$

and it remains to show that

$$\sum\frac{a^2}{2a^2+bc}\le 1.$$

For the nontrivial case $a,b,c>0$, this is equivalent to

$$\sum\frac{1}{2+bc/a^2}\le 1,$$

which follows immediately from P 1.2-(b). The equality holds for $a=b=c$, and for $a=0$ and $b=c$ (or any cyclic permutation).

(b) By the Cauchy-Schwarz inequality, we have

$$\frac{49}{13a^3+(b+c)^3}=\frac{(1+6)^2}{(a^3+b^3+c^3)+12a^3+3bc(b+c))}$$
$$\le\frac{1}{a^3+b^3+c^3}+\frac{36}{12a^3+3bc(b+c)},$$

hence

$$\sum\frac{49a^3}{13a^3+(b+c)^3}\le\sum\frac{a^3}{a^3+b^3+c^3}+\sum\frac{36a^3}{12a^3+3bc(b+c)}$$
$$=1+\sum\frac{12a^3}{4a^3+bc(b+c)}.$$

Thus, it suffices to show that

$$\sum\frac{2a^3}{4a^3+bc(b+c)}\le 1.$$

For the nontrivial case $a,b,c>0$, this is equivalent to

$$\sum\frac{1}{2+bc(b+c)/(2a^3)}\le 1.$$

Since

$$\prod bc(b+c)/(2a^3)\ge\prod bc\sqrt{bc}/a^3=1,$$

the inequality follows immediately from P 1.2-(b). The equality holds for $a=b=c$, and for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.93. *If a, b, c are nonnegative real numbers, then*

$$\frac{b^2+c^2-a^2}{2a^2+(b+c)^2}+\frac{c^2+a^2-b^2}{2b^2+(c+a)^2}+\frac{a^2+b^2-c^2}{2c^2+(a+b)^2}\geq\frac{1}{2}.$$

(Vasile Cîrtoaje, 2011)

Solution. We apply the SOS method. Write the inequality as follows:

$$\sum\left[\frac{b^2+c^2-a^2}{2a^2+(b+c)^2}-\frac{1}{6}\right]\geq 0,$$

$$\sum\frac{5(b^2+c^2-2a^2)+2(a^2-bc)}{2a^2+(b+c)^2}\geq 0,$$

$$\sum\frac{5(b^2-a^2)+5(c^2-a^2)+(a-b)(a+c)+(a-c)(a+b)}{2a^2+(b+c)^2}\geq 0,$$

$$\sum\frac{(b-a)[5(b+a)-(a+c)]}{2a^2+(b+c)^2}+\sum\frac{(c-a)[5(c+a)-(a+b)]}{2a^2+(b+c)^2}\geq 0,$$

$$\sum\frac{(b-a)[5(b+a)-(a+c)]}{2a^2+(b+c)^2}+\sum\frac{(a-b)[5(a+b)-(b+c)]}{2b^2+(c+a)^2}\geq 0,$$

$$\sum(a-b)^2[2c^2+(a+b)^2][2(a^2+b^2)+c^2+3ab-3c(a+b)]\geq 0,$$

$$\sum(b-c)^2R_aS_a\geq 0,$$

where

$$R_a=2a^2+(b+c)^2,\qquad S_a=a^2+2(b^2+c^2)+3bc-3a(b+c).$$

Without loss of generality, assume that $a\geq b\geq c$. We have

$$S_b=b^2+2(c^2+a^2)+3ca-3b(c+a)=(a-b)(2a-b)+2c^2+3c(a-b)\geq 0,$$

$$S_c=c^2+2(a^2+b^2)+3ab-3c(a+b)\geq 7ab-3c(a+b)\geq 3a(b-c)+3b(a-c)\geq 0,$$

$$S_a+S_b=3(a-b)^2+4c^2\geq 0.$$

Since

$$\sum(b-c)^2R_aS_a\geq(b-c)^2R_aS_a+(c-a)^2R_bS_b$$

$$=(b-c)^2R_a(S_a+S_b)+[(c-a)^2R_b-(b-c)^2R_a]S_b,$$

it suffices to prove that

$$(a-c)^2R_b\geq(b-c)^2R_a.$$

We can get this by multiplying the inequalities

$$b^2(a-c)^2\geq a^2(b-c)^2$$

and

$$a^2R_b\geq b^2R_a.$$

The equality holds for $a=b=c$, and for $a=b$ and $c=0$ (or any cyclic permutation).

□

P 1.94. *If a, b, c are nonnegative real numbers, no two of which are zero, such that $a+b+c = 3$, then*

$$\frac{a+b}{2a^2+3ab+2b^2}+\frac{b+c}{2b^2+3bc+2c^2}+\frac{c+a}{2c^2+3ca+2a^2}\ge\frac{6}{7}.$$

(Vasile Cîrtoaje, 2023)

Solution. Denote

$$p = a+b+c, \qquad q = ab+bc+ca, \qquad r = abc,$$

and write the inequality in the homogeneous form

$$\sum\frac{p-a}{p^2-2q-a^2+kbc}\ge\frac{18}{(2+k)p},$$

$$(2+k)p\sum(p^2-2q-b^2+kca)(p^2-2q-c^2+kab)\ge 18\prod(p^2-2q-a^2+kbc),$$

$$Ar^2+B(p,q)r+C(p,q)\ge 0,$$

where the highest coefficient A is equal to the highest coefficient of the expression $-18P_2(a,b,c)$, where

$$P_2(a,b,c) = (-a^2+kbc)(-b^2+kca)(-c^2+kab).$$

According to Remark 2 from P 2.75 in Volume 1, we have

$$A = -18P_2(1,1,1) = -18(-1+k)^3 < 0.$$

Thus, for fixed p and q, the function $f(r) = Ar^2+B(p,q)r+C(p,q)$ is a quadratic concave function which has the minimum value when r is maximum or minimum. Since $A \le 0$, according to P 3.76-(a) in Volume 1, it suffices to prove the homogeneous inequality

$$\frac{a+b}{2a^2+3ab+2b^2}+\frac{b+c}{2b^2+3bc+2c^2}+\frac{c+a}{2c^2+3ca+2a^2}\ge\frac{18}{7(a+b+c)}.$$

for $b = c = 1$, and for $a = 0$.

For $b = c = 1$, the homogeneous inequality becomes

$$\frac{a+1}{2a^2+3a+2}+\frac{1}{7}\ge\frac{9}{7(a+2)}.$$

$$a(a-1)^2\ge 0.$$

For $a = 0$, due to homogeneity, we may consider $c = 1$. The homogeneous inequality becomes

$$\frac{1}{2b}+\frac{b+1}{2b^2+3b+2}+\frac{1}{2}\ge\frac{18}{7(b+1)}.$$

$$14b^4-9b^3-10b^2-9b+14\ge 0,$$

$$(b-1)^2(14b^2+19b+14)\ge 0.$$

The equality occurs for $a = b = c = 1$, and also for $a = 0$ and $b = c = \frac{3}{2}$ (or any cyclic permutation).

□

P 1.95. *Let a, b, c be nonnegative real numbers, no two of which are zero. If $k \ge 3 + \sqrt{7}$, then*

(a) $$\frac{a}{a^2 + kbc} + \frac{b}{b^2 + kca} + \frac{c}{c^2 + kab} \ge \frac{9}{(1+k)(a+b+c)};$$

(b) $$\frac{1}{ka^2 + bc} + \frac{1}{kb^2 + ca} + \frac{1}{kc^2 + ab} \ge \frac{9}{(k+1)(ab+bc+ca)}.$$

(Vasile Cîrtoaje, 2005)

Solution. (a) Assume that $a = \max\{a, b, c\}$. Setting

$$t = \frac{b+c}{2}, \qquad t \le a,$$

by the Cauchy-Schwarz inequality, we get

$$\frac{b}{b^2 + kca} + \frac{c}{c^2 + kab} \ge \frac{(b+c)^2}{b(b^2 + kca) + c(c^2 + kab)} = \frac{4t^2}{8t^3 - 6bct + 2kabc}$$
$$= \frac{2t^2}{4t^3 + (ka - 3t)bc} \ge \frac{2t^2}{4t^3 + (ka - 3t)t^2} = \frac{2}{t + ka}.$$

On the other hand,

$$\frac{a}{a^2 + kbc} \ge \frac{a}{a^2 + kt^2}.$$

Therefore, it suffices to prove that

$$\frac{a}{a^2 + kt^2} + \frac{2}{t + ka} \ge \frac{9}{(k+1)(a+2t)},$$

which is equivalent to

$$(a - t)^2[(k^2 - 6k + 2)a + k(4k - 5)t] \ge 0.$$

This inequality is true, since $k^2 - 6k + 2 \ge 0$ and $4k - 5 > 0$. The equality holds for $a = b = c$.

(b) For $a = 0$, the inequality becomes

$$\frac{1}{b^2} + \frac{1}{c^2} \ge \frac{k(8-k)}{(k+1)bc}.$$

We have

$$\frac{1}{b^2} + \frac{1}{c^2} - \frac{k(8-k)}{(k+1)bc} \ge \frac{2}{bc} - \frac{k(8-k)}{(k+1)bc} = \frac{k^2 - 6k + 2}{(k+1)bc} \ge 0.$$

For $a, b, c > 0$, the desired inequality follows from the inequality in (a) by substituting a, b, c with $1/a, 1/b, 1/c$, respectively. The equality holds for $a = b = c$. In the case $k = 3 + \sqrt{7}$, the equality also holds for $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 1.96. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}+\frac{1}{2c^2+ab} \ge \frac{6}{a^2+b^2+c^2+ab+bc+ca}.$$

(Vasile Cîrtoaje, 2005)

Solution. Applying the Cauchy-Schwarz inequality, we have

$$\sum \frac{1}{2a^2+bc} \ge \frac{\left[\sum(b+c)\right]^2}{\sum(b+c)^2(2a^2+bc)} = \frac{4(a+b+c)^2}{\sum(b+c)^2(2a^2+bc)}.$$

Thus, it suffices to show that

$$2(a+b+c)^2(a^2+b^2+c^2+ab+bc+ca) \ge 3\sum(b+c)^2(2a^2+bc),$$

which is equivalent to

$$2\sum a^4 + 3\sum ab(a^2+b^2) + 2abc\sum a \ge 10\sum a^2b^2.$$

This follows by adding Schur's inequality

$$2\sum a^4 + 2abc\sum a \ge 2\sum ab(a^2+b^2)$$

to the inequality

$$5\sum ab(a^2+b^2) \ge 10\sum a^2b^2.$$

The equality holds for $a = b = c$, and also for $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 1.97. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{1}{22a^2+5bc}+\frac{1}{22b^2+5ca}+\frac{1}{22c^2+5ab} \ge \frac{1}{(a+b+c)^2}.$$

(Vasile Cîrtoaje, 2005)

Solution. Applying the Cauchy-Schwarz inequality, we have

$$\sum \frac{1}{22a^2+5bc} \ge \frac{\left[\sum(b+c)\right]^2}{\sum(b+c)^2(22a^2+5bc)} = \frac{4(a+b+c)^2}{\sum(b+c)^2(22a^2+5bc)}.$$

Thus, it suffices to show that

$$4(a+b+c)^4 \ge \sum(b+c)^2(22a^2+5bc),$$

which is equivalent to

$$4\sum a^4 + 11\sum ab(a^2+b^2) + 4abc\sum a \ge 30\sum a^2b^2.$$

This follows by adding Schur's inequality

$$4\sum a^4 + 4abc\sum a \ge 4\sum ab(a^2+b^2)$$

to the inequality

$$15\sum ab(a^2+b^2) \ge 30\sum a^2b^2.$$

The equality holds for $a=b=c$.

□

P 1.98. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{1}{2a^2+bc} + \frac{1}{2b^2+ca} + \frac{1}{2c^2+ab} \ge \frac{8}{(a+b+c)^2}.$$

(Vasile Cîrtoaje, 2005)

First Solution. Applying the Cauchy-Schwarz inequality, we have

$$\sum \frac{1}{2a^2+bc} \ge \frac{\left[\sum(b+c)\right]^2}{\sum(b+c)^2(2a^2+bc)} = \frac{4(a+b+c)^2}{\sum(b+c)^2(2a^2+bc)}.$$

Thus, it suffices to show that

$$(a+b+c)^4 \ge 2\sum(b+c)^2(2a^2+bc),$$

which is equivalent to

$$\sum a^4 + 2\sum ab(a^2+b^2) + 4abc\sum a \ge 6\sum a^2b^2.$$

We will prove the sharper inequality

$$\sum a^4 + 2\sum ab(a^2+b^2) + abc\sum a \ge 6\sum a^2b^2.$$

This follows by adding Schur's inequality

$$\sum a^4 + abc\sum a \ge \sum ab(a^2+b^2)$$

to the inequality

$$3\sum ab(a^2+b^2) \ge 6\sum a^2b^2.$$

The equality holds for $a=0$ and $b=c$ (or any cyclic permutation).

Second Solution. Without loss of generality, we may assume that $a \ge b \ge c$. Since the equality holds for $c = 0$ and $a = b$, when

$$\frac{1}{2a^2+bc} = \frac{1}{2b^2+ca} = \frac{1}{4c^2+2ab},$$

write the inequality as

$$\frac{1}{2a^2+bc} + \frac{1}{2b^2+ca} + \frac{1}{4c^2+2ab} + \frac{1}{4c^2+2ab} \ge \frac{8}{(a+b+c)^2},$$

then apply the AM-HM inequality. Thus, it suffices to prove that

$$\frac{16}{(2a^2+bc)+(2b^2+ca)+(4c^2+2ab)+(4c^2+2ab)} \ge \frac{8}{(a+b+c)^2},$$

which is equivalent to the obvious inequality

$$c(a+b-2c) \ge 0.$$

□

P 1.99. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{1}{a^2+bc} + \frac{1}{b^2+ca} + \frac{1}{c^2+ab} \ge \frac{12}{(a+b+c)^2}.$$

(Vasile Cîrtoaje, 2005)

Solution. Write the inequality such that the numerators of the fractions are nonnegative and as small as possible:

$$\sum \left[\frac{1}{a^2+bc} - \frac{1}{(a+b+c)^2}\right] \ge \frac{9}{(a+b+c)^2},$$

$$\sum \frac{(a+b+c)^2 - a^2 - bc}{a^2+bc} \ge 9.$$

Assuming that $a + b + c = 1$, the inequality becomes

$$\sum \frac{1-a^2-bc}{a^2+bc} \ge 9.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{1-a^2-bc}{a^2+bc} \ge \frac{\left[\sum(1-a^2-bc)\right]^2}{\sum(1-a^2-bc)(a^2+bc)}.$$

Then, it suffices to prove that

$$\left(3-\sum a^2-\sum bc\right)^2 \ge 9\sum(a^2+bc)-9\sum(a^2+bc)^2,$$

which is equivalent to

$$(1-4q)(4-7q)+36abc \ge 0, \qquad q=ab+bc+ca.$$

For $q \le 1/4$, this inequality is clearly true. Consider further that $q > 1/4$. By Schur's inequality of degree three

$$(a+b+c)^3+9abc \ge 4(a+b+c)(ab+bc+ca),$$

we get $1+9abc \ge 4q$, and hence $36abc \ge 16q-4$. Thus,

$$(1-4q)(4-7q)+36abc \ge (1-4q)(4-7q)+16q-4 = 7q(4q-1) > 0.$$

The equality holds for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.100. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

(a) $$\frac{1}{a^2+2bc}+\frac{1}{b^2+2ca}+\frac{1}{c^2+2ab} \ge \frac{1}{a^2+b^2+c^2}+\frac{2}{ab+bc+ca};$$

(b) $$\frac{a(b+c)}{a^2+2bc}+\frac{b(c+a)}{b^2+2ca}+\frac{c(a+b)}{c^2+2ab} \ge 1+\frac{ab+bc+ca}{a^2+b^2+c^2}.$$

(Darij Grinberg and Vasile Cîrtoaje, 2005)

Solution. (a) Write the inequality as

$$\frac{\sum(b^2+2ca)(c^2+2ab)}{(a^2+2bc)(b^2+2ca)(c^2+2ab)} \ge \frac{ab+bc+ca+2a^2+2b^2+2c^2}{(a^2+b^2+c^2)(ab+bc+ca)}.$$

Since

$$\sum(b^2+2ca)(c^2+2ab) = (ab+bc+ca)(ab+bc+ca+2a^2+2b^2+2c^2),$$

it suffices to show that

$$(a^2+b^2+c^2)(ab+bc+ca)^2 \ge (a^2+2bc)(b^2+2ca)(c^2+2ab),$$

which is just the inequality (a) in P 2.16 in Volume 1. The equality holds for $a=b$, or $b=c$, or $c=a$.

(b) Write the inequality in (a) as

$$\sum \frac{ab+bc+ca}{a^2+2bc} \ge 2 + \frac{ab+bc+ca}{a^2+b^2+c^2},$$

or

$$\sum \frac{a(b+c)}{a^2+2bc} + \sum \frac{bc}{a^2+2bc} \ge 2 + \frac{ab+bc+ca}{a^2+b^2+c^2}.$$

The desired inequality follows by adding this inequality to

$$1 \ge \sum \frac{bc}{a^2+2bc}.$$

The last inequality is equivalent to

$$\sum \frac{a^2}{a^2+2bc} \ge 1,$$

which follows by applying the AM-GM inequality as follows:

$$\sum \frac{a^2}{a^2+2bc} \ge \sum \frac{a^2}{a^2+b^2+c^2} = 1.$$

The equality holds for $a = b = c$.

□

P 1.101. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

(a) $$\frac{a}{a^2+2bc} + \frac{b}{b^2+2ca} + \frac{c}{c^2+2ab} \le \frac{a+b+c}{ab+bc+ca};$$

(b) $$\frac{a(b+c)}{a^2+2bc} + \frac{b(c+a)}{b^2+2ca} + \frac{c(a+b)}{c^2+2ab} \le 1 + \frac{a^2+b^2+c^2}{ab+bc+ca}.$$

(Vasile Cîrtoaje, 2008)

Solution. (a) Use the SOS method. Write the inequality as

$$\sum a\left(1 - \frac{ab+bc+ca}{a^2+2bc}\right) \ge 0,$$

$$\sum \frac{a(a-b)(a-c)}{a^2+2bc} \ge 0.$$

Assume that $a \ge b \ge c$. Since $(c-a)(c-b) \ge 0$, it suffices to show that

$$\frac{a(a-b)(a-c)}{a^2+2bc} + \frac{b(b-a)(b-c)}{b^2+2ca} \ge 0.$$

This inequality is equivalent to

$$c(a-b)^2[2a(a-c)+2b(b-c)+3ab] \ge 0,$$

which is clearly true. The equality holds for $a=b=c$, and for $a=b$ and $c=0$ (or any cyclic permutation).

(b) Since

$$\frac{a(b+c)}{a^2+2bc} = \frac{a(a+b+c)}{a^2+2bc} - \frac{a^2}{a^2+2bc},$$

we can write the inequality as

$$(a+b+c)\sum \frac{a}{a^2+2bc} \le 1 + \frac{a^2+b^2+c^2}{ab+bc+ca} + \sum \frac{a^2}{a^2+2bc}.$$

According to the inequality in (a), it suffices to show that

$$\frac{(a+b+c)^2}{ab+bc+ca} \le 1 + \frac{a^2+b^2+c^2}{ab+bc+ca} + \sum \frac{a^2}{a^2+2bc},$$

which is equivalent to

$$\sum \frac{a^2}{a^2+2bc} \ge 1.$$

Indeed,

$$\sum \frac{a^2}{a^2+2bc} \ge \sum \frac{a^2}{a^2+b^2+c^2} = 1.$$

The equality holds for $a=b=c$.

□

P 1.102. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

(a) $$\frac{a}{2a^2+bc} + \frac{b}{2b^2+ca} + \frac{c}{2c^2+ab} \ge \frac{a+b+c}{a^2+b^2+c^2};$$

(b) $$\frac{b+c}{2a^2+bc} + \frac{c+a}{2b^2+ca} + \frac{a+b}{2c^2+ab} \ge \frac{6}{a+b+c}.$$

(Vasile Cîrtoaje, 2008)

Solution. Assume that

$$a \ge b \ge c.$$

(a) Multiplying by $a+b+c$, we can write the inequality as follows:

$$\sum \frac{a(a+b+c)}{2a^2+bc} \ge \frac{(a+b+c)^2}{a^2+b^2+c^2},$$

$$3-\frac{(a+b+c)^2}{a^2+b^2+c^2}\ge\sum\left[1-\frac{a(a+b+c)}{2a^2+bc}\right],$$

$$2\sum(a-b)(a-c)\ge(a^2+b^2+c^2)\sum\frac{(a-b)(a-c)}{2a^2+bc},$$

$$\sum\frac{3a^2-(b-c)^2}{2a^2+bc}(a-b)(a-c)\ge 0,$$

$$3f(a,b,c)+(a-b)(b-c)(c-a)g(a,b,c)\ge 0,$$

where

$$f(a,b,c)=\sum\frac{a^2(a-b)(a-c)}{2a^2+bc},\quad g(a,b,c)=\sum\frac{b-c}{2a^2+bc}.$$

It suffices to show that $f(a,b,c)\ge 0$ and $g(a,b,c)\le 0$. We have

$$\begin{aligned}f(a,b,c)&\ge\frac{a^2(a-b)(a-c)}{2a^2+bc}+\frac{b^2(b-a)(b-c)}{2b^2+ca}\\&\ge\frac{a^2(a-b)(b-c)}{2a^2+bc}+\frac{b^2(b-a)(b-c)}{2b^2+ca}\\&=\frac{a^2c(a-b)^2(b-c)(a^2+ab+b^2)}{(2a^2+bc)(2b^2+ca)}\ge 0.\end{aligned}$$

Also,

$$\begin{aligned}g(a,b,c)&=\frac{b-c}{2a^2+bc}-\frac{(a-b)+(b-c)}{2b^2+ca}+\frac{a-b}{2c^2+ab}\\&=(a-b)\left(\frac{1}{2c^2+ab}-\frac{1}{2b^2+ca}\right)+(b-c)\left(\frac{1}{2a^2+bc}-\frac{1}{2b^2+ca}\right)\\&=\frac{(a-b)(b-c)}{2b^2+ca}\left[\frac{2b+2c-a}{2c^2+ab}-\frac{2b+2a-c}{2a^2+bc}\right]=\\&=\frac{2(a-b)(b-c)(c-a)(a^2+b^2+c^2-ab-bc-ca)}{(2a^2+bc)(2b^2+ca)(2c^2+ab)}\le 0.\end{aligned}$$

The equality holds for $a=b=c$, and for $a=b$ and $c=0$ (or any cyclic permutation).

(b) We apply the SOS method. Write the inequality as follows:

$$\sum\left[\frac{(b+c)(a+b+c)}{2a^2+bc}-2\right]\ge 0,$$

$$\sum\frac{(b^2+ab-2a^2)+(c^2+ca-2a^2)}{2a^2+bc}\ge 0,$$

$$\sum\frac{(b-a)(b+2a)+(c-a)(c+2a)}{2a^2+bc}\ge 0,$$

$$\sum\frac{(b-a)(b+2a)}{2a^2+bc}+\sum\frac{(a-b)(a+2b)}{2b^2+ca}\ge 0,$$

$$\sum (a-b)\left(\frac{a+2b}{2b^2+ca}-\frac{b+2a}{2a^2+bc}\right) \ge 0,$$

$$\sum (a-b)^2(2c^2+ab)(a^2+b^2+3ab-ac-bc) \ge 0.$$

It suffices to show that

$$\sum (a-b)^2(2c^2+ab)(a^2+b^2+2ab-ac-bc) \ge 0,$$

which is equivalent to

$$\sum (a-b)^2(2c^2+ab)(a+b)(a+b-c) \ge 0.$$

This inequality is true if

$$(b-c)^2(2a^2+bc)(b+c)(b+c-a)+(c-a)^2(2b^2+ca)(c+a)(c+a-b) \ge 0;$$

that is,

$$(a-c)^2(2b^2+ca)(a+c)(a+c-b) \ge (b-c)^2(2a^2+bc)(b+c)(a-b-c).$$

Since

$$a+c \ge b+c, \qquad a+c-b \ge a-b-c,$$

it is enough to prove that

$$(a-c)^2(2b^2+ca) \ge (b-c)^2(2a^2+bc).$$

We can obtain this inequality by multiplying the inequalities

$$b^2(a-c)^2 \ge a^2(b-c)^2$$

and

$$a^2(2b^2+ca) \ge b^2(2a^2+bc).$$

The equality holds for $a=b=c$, and for $a=b$ and $c=0$ (or any cyclic permutation).

□

P 1.103. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{a(b+c)}{a^2+bc}+\frac{b(c+a)}{b^2+ca}+\frac{c(a+b)}{c^2+ab} \ge \frac{(a+b+c)^2}{a^2+b^2+c^2}.$$

(Pham Huu Duc, 2006)

Solution. Assume that $a \ge b \ge c$ and write the inequality as follows:

$$3 - \frac{(a+b+c)^2}{a^2+b^2+c^2} \ge \sum \left(1 - \frac{ab+ac}{a^2+bc}\right),$$

$$2\sum (a-b)(a-c) \ge (a^2+b^2+c^2)\sum \frac{(a-b)(a-c)}{a^2+bc},$$

$$\sum \frac{(a-b)(a-c)(a+b-c)(a-b+c)}{a^2+bc} \ge 0.$$

It suffices to show that

$$\frac{(b-c)(b-a)(b+c-a)(b-c+a)}{b^2+ca} + \frac{(c-a)(c-b)(c+a-b)(c-a+b)}{c^2+ab} \ge 0,$$

which is equivalent to the obvious inequality

$$\frac{(b-c)^2(c-a+b)^2(a^2+bc)}{(b^2+ca)(c^2+ab)} \ge 0.$$

The equality holds for $a = b = c$, and for $a = b$ and $c = 0$ (or any cyclic permutation).

□

P 1.104. *Let a, b, c be nonnegative real numbers, no two of which are zero. If $k > 0$, then*

$$\frac{b^2+c^2+\sqrt{3}bc}{a^2+kbc} + \frac{c^2+a^2+\sqrt{3}ca}{b^2+kca} + \frac{a^2+b^2+\sqrt{3}ab}{c^2+kab} \ge \frac{3(2+\sqrt{3})}{1+k}.$$

(Vasile Cîrtoaje, 2013)

Solution. We use the *highest coefficient method.* Write the inequality in the form $f_6(a,b,c) \ge 0$, where

$$f_6(a,b,c) = (1+k)\sum (b^2+c^2+\sqrt{3}bc)(b^2+kca)(c^2+kab)$$
$$-3(2+\sqrt{3}\,)(a^2+kbc)(b^2+kca)(c^2+kab).$$

Clearly, $f_6(a,b,c)$ has the same highest coefficient A as

$$(1+k)P_2(a,b,c) - 3(2+\sqrt{3}\,)P_3(a,b,c),$$

where

$$P_2(a,b,c) = \sum (\sqrt{3}bc - a^2)(b^2+kca)(c^2+kab),$$

$$P_3(a,b,c) = (a^2+kbc)(b^2+kca)(c^2+kab).$$

According to Remark 2 from the proof of P 2.75 in Volume 1, we have

$$\begin{aligned} A &= (1+k)P_2(1,1,1) - 3(2+\sqrt{3}\,)P_3(1,1,1) \\ &= 3(\sqrt{3}-1)(1+k)^3 - 3(2+\sqrt{3}\,)(1+k)^3 = -9(1+k)^3. \end{aligned}$$

Since $A \le 0$, according to P 3.76-(a) in Volume 1, it suffices to prove the original inequality for $b = c = 1$ and for $a = 0$.

In the first case ($b = c = 1$), the inequality is equivalent to

$$\frac{2+\sqrt{3}}{a^2+k} + \frac{2(a^2+\sqrt{3}a+1)}{ka+1} \ge \frac{3(2+\sqrt{3})}{1+k},$$

$$\frac{2(a^2+\sqrt{3}a+1)}{ka+1} \ge \frac{(2+\sqrt{3})(3a^2+2k-1)}{(k+1)(a^2+k)},$$

$$(a-1)^2\left[(k+1)a^2 - \left(1+\frac{\sqrt{3}}{2}\right)(k-2)a + \left(k - \frac{1+\sqrt{3}}{2}\right)^2\right] \ge 0.$$

For the nontrivial case $k > 2$, we have

$$(k+1)a^2 + \left(k - \frac{1+\sqrt{3}}{2}\right)^2 \ge 2\sqrt{k+1}\left(k - \frac{1+\sqrt{3}}{2}\right)a$$

$$\ge 2\sqrt{3}\left(k - \frac{1+\sqrt{3}}{2}\right)a \ge \left(1+\frac{\sqrt{3}}{2}\right)(k-2)a.$$

In the second case ($a = 0$), the original inequality can be written as

$$\frac{1}{k}\left(\frac{b}{c}+\frac{c}{b}+\sqrt{3}\right) + \left(\frac{b^2}{c^2}+\frac{c^2}{b^2}\right) \ge \frac{3(2+\sqrt{3})}{1+k}.$$

It suffices to show that

$$\frac{1}{k}(2+\sqrt{3}) + 2 \ge \frac{3(2+\sqrt{3})}{1+k},$$

which is equivalent to

$$\left(k - \frac{1+\sqrt{3}}{2}\right)^2 \ge 0.$$

The equality holds for $a = b = c$. If $k = \dfrac{1+\sqrt{3}}{2}$, then the equality holds also for $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 1.105. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{1}{a^2+b^2} + \frac{1}{b^2+c^2} + \frac{1}{c^2+a^2} + \frac{8}{a^2+b^2+c^2} \ge \frac{6}{ab+bc+ca}.$$

(Vasile Cîrtoaje, 2013)

Solution. Multiplying by $a^2+b^2+c^2$, the inequality becomes

$$\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2}+11 \ge \frac{6(a^2+b^2+c^2)}{ab+bc+ca}.$$

Since

$$\left(\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2}\right)(a^2b^2+b^2c^2+c^2a^2) =$$

$$= a^4+b^4+c^4+a^2b^2c^2\left(\frac{1}{a^2+b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}\right) \ge a^4+b^4+c^4,$$

it suffices to show that

$$\frac{a^4+b^4+c^4}{a^2b^2+b^2c^2+c^2a^2}+11 \ge \frac{6(a^2+b^2+c^2)}{ab+bc+ca},$$

which is equivalent to

$$\frac{(a^2+b^2+c^2)^2}{a^2b^2+b^2c^2+c^2a^2}+9 \ge \frac{6(a^2+b^2+c^2)}{ab+bc+ca}.$$

Clearly, it is enough to prove that

$$\left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^2+9 \ge \frac{6(a^2+b^2+c^2)}{ab+bc+ca},$$

which is

$$\left(\frac{a^2+b^2+c^2}{ab+bc+ca}-3\right)^2 \ge 0.$$

The equality holds for $a=0$ and $\frac{b}{c}+\frac{c}{b}=3$ (or any cyclic permutation).

□

P 1.106. *If a,b,c are the lengths of the sides of a triangle, then*

$$\frac{a(b+c)}{a^2+2bc}+\frac{b(c+a)}{b^2+2ca}+\frac{c(a+b)}{c^2+2ab} \le 2.$$

(Vo Quoc Ba Can and Vasile Cîrtoaje, 2010)

FirstSolution. Write the inequality as

$$\sum\left(1-\frac{ab+ac}{a^2+2bc}\right) \ge 1, \qquad \sum\frac{a^2+2bc-ab-ac}{a^2+2bc} \ge 1.$$

Since

$$a^2+2bc-ab-ac = bc-(a-c)(b-a) \ge |a-c||b-a|-(a-c)(b-a) \ge 0,$$

by the Cauchy-Schwarz inequality, we have

$$\sum \frac{a^2+2bc-ab-ac}{a^2+2bc} \geq \frac{\left[\sum(a^2+2bc-ab-ac)\right]^2}{\sum(a^2+2bc)(a^2+2bc-ab-ac)}.$$

Thus, it suffices to prove that

$$(a^2+b^2+c^2)^2 \geq \sum(a^2+2bc)(a^2+2bc-ab-ac),$$

which reduces to the obvious inequality

$$ab(a-b)^2+bc(b-c)^2+ca(c-a)^2 \geq 0.$$

The equality holds for an equilateral triangle, and for a degenerate triangle with $a=0$ and $b=c$ (or any cyclic permutation).

Second Solution (by *Nguyen Van Huyen*). Since

$$a^2+2bc-ab-ac = bc-(a-c)(b-a) \geq |a-c||b-a|-(a-c)(b-a) \geq 0,$$

we have

$$\frac{a(b+c)}{a^2+2bc} \leq \frac{a(b+c)+(b-c)^2}{a^2+2bc+(b-c)^2} = \frac{b^2+c^2+a(b+c)-2bc}{a^2+b^2+c^2}.$$

Therefore,

$$\sum \frac{a(b+c)}{a^2+2bc} \leq \sum \frac{b^2+c^2+a(b+c)-2bc}{a^2+b^2+c^2} = \frac{2(a^2+b^2+c^2)}{a^2+b^2+c^2} = 2.$$

□

P 1.107. *If a,b,c are real numbers, then*

$$\frac{a^2-bc}{2a^2+b^2+c^2}+\frac{b^2-ca}{2b^2+c^2+a^2}+\frac{c^2-ab}{2c^2+a^2+b^2} \geq 0.$$

(Nguyen Anh Tuan, 2005)

First Solution. Rewrite the inequality as

$$\sum \left(\frac{1}{2}-\frac{a^2-bc}{2a^2+b^2+c^2}\right) \leq \frac{3}{2},$$

$$\sum \frac{(b+c)^2}{2a^2+b^2+c^2} \leq 3.$$

If two of a,b,c are zero, then the inequality is trivial. Otherwise, applying the Cauchy-Schwarz inequality, we get

$$\sum \frac{(b+c)^2}{2a^2+b^2+c^2} = \sum \frac{(b+c)^2}{(a^2+b^2)+(a^2+c^2)} \le \sum \left(\frac{b^2}{a^2+b^2}+\frac{c^2}{a^2+c^2}\right)$$
$$= \sum \frac{b^2}{a^2+b^2} + \sum \frac{a^2}{b^2+a^2} = 3.$$

The equality holds for $a=b=c$.

Second Solution. Use the SOS method. We have

$$2\sum \frac{a^2-bc}{2a^2+b^2+c^2} = \sum \frac{(a-b)(a+c)+(a-c)(a+b)}{2a^2+b^2+c^2}$$
$$= \sum \frac{(a-b)(a+c)}{2a^2+b^2+c^2} + \sum \frac{(b-a)(b+c)}{2b^2+c^2+a^2}$$
$$= \sum (a-b)\left(\frac{a+c}{2a^2+b^2+c^2} - \frac{b+c}{2b^2+c^2+a^2}\right)$$
$$= (a^2+b^2+c^2-ab-bc-ca)\sum \frac{(a-b)^2}{(2a^2+b^2+c^2)(2b^2+c^2+a^2)} \ge 0.$$

□

P 1.108. *If a,b,c are nonnegative real numbers, then*

$$\frac{3a^2-bc}{2a^2+b^2+c^2} + \frac{3b^2-ca}{2b^2+c^2+a^2} + \frac{3c^2-ab}{2c^2+a^2+b^2} \le \frac{3}{2}.$$

(Vasile Cîrtoaje, 2008)

First Solution. Write the inequality as

$$\sum \left(\frac{3}{2} - \frac{3a^2-bc}{2a^2+b^2+c^2}\right) \ge 3,$$

$$\sum \frac{8bc+3(b-c)^2}{2a^2+b^2+c^2} \ge 6.$$

By the Cauchy-Schwarz inequality, we have

$$8bc+3(b-c)^2 \ge \frac{[4bc+(b-c)^2]^2}{2bc+\frac{1}{3}(b-c)^2} = \frac{2(b+c)^4}{b^2+c^2+4bc}.$$

Therefore, it suffices to prove that

$$\sum \frac{(b+c)^4}{(2a^2+b^2+c^2)(b^2+c^2+4bc)} \ge 2.$$

Using again the Cauchy-Schwarz inequality, we get

$$\sum \frac{(b+c)^4}{(2a^2+b^2+c^2)(b^2+c^2+4bc)} \geq \frac{\left[\sum(b+c)^2\right]^2}{\sum(2a^2+b^2+c^2)(b^2+c^2+4bc)} = 2.$$

The equality holds for $a = b = c$, for $a = 0$ and $b = c$ (or any cyclic permutation), and for $b = c = 0$ (or any cyclic permutation).

Second Solution. Use the SOS method. Write the inequality as

$$\sum \left(\frac{1}{2} - \frac{3a^2 - bc}{2a^2+b^2+c^2} \right) \geq 0,$$

$$\sum \frac{(b+c+2a)(b+c-2a)}{2a^2+b^2+c^2} \geq 0,$$

$$\sum \frac{(b+c+2a)(b-a)+(b+c+2a)(c-a)}{2a^2+b^2+c^2} \geq 0,$$

$$\sum \frac{(b+c+2a)(b-a)}{2a^2+b^2+c^2} + \sum \frac{(c+a+2b)(a-b)}{2b^2+c^2+a^2} \geq 0,$$

$$\sum (a-b) \left(\frac{c+a+2b}{2b^2+c^2+a^2} - \frac{b+c+2a}{2a^2+b^2+c^2} \right) \geq 0,$$

$$\sum (3ab+bc+ca-c^2)(2c^2+a^2+b^2)(a-b)^2 \geq 0.$$

Clearly, it suffices to show that

$$\sum c(a+b-c)(2c^2+a^2+b^2)(a-b)^2 \geq 0.$$

Assume that $a \geq b \geq c$. It is enough to prove that

$$a(b+c-a)(2a^2+b^2+c^2)(b-c)^2 + b(c+a-b)(2b^2+c^2+a^2)(c-a)^2 \geq 0;$$

that is,

$$b(c+a-b)(2b^2+c^2+a^2)(a-c)^2 \geq a(a-b-c)(2a^2+b^2+c^2)(b-c)^2.$$

Since $c+a-b \geq a-b-c$, it suffices to prove that

$$b(2b^2+c^2+a^2)(a-c)^2 \geq a(2a^2+b^2+c^2)(b-c)^2.$$

We can obtain this inequality by multiplying the inequalities

$$b^2(a-c)^2 \geq a^2(b-c)^2$$

and

$$a(2b^2+c^2+a^2) \geq b(2a^2+b^2+c^2).$$

The last inequality is equivalent to

$$(a-b)[(a-b)^2+ab+c^2] \geq 0.$$

□

P 1.109. *If a, b, c are nonnegative real numbers, then*

$$\frac{(b+c)^2}{4a^2+b^2+c^2}+\frac{(c+a)^2}{4b^2+c^2+a^2}+\frac{(a+b)^2}{4c^2+a^2+b^2} \ge 2.$$

(Vasile Cîrtoaje, 2005)

Solution. By the Cauchy-Schwarz inequality, we have

$$\sum \frac{(b+c)^2}{4a^2+b^2+c^2} \ge \frac{\left[\sum (b+c)^2\right]^2}{\sum (b+c)^2(4a^2+b^2+c^2)}$$

$$= 2\c{d}ot \frac{\sum a^4 + 3\sum a^2b^2 + 4abc\sum a + 2\sum ab(a^2+b^2)}{\sum a^4 + 5\sum a^2b^2 + 4abc\sum a + \sum ab(a^2+b^2)} \ge 2$$

because

$$\sum ab(a^2+b^2) \ge 2\sum a^2b^2.$$

The equality holds for $a = b = c$, and for $b = c = 0$ (or any cyclic permutation).

□

P 1.110. *If a, b, c are positive real numbers, then*

(a) $$\sum \frac{1}{11a^2+2b^2+2c^2} \le \frac{3}{5(ab+bc+ca)};$$

(b) $$\sum \frac{1}{4a^2+b^2+c^2} \le \frac{1}{2(a^2+b^2+c^2)} + \frac{1}{ab+bc+ca}.$$

(Vasile Cîrtoaje, 2008)

Solution. We will prove that

$$\sum \frac{k+2}{ka^2+b^2+c^2} \le \frac{11-2k}{a^2+b^2+c^2} + \frac{2(k-1)}{ab+bc+ca}$$

for any $k > 1$. Due to homogeneity, we may assume that $a^2+b^2+c^2 = 3$. On this hypothesis, we need to show that

$$\sum \frac{k+2}{(k-1)a^2+3} \le \frac{11-2k}{3} + \frac{2(k-1)}{ab+bc+ca}.$$

Using the substitution $m = 3/(k-1)$, $m > 0$, the inequality can be written as

$$m(m+1)\sum \frac{1}{a^2+m} \le 3m-2+\frac{6}{ab+bc+ca}.$$

By the Cauchy-Schwarz inequality, we have

$$(a^2+m)[m+(m+1-a)^2] \ge [a\sqrt{m}+\sqrt{m}(m+1-a)]^2 = m(m+1)^2,$$

and hence

$$\frac{m(m+1)}{a^2+m} \le \frac{a^2-1}{m+1} + m + 2 - 2a,$$

$$m(m+1)\sum \frac{1}{a^2+m} \le 3(m+2) - 2\sum a.$$

Thus, it suffices to show that

$$3(m+2) - 2\sum a \le 3m - 2 + \frac{6}{ab+bc+ca};$$

that is,

$$(4-a-b-c)(ab+bc+ca) \le 3.$$

Let $p = a+b+c$. Since

$$2(ab+bc+ca) = (a+b+c)^2 - (a^2+b^2+c^2) = p^2 - 3,$$

we get

$$\begin{aligned} 6 - 2(4-a-b-c)(ab+bc+ca) &= 6 - (4-p)(p^2-3) \\ &= (p-3)^2(p+2) \ge 0. \end{aligned}$$

This completes the proof. The equality holds for $a = b = c$.

□

P 1.111. *If a, b, c are nonnegative real numbers such that $ab+bc+ca = 3$, then*

$$\frac{\sqrt{a}}{b+c} + \frac{\sqrt{b}}{c+a} + \frac{\sqrt{c}}{a+b} \ge \frac{3}{2}.$$

(Vasile Cîrtoaje, 2006)

Solution. By the Cauchy-Schwarz inequality, we have

$$\sum \frac{\sqrt{a}}{b+c} \ge \frac{\left(\sum a^{3/4}\right)^2}{\sum a(b+c)} = \frac{1}{6}\left(\sum a^{3/4}\right)^2.$$

Thus, it suffices to show that

$$a^{3/4} + b^{3/4} + c^{3/4} \ge 3,$$

which follows immediately from Remark 1 from the proof of the inequality in P 3.33 in Volume 1. The equality occurs for $a = b = c = 1$.

Remark. Analogously, using Remark 2 from the proof of P 3.33 in Volume 1, we can prove that

$$\frac{a^k}{b+c} + \frac{b^k}{c+a} + \frac{c^k}{a+b} \ge \frac{3}{2}$$

for all $k \ge 3 - \dfrac{4\ln 2}{\ln 3} \approx 0.476$. For $k = 3 - \dfrac{4\ln 2}{\ln 3}$, the equality occurs for $a = b = c = 1$, and also for $a = 0$ and $b = c = \sqrt{3}$ (or any cyclic permutation).

□

P 1.112. *If a, b, c are nonnegative real numbers such that $ab+bc+ca \ge 3$, then*

$$\frac{1}{2+a}+\frac{1}{2+b}+\frac{1}{2+c} \ge \frac{1}{1+b+c}+\frac{1}{1+c+a}+\frac{1}{1+a+b}.$$

(Vasile Cîrtoaje, 2014)

Solution. Consider $c=\min\{a,b,c\}$, and denote

$$E(a,b,c)=\frac{1}{2+a}+\frac{1}{2+b}+\frac{1}{2+c}-\frac{1}{1+b+c}-\frac{1}{1+c+a}-\frac{1}{1+a+b}.$$

If $c \ge 1$, the desired inequality $E(a,b,c) \ge 0$ follows by summing the obvious inequalities

$$\frac{1}{2+a} \ge \frac{1}{1+c+a},$$

$$\frac{1}{2+b} \ge \frac{1}{1+a+b},$$

$$\frac{1}{2+c} \ge \frac{1}{1+b+c}.$$

Consider further that $c<1$. From

$$E(a,b,c)=-\frac{1-c}{(2+a)(1+c+a)}-\frac{1}{1+a+b}+\frac{1}{2+b}+\frac{1}{2+c}-\frac{1}{1+b+c}$$

and

$$E(a,b,c)=-\frac{1-c}{(2+b)(1+b+c)}-\frac{1}{1+a+b}+\frac{1}{2+a}+\frac{1}{2+c}-\frac{1}{1+c+a},$$

it follows that $E(a,b,c)$ is increasing in a and b. Based on this result, it suffices to prove the desired inequality only for

$$ab+bc+ca=3.$$

Applying the AM-GM inequality, we get

$$3=ab+bc+ca \ge 3(abc)^{2/3}, \qquad abc \le 1,$$

$$a+b+c \ge 3\sqrt[3]{abc} \ge 3.$$

We will show that

$$\frac{1}{2+a}+\frac{1}{2+b}+\frac{1}{2+c} \ge 1 \ge \frac{1}{1+b+c}+\frac{1}{1+c+a}+\frac{1}{1+a+b}.$$

By direct calculation, we can show that the left inequality is equivalent to $abc \le 1$, while the right inequality is equivalent to $a+b+c \ge 2+abc$. Clearly, these are true and the proof is completed. The equality occurs for $a=b=c=1$.

□

P 1.113. *If a, b, c are the lengths of the sides of a triangle, then*

(a) $$\frac{a^2-bc}{3a^2+b^2+c^2}+\frac{b^2-ca}{3b^2+c^2+a^2}+\frac{c^2-ab}{3c^2+a^2+b^2}\le 0;$$

(b) $$\frac{a^4-b^2c^2}{3a^4+b^4+c^4}+\frac{b^4-c^2a^2}{3b^4+c^4+a^4}+\frac{c^4-a^2b^2}{3c^4+a^4+b^4}\le 0.$$

(Nguyen Anh Tuan and Vasile Cîrtoaje, 2006)

Solution. (a) Apply the SOS method. We have

$$2\sum\frac{a^2-bc}{3a^2+b^2+c^2}=\sum\frac{(a-b)(a+c)+(a-c)(a+b)}{3a^2+b^2+c^2}$$
$$=\sum\frac{(a-b)(a+c)}{3a^2+b^2+c^2}+\sum\frac{(b-a)(b+c)}{3b^2+c^2+a^2}$$
$$=\sum(a-b)\left(\frac{a+c}{3a^2+b^2+c^2}-\frac{b+c}{3b^2+c^2+a^2}\right)$$
$$=(a^2+b^2+c^2-2ab-2bc-2ca)\sum\frac{(a-b)^2}{(3a^2+b^2+c^2)(3b^2+c^2+a^2)}.$$

Since

$$a^2+b^2+c^2-2ab-2bc-2ca=a(a-b-c)+b(b-c-a)+c(c-a-b)\le 0,$$

the conclusion follows. The equality holds for an equilateral triangle, and for a degenerate triangle with $a=0$ and $b=c$ (or any cyclic permutation).

(b) Using the same way as above, we get

$$2\sum\frac{a^4-b^2c^2}{3a^4+b^4+c^4}=A\sum\frac{(a^2-b^2)^2}{(3a^4+b^4+c^4)(3b^4+c^4+a^4)},$$

where

$$A=a^4+b^4+c^4-2a^2b^2-2b^2c^2-2c^2a^2$$
$$=-(a+b+c)(a+b-c)(b+c-a)(c+a-b)\le 0.$$

The equality holds for an equilateral triangle, and for a degenerate triangle with $a=b+c$ (or any cyclic permutation).

□

P 1.114. *If a, b, c are the lengths of the sides of a triangle, then*

$$\frac{bc}{4a^2+b^2+c^2}+\frac{ca}{4b^2+c^2+a^2}+\frac{ab}{4c^2+a^2+b^2}\ge\frac{1}{2}.$$

(Vasile Cîrtoaje and Vo Quoc Ba Can, 2010)

Solution. We apply the SOS method. Write the inequality as

$$\sum \left(\frac{2bc}{4a^2+b^2+c^2} - \sum \frac{b^2c^2}{a^2b^2+b^2c^2+c^2a^2} \right) \geq 0,$$

$$\sum \frac{bc(2a^2-bc)(b-c)^2}{4a^2+b^2+c^2} \geq 0.$$

Without loss of generality, assume that $a \geq b \geq c$. Then, it suffices to prove that

$$\frac{c(2b^2-ca)(c-a)^2}{4b^2+c^2+a^2} + \frac{b(2c^2-ab)(a-b)^2}{4c^2+a^2+b^2} \geq 0.$$

Since

$$2b^2 - ca \geq c(b+c) - ca = c(b+c-a) \geq 0$$

and

$$\begin{aligned}(2b^2-ca)+(2c^2-ab) &= 2(b^2+c^2) - a(b+c) \geq (b+c)^2 - a(b+c) \\ &= (b+c)(b+c-a) \geq 0,\end{aligned}$$

it is enough to show that

$$\frac{c(a-c)^2}{4b^2+c^2+a^2} \geq \frac{b(a-b)^2}{4c^2+a^2+b^2}.$$

This follows by multiplying the inequalities

$$c^2(a-c)^2 \geq b^2(a-b)^2$$

and

$$\frac{b}{4b^2+c^2+a^2} \geq \frac{c}{4c^2+a^2+b^2}.$$

These inequalities are true, since

$$c(a-c) - b(a-b) = (b-c)(b+c-a) \geq 0,$$

$$b(4c^2+a^2+b^2) - c(4b^2+c^2+a^2) = (b-c)[(b-c)^2+a^2-bc] \geq 0.$$

The equality occurs for an equilateral triangle, and for a degenerate triangle with $a = b$ and $c = 0$ (or any cyclic permutation).

□

P 1.115. *If a, b, c are the lengths of the sides of a triangle, then*

$$\frac{1}{b^2+c^2} + \frac{1}{c^2+a^2} + \frac{1}{a^2+b^2} \leq \frac{9}{2(ab+bc+ca)}.$$

(Vo Quoc Ba Can, 2008)

Solution. Apply the SOS method. Write the inequality as

$$\sum\left[\frac{3}{2}-\frac{ab+bc+ca}{b^2+c^2}\right]\ge 0,$$

$$\sum\frac{3(b^2+c^2)-2(ab+bc+ca)}{b^2+c^2}\ge 0,$$

$$\sum\frac{3b(b-a)+3c(c-a)+c(a-b)+b(a-c)}{b^2+c^2}\ge 0,$$

$$\sum\frac{(a-b)(c-3b)+(a-c)(b-3c)}{b^2+c^2}\ge 0,$$

$$\sum\frac{(a-b)(c-3b)}{b^2+c^2}+\sum\frac{(b-a)(c-3a)}{c^2+a^2}\ge 0,$$

$$\sum(a^2+b^2)(a-b)^2(ca+cb+3c^2-3ab)\ge 0.$$

Without loss of generality, assume that $a\ge b\ge c$. Since

$$ab+ac+3a^2-3bc>0,$$

it suffices to prove that

$$(a^2+b^2)(a-b)^2(ca+cb+3c^2-3ab)+(a^2+c^2)(a-c)^2(ab+bc+3b^2-3ac)\ge 0,$$

or, equivalently,

$$(a^2+c^2)(a-c)^2(ab+bc+3b^2-3ac)\ge(a^2+b^2)(a-b)^2(3ab-3c^2-ca-cb).$$

Since

$$\begin{aligned}ab+bc+3b^2-3ac&=a\left(\frac{bc+3b^2}{a}+b-3c\right)\\&\ge a\left(\frac{bc+3b^2}{b+c}+b-3c\right)\\&=\frac{a(b-c)(4b+3c)}{b+c}\ge 0\end{aligned}$$

and

$$\begin{aligned}(ab+bc+3b^2-3ac)-(3ab-3c^2-ca-cb)&=3(b^2+c^2)+2bc-2a(b+c)\\&\ge 3(b^2+c^2)+2bc-2(b+c)^2\\&=(b-c)^2\ge 0,\end{aligned}$$

it suffices to show that

$$(a^2+c^2)(a-c)^2\ge(a^2+b^2)(a-b)^2.$$

This is equivalent to $(b-c)A \ge 0$, where

$$\begin{aligned} A &= 2a^3 - 2a^2(b+c) + 2a(b^2+bc+c^2) - (b+c)(b^2+c^2) \\ &= 2a\left(a - \frac{b+c}{2}\right)^2 + \frac{a(3b^2+2bc+3c^2)}{2} - (b+c)(b^2+c^2) \\ &\ge \frac{b(3b^2+2bc+3c^2)}{2} - (b+c)(b^2+c^2) \\ &= \frac{(b-c)(b^2+bc+2c^2)}{2} \ge 0. \end{aligned}$$

The equality occurs for an equilateral triangle, and for a degenerate triangle with $a/2 = b = c$ (or any cyclic permutation).

□

P 1.116. *If a, b, c are the lengths of the sides of a triangle, then*

(a)
$$\left|\frac{a+b}{a-b} + \frac{b+c}{b-c} + \frac{c+a}{c-a}\right| > 5;$$

(b)
$$\left|\frac{a^2+b^2}{a^2-b^2} + \frac{b^2+c^2}{b^2-c^2} + \frac{c^2+a^2}{c^2-a^2}\right| \ge 3.$$

(Vasile Cîrtoaje, 2003)

Solution. Since the inequalities are symmetric, we consider

$$a > b > c.$$

(a) Let $x = a-c$ and $y = b-c$. From $a > b > c$ and $a \le b+c$, it follows

$$x > y > 0, \qquad c \ge x - y.$$

We have

$$\begin{aligned} \frac{a+b}{a-b} + \frac{b+c}{b-c} + \frac{c+a}{c-a} &= \frac{2c+x+y}{x-y} + \frac{2c+y}{y} - \frac{2c+x}{x} \\ &= 2c\left(\frac{1}{x-y} + \frac{1}{y} - \frac{1}{x}\right) + \frac{x+y}{x-y} \\ &> \frac{2c}{y} + \frac{x+y}{x-y} \ge \frac{2(x-y)}{y} + \frac{x+y}{x-y} \\ &= 2\left(\frac{x-y}{y} + \frac{y}{x-y}\right) + 1 \ge 5. \end{aligned}$$

(b) We will show that

$$\frac{a^2+b^2}{a^2-b^2} + \frac{b^2+c^2}{b^2-c^2} + \frac{c^2+a^2}{c^2-a^2} \ge 3;$$

that is,

$$\frac{b^2}{a^2-b^2}+\frac{c^2}{b^2-c^2}\ge\frac{a^2}{a^2-c^2}.$$

Since

$$\frac{a^2}{a^2-c^2}\le\frac{(b+c)^2}{a^2-c^2},$$

it suffices to prove that

$$\frac{b^2}{a^2-b^2}+\frac{c^2}{b^2-c^2}\ge\frac{(b+c)^2}{a^2-c^2}.$$

This is equivalent to each of the following inequalities:

$$b^2\left(\frac{1}{a^2-b^2}-\frac{1}{a^2-c^2}\right)+c^2\left(\frac{1}{b^2-c^2}-\frac{1}{a^2-c^2}\right)\ge\frac{2bc}{a^2-c^2},$$

$$\frac{b^2(b^2-c^2)}{a^2-b^2}+\frac{c^2(a^2-b^2)}{b^2-c^2}\ge 2bc,$$

$$[b(b^2-c^2)-c(a^2-b^2)]^2\ge 0.$$

This completes the proof. If $a>b>c$, then the equality holds for a degenerate triangle with $a=b+c$ and $b/c=x_1$, where $x_1\approx 1.5321$ is the positive root of the equation $x^3-3x-1=0$.

□

P 1.117. *If a,b,c are the lengths of the sides of a triangle, then*

$$\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}+3\ge 6\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right).$$

Solution. We apply the SOS method. Write the inequality as

$$\sum\frac{b+c}{a}-6\ge 3\left(\sum\frac{2a}{b+c}-3\right).$$

Since

$$\sum\frac{b+c}{a}-6=\sum\left(\frac{b}{c}+\frac{c}{b}\right)-6=\sum\frac{(b-c)^2}{bc}$$

and

$$\begin{aligned}\sum\frac{2a}{b+c}-3&=\sum\frac{2a-b-c}{b+c}=\sum\frac{a-b}{b+c}+\sum\frac{a-c}{b+c}\\&=\sum\frac{a-b}{b+c}+\sum\frac{b-a}{c+a}=\sum\frac{(a-b)^2}{(b+c)(c+a)}\\&=\sum\frac{(b-c)^2}{(c+a)(a+b)},\end{aligned}$$

we can rewrite the inequality as

$$\sum a(b+c)(b-c)^2 S_a \geq 0,$$

where

$$S_a = a(a+b+c) - 2bc.$$

Without loss of generality, assume that $a \geq b \geq c$. Since $S_a > 0$,

$$S_b = b(a+b+c) - 2ca = (b-c)(a+b+c) + c(b+c-a) \geq 0$$

and

$$\begin{aligned}\sum a(b+c)(b-c)^2 S_a &\geq b(c+a)(c-a)^2 S_b + c(a+b)(a-b)^2 S_c \\ &\geq (a-b)^2[b(c+a)S_b + c(a+b)S_c],\end{aligned}$$

it suffices to prove that

$$b(c+a)S_b + c(a+b)S_c \geq 0.$$

This is equivalent to each of the following inequalities

$$(a+b+c)[a(b^2+c^2) + bc(b+c)] \geq 2abc(2a+b+c),$$

$$a(a+b+c)(b-c)^2 + (a+b+c)[2abc + bc(b+c)] \geq 2abc(2a+b+c),$$

$$a(a+b+c)(b-c)^2 + bc(2a+b+c)(b+c-a) \geq 0.$$

Since the last inequality is true, the proof is completed. The equality occurs for an equilateral triangle, and for a degenerate triangle with $a/2 = b = c$ (or any cyclic permutation).

□

P 1.118. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\sum \frac{3a(b+c) - 2bc}{(b+c)(2a+b+c)} \geq \frac{3}{2}.$$

(Vasile Cîrtoaje, 2009)

Solution. Use the SOS method. Write the inequality as follows:

$$\sum \left[\frac{3a(b+c) - 2bc}{(b+c)(2a+b+c)} - \frac{1}{2}\right] \geq 0,$$

$$\sum \frac{4a(b+c) - 6bc - b^2 - c^2}{(b+c)(2a+b+c)} \geq 0,$$

$$\sum \frac{b(a-b) + c(a-c) + 3b(a-c) + 3c(a-b)}{(b+c)(2a+b+c)} \geq 0,$$

$$\sum \frac{(a-b)(b+3c)+(a-c)(c+3b)}{(b+c)(2a+b+c)} \ge 0,$$

$$\sum \frac{(a-b)(b+3c)}{(b+c)(2a+b+c)} + \sum \frac{(b-a)(a+3c)}{(c+a)(2b+c+a)} \ge 0,$$

$$\sum (a-b)\left[\frac{b+3c}{(b+c)(2a+b+c)} - \frac{a+3c}{(c+a)(2b+c+a)}\right] \ge 0,$$

$$(a-b)(b-c)(c-a)\sum (a^2-b^2)(a+b+2c) \ge 0.$$

Since

$$\sum (a^2-b^2)(a+b+2c) = (a-b)(b-c)(c-a),$$

the conclusion follows. The equality holds for $a=b$, or $b=c$, or $c=a$.

□

P 1.119. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\sum \frac{a(b+c)-2bc}{(b+c)(3a+b+c)} \ge 0.$$

(Vasile Cîrtoaje, 2009)

Solution. We apply the SOS method. Since

$$\sum \frac{a(b+c)-2bc}{(b+c)(3a+b+c)} = \sum \frac{b(a-c)+c(a-b)}{(b+c)(3a+b+c)}$$

$$= \sum \frac{c(b-a)}{(c+a)(3b+c+a)} + \sum \frac{c(a-b)}{(b+c)(3a+b+c)}$$

$$= \sum \frac{c(a+b-c)(a-b)^2}{(b+c)(c+a)(3a+b+c)(3b+c+a)},$$

the inequality is equivalent to

$$\sum c(a+b)(3c+a+b)(a+b-c)(a-b)^2 \ge 0.$$

Without loss of generality, assume that $a \ge b \ge c$. Since $a+b-c \ge 0$, it suffices to show that

$$b(c+a)(3b+c+a)(c+a-b)(a-c)^2 \ge a(b+c)(3a+b+c)(a-b-c)(b-c)^2.$$

This is true since

$$c+a-b \ge a-b-c,$$

$$b^2(a-c)^2 \ge a^2(b-c)^2,$$

$$c+a \ge b+c,$$

$$a(3b+c+a) \ge b(3a+b+c).$$

The equality holds for $a=b=c$, and for $a=b$ and $c=0$ (or any cyclic permutation).

□

P 1.120. *Let a, b, c be positive real numbers such that $a^2+b^2+c^2 \geq 3$. Prove that*

$$\frac{a^5-a^2}{a^5+b^2+c^2}+\frac{b^5-b^2}{b^5+c^2+a^2}+\frac{c^5-c^2}{c^5+a^2+b^2} \geq 0.$$

(Vasile Cîrtoaje, 2005)

Solution. The inequality is equivalent to

$$\frac{1}{a^5+b^2+c^2}+\frac{1}{b^5+c^2+a^2}+\frac{1}{c^5+a^2+b^2} \leq \frac{3}{a^2+b^2+c^2}.$$

Setting $a = tx$, $b = ty$ and $c = tz$, where

$$x, y, z > 0, \qquad x^2+y^2+z^2 = 3,$$

the condition $a^2+b^2+c^2 \geq 3$ implies $t \geq 1$, and the inequality becomes

$$\frac{1}{t^3x^5+y^2+z^2}+\frac{1}{t^3y^5+z^2+x^2}+\frac{1}{t^3z^5+x^2+y^2} \leq 1.$$

We see that it suffices to prove this inequality for $t = 1$, when it becomes

$$\frac{1}{x^5-x^2+3}+\frac{1}{y^5-y^2+3}+\frac{1}{z^5-z^2+3} \leq 1.$$

Without loss of generality, assume that $x \geq y \geq z$. There are two cases to consider.

Case 1: $z \leq y \leq x \leq \sqrt{2}$. The desired inequality follows by adding the inequalities

$$\frac{1}{x^5-x^2+3} \leq \frac{3-x^2}{6}, \quad \frac{1}{y^5-y^2+3} \leq \frac{3-y^2}{6}, \quad \frac{1}{z^5-z^2+3} \leq \frac{3-z^2}{6}.$$

We have

$$\frac{1}{x^5-x^2+3}-\frac{3-x^2}{6} = \frac{(x-1)^2(x^5+2x^4-3x^2-6x-3)}{6(x^5-x^2+3)} \leq 0$$

since

$$\begin{aligned} x^5+2x^4-3x^2-6x-3 &= x^2\left(x^3+2x^2-3-\frac{6}{x}-\frac{3}{x^2}\right) \\ &\leq x^2\left(2\sqrt{2}+4-3-3\sqrt{2}-\frac{3}{2}\right) \\ &= -x^2(\sqrt{2}+\frac{1}{2}) < 0. \end{aligned}$$

Case 2: $x > \sqrt{2}$. From $x^2+y^2+z^2 = 3$, it follows that $y^2+z^2 < 1$. Since

$$\frac{1}{x^5-x^2+3} < \frac{1}{(2\sqrt{2}-1)x^2+3} < \frac{1}{2(2\sqrt{2}-1)+3} < \frac{1}{6}$$

and

$$\frac{1}{y^5-y^2+3}+\frac{1}{z^5-z^2+3}<\frac{1}{3-y^2}+\frac{1}{3-z^2},$$

it suffices to prove that

$$\frac{1}{3-y^2}+\frac{1}{3-z^2}\le\frac{5}{6}.$$

Indeed, we have

$$\frac{1}{3-y^2}+\frac{1}{3-z^2}-\frac{5}{6}=\frac{9(y^2+z^2-1)-5y^2z^2}{6(3-y^2)(3-z^2)}<0,$$

which completes the proof. The equality occurs for $a=b=c=1$.

Remark. Since $abc\ge 1$ involves $a^2+b^2+c^2\ge 3\sqrt[3]{a^2b^2c^2}\ge 3$, the inequality is also true under the condition $abc\ge 1$. A proof of this inequality (which is a problem from IMO-2005 - proposed by *Hojoo Lee*) is the following:

$$\sum\frac{a^5-a^2}{a^5+b^2+c^2}\ge\sum\frac{a^5-a^2}{a^5+a^3(b^2+c^2)}=\frac{1}{a^2+b^2+c^2}\sum\left(a^2-\frac{1}{a}\right),$$

$$\sum\left(a^2-\frac{1}{a}\right)\ge\sum(a^2-bc)=\frac{1}{2}\sum(a-b)^2\ge 0.$$

□

P 1.121. *Let a,b,c be positive real numbers such that $a^2+b^2+c^2=a^3+b^3+c^3$. Prove that*

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\ge\frac{3}{2}.$$

(Pham Huu Duc, 2008)

First Solution. By the Cauchy-Schwarz inequality, we have

$$\sum\frac{a^2}{b+c}\ge\frac{\left(\sum a^3\right)^2}{\sum a^4(b+c)}=\frac{\left(\sum a^3\right)\left(\sum a^2\right)}{\left(\sum a^3\right)\left(\sum ab\right)-abc\sum a^2}.$$

Therefore, it is enough to show that

$$2\left(\sum a^3\right)\left(\sum a^2\right)+3abc\sum a^2\ge 3\left(\sum a^3\right)\left(\sum ab\right).$$

Write this inequality as follows:

$$3\left(\sum a^3\right)\left(\sum a^2-\sum ab\right)-\left(\sum a^3-3abc\right)\left(\sum a^2\right)\ge 0,$$

$$3\left(\sum a^3\right)\left(\sum a^2-\sum ab\right)-\left(\sum a\right)\left(\sum a^2-\sum ab\right)\left(\sum a^2\right)\ge 0,$$

$$\left(\sum a^2 - \sum ab\right)\left[3\sum a^3 - \left(\sum a\right)\left(\sum a^2\right)\right] \ge 0.$$

The last inequality is true since

$$2\left(\sum a^2 - \sum ab\right) = \sum (a-b)^2 \ge 0$$

and

$$3\sum a^3 - \left(\sum a\right)\left(\sum a^2\right) = \sum (a^3+b^3) - \sum ab(a+b)$$
$$= \sum (a+b)(a-b)^2 \ge 0.$$

The equality occurs for $a = b = c = 1$.

Second Solution. Write the inequality in the homogeneous form $A \ge B$, where

$$A = 2\sum \frac{a^2}{b+c} - \sum a, \quad B = \frac{3(a^3+b^3+c^3)}{a^2+b^2+c^2} - \sum a.$$

Since

$$A = \sum \frac{a(a-b)+a(a-c)}{b+c} = \sum \frac{a(a-b)}{b+c} + \sum \frac{b(b-a)}{c+a}$$
$$= (a+b+c)\sum \frac{(a-b)^2}{(b+c)(c+a)}$$

and

$$B = \frac{\sum (a^3+b^3) - \sum ab(a+b)}{a^2+b^2+c^2} = \frac{\sum (a+b)(a-b)^2}{a^2+b^2+c^2},$$

we can write the inequality as

$$\sum \left[\frac{a+b+c}{(b+c)(c+a)} - \frac{a+b}{a^2+b^2+c^2}\right](a-b)^2 \ge 0,$$

$$(a^3+b^3+c^3-2abc)\sum \frac{(a-b)^2}{(b+c)(c+a)} \ge 0.$$

Since $a^3+b^3+c^3 \ge 3abc$, the conclusion follows.

□

P 1.122. *If $a, b, c \in [0,1]$, then*

$$\frac{a}{bc+2} + \frac{b}{ca+2} + \frac{c}{ab+2} \le 1.$$

(Vasile Cîrtoaje, 2010)

Solution. (a) ***First Solution***. It suffices to show that

$$\frac{a}{abc+2}+\frac{b}{abc+2}+\frac{c}{abc+2}\le 1,$$

which is equivalent to

$$abc+2\ge a+b+c.$$

We have

$$abc+2-a-b-c=(1-b)(1-c)+(1-a)(1-bc)\ge 0.$$

The equality holds for $a=b=c=1$, and for $a=0$ and $b=c=1$ (or any cyclic permutation).

Second Solution. Assume that $a=\max\{a,b,c\}$. It suffices to show that

$$\frac{a}{bc+2}+\frac{b}{bc+2}+\frac{c}{bc+2}\le 1.$$

that is,

$$a+b+c)\le 2+bc.$$

We have

$$2+bc-a-b-c)=1-a+(1-b)(1-c)\ge 0.$$

□

P 1.123. *Let a,b,c be positive real numbers such that $a+b+c=2$. Prove that*

$$5(1-ab-bc-ca)\left(\frac{1}{1-ab}+\frac{1}{1-bc}+\frac{1}{1-ca}\right)+9\ge 0.$$

(Vasile Cîrtoaje, 2011)

Solution. Write the inequality as

$$24-\frac{5a(b+c)}{1-bc}-\frac{5b(c+a)}{1-ca}-\frac{5c(a+b)}{1-ab}\ge 0.$$

Since

$$4(1-bc)\ge 4-(b+c)^2=(a+b+c)^2-(b+c)^2=a(a+2b+2c),$$

it suffices to show that

$$6-5\left(\frac{b+c}{a+2b+2c}-\frac{c+a}{b+2c+2a}-\frac{a+b}{c+2a+2b}\right)\ge 0,$$

which is equivalent to

$$\sum 5\left(1-\frac{b+c}{a+2b+2c}\right)\ge 9,$$

$$5(a+b+c)\sum \frac{1}{a+2b+2c} \ge 9,$$

$$\left[\sum(a+2b+2c)\right]\left(\sum \frac{1}{a+2b+2c}\right) \ge 9.$$

The last inequality follows immediately from the AM-HM inequality. The equality holds for $a=b=c=2/3$.

□

P 1.124. *Let a,b,c be nonnegative real numbers such that $a+b+c=2$. Prove that*

$$\frac{2-a^2}{2-bc}+\frac{2-b^2}{2-ca}+\frac{2-c^2}{2-ab} \le 3.$$

(Vasile Cîrtoaje, 2011)

First Solution. Write the inequality as follows:

$$\sum\left(1-\frac{2-a^2}{2-bc}\right) \ge 0,$$

$$\sum \frac{a^2-bc}{2-bc} \ge 0,$$

$$\sum(a^2-bc)(2-ca)(2-ab) \ge 0,$$

$$\sum(a^2-bc)[4-2a(b+c)+a^2bc] \ge 0,$$

$$4\sum(a^2-bc)-2\sum a(b+c)(a^2-bc)+abc\sum a(a^2-bc) \ge 0.$$

By virtue of the AM-GM inequality,

$$\sum a(a^2-bc) = a^3+b^3+c^3-3abc \ge 0.$$

Then, it suffices to prove that

$$2\sum(a^2-bc) \ge \sum a(b+c)(a^2-bc).$$

Indeed, we have

$$\sum a(b+c)(a^2-bc) = \sum a^3(b+c)-abc\sum(b+c)$$

$$=\sum a(b^3+c^3)-abc\sum(b+c) = \sum a(b+c)(b-c)^2$$

$$\le \sum\left[\frac{a+(b+c)}{2}\right]^2(b-c)^2 = \sum(b-c)^2 = 2\sum(a^2-bc).$$

The equality holds for $a = b = c = 2/3$, and for $a = 0$ and $b = c = 1$ (or any cyclic permutation).

Second Solution. We apply the SOS method. Write the inequality as follows:

$$\sum \frac{a^2 - bc}{2 - bc} \ge 0,$$

$$\sum \frac{(a-b)(a+c) + (a-c)(a+b)}{2 - bc} \ge 0,$$

$$\sum \frac{(a-b)(a+c)}{2-bc} + \sum \frac{(b-a)(b+c)}{2-ca} \ge 0,$$

$$\sum \frac{(a-b)^2[2 - c(a+b) - c^2]}{(2-bc)(2-ca)} \ge 0,$$

$$\sum (a-b)^2(2-ab)(1-c) \ge 0.$$

Assuming that $a \ge b \ge c$, it suffices to prove that

$$(b-c)^2(2-bc)(1-a) + (c-a)^2(2-ca)(1-b) \ge 0.$$

Since

$$2(1-b) = a - b + c \ge 0, \qquad (c-a)^2 \ge (b-c)^2,$$

it suffices to show that

$$(2-bc)(1-a) + (2-ca)(1-b) \ge 0.$$

We have

$$(2-bc)(1-a) + (2-ca)(1-b) = 4 - 2(a+b) - c(a+b) + 2abc$$

$$\ge 4 - (a+b)(2+c) \ge 4 - \left[\frac{(a+b)+(2+c)}{2}\right]^2 = 0.$$

□

P 1.125. *Let a, b, c be nonnegative real numbers such that $a + b + c = 3$. Prove that*

$$\frac{3+5a^2}{3-bc} + \frac{3+5b^2}{3-ca} + \frac{3+5c^2}{3-ab} \ge 12.$$

(Vasile Cîrtoaje, 2010)

Solution. Use the SOS method. Write the inequality as follows:

$$\sum\left(\frac{3+5a^2}{3-bc}-4\right)\ge 0,$$

$$\sum\frac{5a^2+4bc-9}{3-bc}\ge 0,$$

$$\sum\frac{5a^2+4bc-(a+b+c)^2}{3-bc}\ge 0,$$

$$\sum\frac{4a^2-b^2-c^2-2ab+2bc-2ca}{3-bc}\ge 0,$$

$$\sum\frac{2a^2-b^2-c^2+2(a-b)(a-c)}{3-bc}\ge 0,$$

$$\sum\frac{(a-b)(a+b)+(a-c)(a+c)+2(a-b)(a-c)}{3-bc}\ge 0,$$

$$\sum\frac{[(a-b)(a+b)+(a-b)(a-c)]+[(a-c)(a+c)+(a-c)(a-b)]}{3-bc}\ge 0,$$

$$\sum\frac{(a-b)(2a+b-c)+(a-c)(2a+c-b)}{3-bc}\ge 0,$$

$$\sum\frac{(a-b)(2a+b-c)}{3-bc}+\sum\frac{(b-a)(2b+a-c)}{3-ca}\ge 0,$$

$$\sum\frac{(a-b)^2[3-2c(a+b)+c^2]}{(3-bc)(3-ca)}\ge 0,$$

$$\sum\frac{(a-b)^2(c-1)^2}{(3-bc)(3-ca)}\ge 0.$$

The equality holds for $a=b=c=1$.

□

P 1.126. *Let a,b,c be nonnegative real numbers such that $a+b+c=2$. If*

$$\frac{-1}{7}\le m\le\frac{7}{8},$$

then

$$\frac{a^2+m}{3-2bc}+\frac{b^2+m}{3-2ca}+\frac{c^2+m}{3-2ab}\ge\frac{3(4+9m)}{19}.$$

(Vasile Cîrtoaje, 2010)

Solution. We apply the SOS method. Write the inequality as

$$\sum \left(\frac{a^2+m}{3-2bc} - \frac{4+9m}{19} \right) \ge 0,$$

$$\sum \frac{19a^2 + 2(4+9m)bc - 12 - 8m}{3-2bc} \ge 0.$$

Since

$$\begin{aligned}
19a^2 + 2(4+9m)bc - 12 - 8m &= \\
&= 19a^2 + 2(4+9m)bc - (3+2m)(a+b+c)^2 \\
&= (16-2m)a^2 - (3+2m)(b^2+c^2+2ab+2ac) + 2(1+7m)bc \\
&= (3+2m)(2a^2-b^2-c^2) + 2(5-3m)(a^2+bc-ab-ac) + (4-10m)(ab+ac-2bc) \\
&= (3+2m)(a^2-b^2) + (5-3m)(a-b)(a-c) + (4-10m)c(a-b) \\
&\quad +(3+2m)(a^2-c^2) + (5-3m)(a-c)(a-b) + (4-10m)b(a-c) \\
&= (a-b)B + (a-c)C,
\end{aligned}$$

where

$$B = (8-m)a + (3+2m)b - (1+7m)c,$$

$$C = (8-m)a + (3+2m)c - (1+7m)b,$$

the inequality can be written as

$$B_1 + C_1 \ge 0,$$

where

$$B_1 = \sum \frac{(a-b)[(8-m)a + (3+2m)b - (1+7m)c]}{3-2bc},$$

$$C_1 = \sum \frac{(b-a)[(8-m)b + (3+2m)a - (1+7m)c]}{3-2ca}.$$

We have

$$B_1 + C_1 = \sum \frac{(a-b)^2 S_c}{(3-2bc)(3-2ca)},$$

where

$$\begin{aligned}
S_c =& 3(5-3m) - 2(8-m)c(a+b) + 2(1+7m)c^2 \\
=& 6(2m+3)c^2 - 4(8-m)c + 3(5-3m) \\
=& 6(2m+3)\left[c - \frac{8-m}{3(2m+3)}\right]^2 + \frac{(1+7m)(7-8m)}{3(2m+3)}.
\end{aligned}$$

Since $S_c \ge 0$ for $-1/7 \le m \le 7/8$, the proof is completed. The equality holds for $a = b = c = 2/3$. If $m = -1/7$, then the equality holds also for $a = 0$ and $b = c = 1$ (or any cyclic permutation). If $m = 7/8$, then the equality holds also for $a = 1$ and $b = c = 1/2$ (or any cyclic permutation).

Remark. The following more general statement holds:

- *Let a,b,c be nonnegative real numbers such that $a+b+c=3$. If*

$$0<k\le 3, \qquad m_1\le m\le m_2,$$

where

$$m_1=\begin{cases} -\infty, & 0<k\le\dfrac{3}{2} \\ \dfrac{(3-k)(4-k)}{2(3-2k)}, & \dfrac{3}{2}<k\le 3 \end{cases},$$

$$m_2=\frac{36-4k-k^2+4(9-k)\sqrt{3(3-k)}}{72+k},$$

then

$$\frac{a^2+mbc}{9-kbc}+\frac{b^2+mca}{9-kca}+\frac{c^2+mab}{9-kab}\ge\frac{3(1+m)}{9-k},$$

with equality for $a=b=c=1$. If $3/2<k\le 3$ and $m=m_1$, then the equality holds also for

$$a=0, \quad b=c=\frac{3}{2}.$$

If $m=m_2$, then the equality holds also for

$$a=\frac{3k-6+2\sqrt{3(3-k)}}{k}, \quad b=c=\frac{3-\sqrt{3(3-k)}}{k}.$$

The inequalities in P 1.124, P 1.125 and P 1.126 are particular cases of this result (for $k=2$ and $m=m_1=-1$, for $k=3$ and $m=m_2=1/5$, and for $k=8/3$, respectively).

□

P 1.127. *Let a,b,c be nonnegative real numbers such that $a+b+c=3$. Prove that*

$$\frac{47-7a^2}{1+bc}+\frac{47-7b^2}{1+ca}+\frac{47-7c^2}{1+ab}\ge 60.$$

(Vasile Cîrtoaje, 2011)

Solution. We apply the SOS method. Write the inequality as follows:

$$\sum\left(\frac{47-7a^2}{1+bc}-20\right)\ge 0,$$

$$\sum\frac{27-7a^2-20bc}{1+bc}\ge 0,$$

$$\sum \frac{3(a+b+c)^2 - 7a^2 - 20bc}{1+bc} \ge 0,$$

$$\sum \frac{-3(2a^2-b^2-c^2) + 2(a-b)(a-c) + 8(ab-2bc+ca)}{1+bc} \ge 0,$$

$$\sum \frac{-3(a-b)(a+b) + (a-b)(a-c) + 8c(a-b)}{1+bc} +$$

$$+\sum \frac{-3(a-c)(a+c) + (a-c)(a-b) + 8b(a-c)}{1+bc} \ge 0,$$

$$\sum \frac{(a-b)(-2a-3b+7c)}{1+bc} + \sum \frac{(a-c)(-2a-3c+7b)}{1+bc} \ge 0,$$

$$\sum \frac{(a-b)(-2a-3b+7c)}{1+bc} + \sum \frac{(b-a)(-2b-3a+7c)}{1+ca} \ge 0,$$

$$\sum \frac{(a-b)^2[1-2c(a+b)+7c^2]}{(1+bc)(1+ca)} \ge 0,$$

$$\sum \frac{(a-b)^2(3c-1)^2}{(1+bc)(1+ca)} \ge 0,$$

The equality holds for $a=b=c=1$, and for $a=7/3$ and $b=c=1/3$ (or any cyclic permutation).

Remark. The following more general statement holds:

- *Let a,b,c be nonnegative real numbers such that $a+b+c=3$. If*

$$k > 0, \qquad m \ge m_1,$$

where

$$m_1 = \begin{cases} \dfrac{36+4k-k^2+4(9+k)\sqrt{3(3+k)}}{72-k}, & k \ne 72 \\ \dfrac{238}{5}, & k = 72 \end{cases},$$

then

$$\frac{a^2+mbc}{9+kbc} + \frac{b^2+mca}{9+kca} + \frac{c^2+mab}{9+kab} \le \frac{3(1+m)}{9+k},$$

with equality for $a=b=c=1$. If $m=m_1$, then the equality holds also for

$$a = \frac{3k+6-2\sqrt{3(3+k)}}{k}, \quad b = c = \frac{\sqrt{3(3+k)}-3}{k}.$$

The inequality in P 1.127 is a particular case of this result (for $k=9$ and $m=m_1=47/7$).

□

P 1.128. *Let a, b, c be nonnegative real numbers such that $a+b+c=3$. Prove that*

$$\frac{26-7a^2}{1+bc}+\frac{26-7b^2}{1+ca}+\frac{26-7c^2}{1+ab}\le\frac{57}{2}.$$

(Vasile Cîrtoaje, 2011)

Solution. Use the SOS method. Write the inequality as follows:

$$\sum\left(\frac{19}{2}-\frac{26-7a^2}{1+bc}\right)\ge 0,$$

$$\sum\frac{14a^2+19bc-33}{1+bc}\ge 0,$$

$$\sum\frac{42a^2+57bc-11(a+b+c)^2}{1+bc}\ge 0,$$

$$\sum\frac{11(2a^2-b^2-c^2)+9(a-b)(a-c)-13(ab-2bc+ca)}{1+bc}\ge 0,$$

$$\sum\frac{22(a-b)(a+b)+9(a-b)(a-c)-26c(a-b)}{1+bc}+$$

$$+\sum\frac{22(a-c)(a+c)+9(a-c)(a-b)-26b(a-c)}{1+bc}\ge 0,$$

$$\sum\frac{(a-b)(31a+22b-35c)}{1+bc}+\sum\frac{(a-c)(31a+22c-35b)}{1+bc}\ge 0,$$

$$\sum\frac{(a-b)(31a+22b-35c)}{1+bc}+\sum\frac{(b-a)(31b+22a-35c)}{1+ca}\ge 0,$$

$$\sum\frac{(a-b)^2[9+31c(a+b)-35c^2]}{(1+bc)(1+ca)}\ge 0,$$

$$\sum(a-b)^2(1+ab)(1+11c)(3-2c)\ge 0.$$

Assume that $a\ge b\ge c$. Since $3-2c>0$, it suffices to show that

$$(b-c)^2(1+bc)(1+11a)(3-2a)+(c-a)^2(1+ab)(1+11b)(3-2b)\ge 0;$$

that is,

$$(a-c)^2(1+ab)(1+11b)(3-2b)\ge(b-c)^2(1+bc)(1+11a)(2a-3).$$

Since $3-2b=a-b+c\ge 0$, we get this inequality by multiplying the inequalities

$$3-2b\ge 2a-3,$$

$$a(1+ab)\ge b(1+bc),$$

$$a(1+11b)\ge b(1+11a),$$

$$b^2(a-c)^2 \geq a^2(b-c)^2.$$

The equality holds for $a = b = c = 1$, and for $a = b = 3/2$ and $c = 0$ (or any cyclic permutation).

Remark. The following more general statement holds:

- *Let a, b, c be nonnegative real numbers such that $a + b + c = 3$. If*

$$k > 0, \qquad m \leq m_2, \qquad m_2 = \frac{(3+k)(4+k)}{2(3+2k)},$$

then

$$\frac{a^2+mbc}{9+kbc} + \frac{b^2+mca}{9+kca} + \frac{c^2+mab}{9+kab} \geq \frac{3(1+m)}{9+k},$$

with equality for $a = b = c = 1$. When $m = m_2$, the equality holds also for $a = 0$ and $b = c = 3/2$ (or any cyclic permutation).

The inequalities in P 1.128 is a particular cases of this result (for $k = 9$ and $m = m_2 = 26/7$).

□

P 1.129. *If a, b, c are nonnegative real numbers, then*

$$\sum \frac{5a(b+c)-6bc}{a^2+b^2+c^2+bc} \leq 3.$$

(Vasile Cîrtoaje, 2010)

First Solution. Apply the SOS method. If two of a, b, c are zero, then the inequality is trivial. Consider further that

$$a^2+b^2+c^2 = 1, \qquad a \geq b \geq c, \quad b > 0,$$

and write the inequality as follows:

$$\sum \left[1 - \frac{5a(b+c)-6bc}{a^2+b^2+c^2+bc}\right] \geq 0,$$

$$\sum \frac{a^2+b^2+c^2-5a(b+c)+7bc}{a^2+b^2+c^2+bc} \geq 0,$$

$$\sum \frac{(7b+2c-a)(c-a)-(7c+2b-a)(a-b)}{1+bc} \geq 0,$$

$$\sum \frac{(7c+2a-b)(a-b)}{1+ca} - \sum \frac{(7c+2b-a)(a-b)}{1+bc} \geq 0,$$

$$\sum (a-b)^2(1+ab)(3+ac+bc-7c^2) \geq 0.$$

Since

$$3+ac+bc-7c^2=3a^2+3b^2+ac+bc-4c^2>0,$$

it suffices to prove that

$$(1+bc)(3+ab+ac-7a^2)(b-c)^2+(1+ac)(3+ab+bc-7b^2)(a-c)^2\geq 0.$$

Since

$$3+ab+ac-7b^2=3(a^2-b^2)+3c^2+b(a-b)+bc\geq 0$$

and $1+ac\geq 1+bc$, it is enough to show that

$$(3+ab+ac-7a^2)(b-c)^2+(3+ab+bc-7b^2)(a-c)^2\geq 0.$$

From $b(a-c)\geq a(b-c)\geq 0$, we get $b^2(a-c)^2\geq a^2(b-c)^2$, hence

$$b(a-c)^2\geq a(b-c)^2.$$

Thus, it suffices to show that

$$b(3+ab+ac-7a^2)+a(3+ab+bc-7b^2)\geq 0.$$

This is true if

$$b(3+ab-7a^2)+a(3+ab-7b^2)\geq 0.$$

Indeed,

$$b(3+ab-7a^2)+a(3+ab-7b^2)=3(a+b)(1-2ab)\geq 0,$$

since

$$1-2ab=(a-b)^2+c^2\geq 0.$$

The equality holds for $a=b=c$, and for $a=b$ and $c=0$ (or any cyclic permutation).

Second Solution. Without loss of generality, assume that $a^2+b^2+c^2=1$ and $a\leq b\leq c$. Setting

$$p=a+b+c,\qquad q=ab+bc+ca,\qquad r=abc,$$

the inequality becomes

$$\sum\frac{5q-11bc}{1+bc}\leq 3,$$

$$3\prod(1+bc)+\sum(11bc-5q)(1+ca)(1+ab)\geq 0,$$

$$3(1+q+pr+r^2)+11(q+2pr+3r^2)-5q(3+2q+pr)\geq 0,$$

$$36r^2+5(5-q)pr+3-q-10q^2\geq 0.$$

According to P 3.57-(a) in Volume 1, for fixed p and q, the product $r=abc$ is minimum when $b=c$ or $a=0$. Therefore, since $5-q\geq 4>0$, it suffices to prove the original homogeneous inequality for $a=0$, and for $b=c=1$. For $a=0$, the original inequality becomes

$$\frac{-6bc}{b^2+c^2+bc}+\frac{10bc}{b^2+c^2}\leq 3,$$

$$(b-c)^2(3b^2+5bc+3b^2) \ge 0,$$

while for $b=c=1$, the original inequality becomes

$$\frac{10a-6}{a^2+3} + 2\frac{5-a}{a^2+a+2} \le 3,$$

which is equivalent to

$$a(3a+1)(a-1)^2 \ge 0.$$

Remark. Similarly, we can prove the following generalization:

- *Let a,b,c be nonnegative real numbers. If $k>0$, then*

$$\sum \frac{(2k+3)a(b+c)+(k+2)(k-3)bc}{a^2+b^2+c^2+kbc} \le 3k,$$

with equality for $a=b=c$, and for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.130. *Let a,b,c be nonnegative real numbers, no two of which are zero, and let*

$$x = \frac{a^2+b^2+c^2}{ab+bc+ca}.$$

Prove that

(a)
$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{1}{2} \ge x+\frac{1}{x};$$

(b)
$$6\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right) \ge 5x+\frac{4}{x};$$

(c)
$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}-\frac{3}{2} \ge \frac{1}{3}\left(x-\frac{1}{x}\right).$$

(Vasile Cîrtoaje, 2011)

Solution. We will prove the more general inequality

$$\frac{2a}{b+c}+\frac{2b}{c+a}+\frac{2c}{a+b}+1-3k \ge (2-k)x+\frac{2(1-k)}{x},$$

where

$$0 \le k \le k_0, \qquad k_0 = \frac{21+6\sqrt{6}}{25} \approx 1.428.$$

For $k=0$, $k=1/3$ and $k=4/3$, we get the inequalities in (a), (b) and (c), respectively. Let $p=a+b+c$ and $q=ab+bc+ca$. Since $x=(p^2-2q)/q$, we can write the inequality as follows:

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \ge f(p,q),$$

$$\sum \left(\frac{a}{b+c} + 1 \right) \ge 3 + f(p,q),$$

$$\frac{p(p^2+q)}{pq - abc} \ge 3 + f(p,q).$$

According to P 3.57-(a) in Volume 1, for fixed p and q, the product abc is minimum when $b = c$ or $a = 0$. Therefore, it suffices to prove the inequality for $a = 0$, and for $b = c = 1$. For $a = 0$, using the substitution $y = b/c + c/b$, the desired inequality becomes

$$2y + 1 - 3k \ge (2-k)y + \frac{2(1-k)}{y},$$

$$\frac{(y-2)[k(y-1)+1]}{y} \ge 0.$$

Since $y \ge 2$, this inequality is clearly true. For $b = c = 1$, the desired inequality becomes

$$a + \frac{4}{a+1} + 1 - 3k \ge \frac{(2-k)(a^2+2)}{2a+1} + \frac{2(1-k)(2a+1)}{a^2+2},$$

which is equivalent to

$$a(a-1)^2[ka^2 + 3(1-k)a + 6 - 4k] \ge 0.$$

For $0 \le k \le 1$, this is obvious, and for $1 < k \le (21 + 6\sqrt{6})/25$, we have

$$ka^2 + 3(1-k)a + 6 - 4k \ge [2\sqrt{k(6-4k)} + 3(1-k)]a \ge 0.$$

The equality holds for $a = b = c$, and for $a = 0$ and $b = c$ (or any cyclic permutation). If $k = k_0$, then the equality holds also for $(2+\sqrt{6})a = 2b = 2c$ (or any cyclic permutation).

□

P 1.131. *If a, b, c are real numbers, then*

$$\frac{1}{a^2+7(b^2+c^2)} + \frac{1}{b^2+7(c^2+a^2)} + \frac{1}{c^2+7(a^2+b^2)} \le \frac{9}{5(a+b+c)^2}.$$

(Vasile Cîrtoaje, 2008)

Solution. We use the *highest coefficient method.* Let

$$p = a+b+c, \qquad q = ab+bc+ca.$$

Write the inequality as $f_6(a,b,c) \ge 0$, where

$$f_6(a,b,c) = 9\prod(a^2+7b^2+7c^2) - 5p^2\sum(b^2+7c^2+7a^2)(c^2+7a^2+7b^2).$$

Since

$$\prod(a^2+7b^2+7c^2)=\prod[7(p^2-2q)-6a^2],$$

$f_6(a,b,c)$ has the highest coefficient

$$A=9(-6)^3<0.$$

According to P 2.75 in Volume 1, it suffices to prove the original inequality for $b=c=1$, when the inequality reduces to

$$\frac{1}{a^2+14}+\frac{2}{7a^2+8}\le\frac{9}{5(a+2)^2},$$

$$(a-1)^2(a-4)^2\ge 0.$$

Thus, the proof is completed. The equality holds for $a=b=c$, and for $a/4=b=c$ (or any cyclic permutation).

□

P 1.132. *If a,b,c are real numbers, then*

$$\frac{bc}{3a^2+b^2+c^2}+\frac{ca}{3b^2+c^2+a^2}+\frac{ab}{3c^2+a^2+b^2}\le\frac{3}{5}.$$

(Vasile Cîrtoaje and Pham Kim Hung, 2005)

Solution. Use the *highest coefficient method*. Write the inequality as $f_6(a,b,c)\ge 0$, where

$$f_6(a,b,c)=3\prod(3a^2+b^2+c^2)-5\sum bc(3b^2+c^2+a^2)(3c^2+a^2+b^2).$$

Let

$$p=a+b+c,\qquad q=ab+bc+ca.$$

From

$$f_6(a,b,c)=3\prod(2a^2+p^2-2q)-5\sum bc(2b^2+p^2-2q)(2c^2+p^2-2q),$$

it follows that $f_6(a,b,c)$ has the same highest coefficient A as

$$24a^2b^2c^2-20\sum b^3c^3;$$

that is,

$$A=24-60<0.$$

According to P 2.75 in Volume 1, it suffices to prove the original inequality for $b=c=1$, when the inequality is equivalent to

$$\frac{1}{3a^2+2}+\frac{2a}{a^2+4}\le\frac{3}{5},$$

$$(a-1)^2(3a-2)^2 \ge 0.$$

Thus, the proof is completed. The equality holds for $a=b=c$, and for $3a/2=b=c$ (or any cyclic permutation).

Remark. The inequality in P 1.132 is a particular case ($k=3$) of the following more general result (*Vasile Cîrtoaje*, 2008):

- *Let a,b,c be real numbers. If $k>1$, then*

$$\sum \frac{k(k-3)a^2+2(k-1)bc}{ka^2+b^2+c^2} \le \frac{3(k+1)(k-2)}{k+2},$$

with equality for $a=b=c$, and for $ka/2=b=c$ (or any cyclic permutation).

□

P 1.133. *If a,b,c are real numbers such that $a+b+c=3$, then*

$$\frac{1}{8+5(b^2+c^2)}+\frac{1}{8+5(c^2+a^2)}+\frac{1}{8+5(a^2+b^2)} \le \frac{1}{6}.$$

(Vasile Cîrtoaje, 2006)

Solution. Use the *highest coefficient method*. Denote

$$p=a+b+c, \qquad q=ab+bc+ca,$$

and write the inequality in the homogeneous form

$$\frac{1}{8p^2+45(b^2+c^2)}+\frac{1}{8p^2+45(c^2+a^2)}+\frac{1}{8p^2+45(a^2+b^2)} \le \frac{1}{6p^2},$$

which is equivalent to $f_6(a,b,c) \ge 0$, where

$$\begin{aligned} f_6(a,b,c) = &\prod(53p^2-90q-45a^2) \\ &- 6p^2\sum(53p^2-90q-45b^2)(53p^2-90q-45c^2). \end{aligned}$$

Clearly, $f_6(a,b,c)$ has the highest coefficient

$$A=(-45)^3<0.$$

According to P 2.75 in Volume 1, it suffices to prove the homogeneous inequality for $b=c=1$; that is,

$$\frac{1}{8(a+2)^2+90}+\frac{2}{8(a+2)^2+45(1+a^2)} \le \frac{1}{6(a+2)^2}.$$

Using the substitution

$$a+2=3x,$$

the inequality becomes as follows:

$$\frac{1}{72x^2+90}+\frac{2}{72x^2+45+45(3x-2)^2)}\le\frac{1}{54x^2},$$

$$\frac{1}{8x^2+10}+\frac{2}{53x^2-60x+25}\le\frac{1}{6x^2},$$

$$x^4-12x^3+46x^2-60x+25\ge 0,$$

$$(x-1)^2(x-5)^2\ge 0,$$

$$(a-1)^2(a-13)^2\ge 0.$$

The equality holds for $a=b=c=1$, and for $a=13/5$ and $b=c=1/5$ (or any cyclic permutation).

□

P 1.134. *If a,b,c are real numbers, then*

$$\frac{(a+b)(a+c)}{a^2+4(b^2+c^2)}+\frac{(b+c)(b+a)}{b^2+4(c^2+a^2)}+\frac{(c+a)(c+b)}{c^2+4(a^2+b^2)}\le\frac{4}{3}.$$

(Vasile Cîrtoaje, 2008)

Solution. Use the *highest coefficient method*. Let

$$p=a+b+c,\qquad q=ab+bc+ca.$$

Write the inequality as $f_6(a,b,c)\ge 0$, where

$$f_6(a,b,c)=4\prod(a^2+4b^2+4c^2)$$

$$-3\sum(a+b)(a+c)(b^2+4c^2+4a^2)(c^2+4a^2+4b^2)$$

$$=4\prod(4p^2-8q-3a^2)-3\sum(a^2+q)(4p^2-8q-3b^2)(4p^2-8q-3c^2).$$

Thus, $f_6(a,b,c)$ has the highest coefficient

$$A=4(-3)^3-3^4<0.$$

By P 2.75 in Volume 1, it suffices to prove the original inequality for $b=c=1$, when the inequality is equivalent to

$$\frac{(a+1)^2}{a^2+8}+\frac{4(a+1)}{4a^2+5}\le\frac{4}{3},$$

$$(a-1)^2(2a-7)^2\ge 0.$$

The equality holds for $a=b=c$, and for $2a/7=b=c$ (or any cyclic permutation).

□

P 1.135. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\sum \frac{1}{(b+c)(7a+b+c)} \le \frac{1}{2(ab+bc+ca)}.$$

(Vasile Cîrtoaje, 2009)

First Solution. Write the inequality as

$$\sum \left[1 - \frac{4(ab+bc+ca)}{(b+c)(7a+b+c)}\right] \ge 1,$$

$$\sum \frac{(b-c)^2 + 3a(b+c)}{(b+c)(7a+b+c)} \ge 1.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{(b-c)^2 + 3a(b+c)}{(b+c)(7a+b+c)} \ge \frac{4(a+b+c)^4}{\sum[(b-c)^2 + 3a(b+c)](b+c)(7a+b+c)}.$$

Therefore, it suffices to show that

$$4(a+b+c)^4 \ge \sum (b^2 + c^2 - 2bc + 3ca + 3ab)(b+c)(7a+b+c).$$

Write this inequality as

$$\sum a^4 + abc \sum a + 3\sum ab(a^2+b^2) - 8\sum a^2b^2 \ge 0,$$

$$\sum a^4 + abc \sum a - \sum ab(a^2+b^2) + 4\sum ab(a-b)^2 \ge 0.$$

Since

$$\sum a^4 + abc \sum a - \sum ab(a^2+b^2) \ge 0$$

(Schur's inequality of degree four), the conclusion follows. The equality holds for $a = b = c$, and also for $a = 0$ and $b = c$ (or any cyclic permutation).

Second Solution. Use the *highest coefficient method*. We need to prove that $f_6(a,b,c) \ge 0$, where

$$f_6(a,b,c) = \prod (b+c)(7a+b+c)$$

$$-2(ab+bc+ca)\sum (a+b)(a+c)(7b+c+a)(7c+a+b).$$

Let $p = a+b+c$. Clearly, $f_6(a,b,c)$ has the same highest coefficient A as $f(a,b,c)$, where

$$f(a,b,c) = \prod (b+c)(7a+b+c) = \prod (p-a)(p+6a);$$

that is,

$$A = (-6)^3 < 0.$$

Thus, by P 3.76-(a) in Volume 1, it suffices to prove the original inequality for $b = c = 1$, and for $a = 0$.

For $b = c = 1$, the inequality reduces to

$$\frac{1}{2(7a+2)} + \frac{2}{(a+1)(a+8)} \le \frac{1}{2(2a+1)},$$

$$a(a-1)^2 \ge 0.$$

For $a = 0$, the inequality can be written as

$$\frac{1}{(b+c)^2} + \frac{1}{c(7b+c)} + \frac{1}{b(7c+b)} \le \frac{1}{2bc},$$

$$\frac{1}{(b+c)^2} + \frac{b^2+c^2+14bc}{bc[7(b^2+c^2)+50bc]} \le \frac{1}{2bc},$$

$$\frac{1}{x+2} + \frac{x+14}{7x+50} \le \frac{1}{2},$$

where

$$x = \frac{b}{c} + \frac{c}{b}, \qquad x \ge 2.$$

This reduces to the obvious inequality

$$(x-2)(5x+28) \ge 0.$$

□

P 1.136. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\sum \frac{1}{b^2+c^2+4a(b+c)} \le \frac{9}{10(ab+bc+ca)}.$$

(Vasile Cîrtoaje, 2009)

Solution. Use the *highest coefficient method.* Let

$$p = a+b+c, \qquad q = ab+bc+ca.$$

We need to prove that $f_6(a,b,c) \ge 0$, where

$$f_6(a,b,c) = 9\prod[b^2+c^2+4a(b+c)]$$

$$-10(ab+bc+ca)\sum[a^2+b^2+4c(a+b)][a^2+c^2+4b(a+c)]$$

$$= 9\prod(p^2+2q-a^2-4bc) - 10q\sum(p^2+2q-c^2-4ab)(p^2+2q-b^2-4ca).$$

Clearly, $f_6(a,b,c)$ has the same highest coefficient A as $P_3(a,b,c)$, where

$$P_3(a,b,c) = -9\prod(a^2+4bc).$$

According to Remark 2 from the proof of P 2.75 in Volume 1,

$$A = P_3(1,1,1) = -9\cdot 125 < 0.$$

Thus, by P 3.76-(a) in Volume 1, it suffices to prove the original inequality for $b=c=1$, and for $a=0$.

For $b=c=1$, the inequality reduces to

$$\frac{1}{2(4a+1)} + \frac{2}{a^2+4a+5} \le \frac{9}{10(2a+1)},$$

$$a(a-1)^2 \ge 0.$$

For $a=0$, the inequality becomes

$$\frac{1}{b^2+c^2} + \frac{1}{b^2+4bc} + \frac{1}{c^2+4bc} \le \frac{9}{10bc},$$

$$\frac{1}{b^2+c^2} + \frac{b^2+c^2+8bc}{4bc(b^2+c^2)+17b^2c^2} \le \frac{9}{10bc}.$$

$$\frac{1}{x} + \frac{x+8}{4x+17} \le \frac{9}{10},$$

$$(x-2)(26x+85) \ge 0,$$

where

$$x = \frac{b}{c}+\frac{c}{b}, \qquad x \ge 2.$$

The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.137. *Let a,b,c be nonnegative real numbers, no two of which are zero. If $a+b+c=3$, then*

$$\frac{1}{3-ab} + \frac{1}{3-bc} + \frac{1}{3-ca} \le \frac{9}{2(ab+bc+ca)}.$$

(Vasile Cîrtoaje, 2011)

First Solution. We apply the SOS method. Write the inequality as

$$\sum\left(\frac{3}{2} - \frac{ab+bc+ca}{3-bc}\right) \ge 0.$$

$$\sum \frac{9-2a(b+c)-5bc}{3-bc} \ge 0,$$

$$\sum \frac{a^2+b^2+c^2-3bc}{3-bc} \geq 0.$$

Since

$$2(a^2+b^2+c^2-3bc) = 2(a^2-bc)+2(b^2+c^2-ab-ac)+2(ab+ac-2bc)$$

$$= (a-b)(a+c)+(a-c)(a+b)-2b(a-b)-2c(a-c)+2c(a-b)+2b(a-c)$$

$$= (a-b)(a-2b+3c)+(a-c)(a-2c+3b),$$

the required inequality is equivalent to

$$\sum \frac{(a-b)(a-2b+3c)+(a-c)(a-2c+3b)}{3-bc} \geq 0,$$

$$\sum \frac{(a-b)(a-2b+3c)}{3-bc} + \sum \frac{(b-a)(b-2a+3c)}{3-ca} \geq 0,$$

$$\sum \frac{(a-b)^2[9-c(a+b+3c)]}{(3-bc)(3-ca)} \geq 0,$$

$$\sum (a-b)^2(3-ab)(3+c)(3-2c) \geq 0.$$

Without loss of generality, assume that $a \geq b \geq c$. It suffices to prove that

$$(b-c)^2(3-bc)(3+a)(3-2a)+(c-a)^2(3-ca)(3+b)(3-2b) \geq 0,$$

which is equivalent to

$$(a-c)^2(3-ac)(3+b)(3-2b) \geq (b-c)^2(3-bc)(a+3)(2a-3).$$

Since $3-2b = a-b+c \geq 0$, we can obtain this inequality by multiplying the inequalities

$$b^2(a-c)^2 \geq a^2(b-c)^2,$$

$$a(3-ac) \geq b(3-bc),$$

$$a(3+b)(3-2b) \geq b(a+3)(2a-3) \geq 0.$$

We have

$$a(3-ac)-b(3-bc) = (a-b)[3-c(a+b)] = (a-b)(3-3c+c^2)$$
$$\geq (a-b)(3-3c) \geq 0.$$

Also, since $a+b \leq a+b+c = 3$, we have

$$a(3+b)(3-2b)-b(a+3)(2a-3) = 9(a+b)-6ab-2ab(a+b)$$

$$\geq 9(a+b)-12ab \geq 3(a+b)^2-12ab = 3(a-b)^2 \geq 0.$$

The equality holds for $a = b = c = 1$, and for $a = 0$ and $b = c = 3/2$ (or any cyclic permutation).

Second Solution. Write the inequality in the homogeneous form

$$\frac{1}{p^2 - 3ab} + \frac{1}{p^2 - 3bc} + \frac{1}{p^2 - 3ca} \le \frac{3}{2q},$$

where

$$p = a + b + c, \qquad q = ab + bc + ca.$$

We need to prove that $f_6(a, b, c) \ge 0$, where

$$f_6(a, b, c) = 3\prod(p^2 - 3bc) - 2q\sum(p^2 - 3ca)(p^2 - 3ab).$$

Clearly, $f_6(a, b, c)$ has the highest coefficient

$$A = 3(-3)^3 < 0.$$

Thus, by P 3.76-(a) in Volume 1, it suffices to prove the homogeneous inequality for $b = c = 1$, and for $a = 0$.

For $b = c = 1$, the homogeneous inequality reduces to

$$\frac{2}{(a+2)^2 - 3a} + \frac{1}{(a+2)^2 - 3} \le \frac{3}{2(2a+1)},$$

$$\frac{a^2 + 3a + 2}{(a^2 + a + 4)(a^2 + 4a + 1)} \le \frac{3}{2(2a+1)},$$

$$a(a + 3)(a - 1)^2 \ge 0.$$

For $a = 0$, the homogeneous inequality can be written as

$$\frac{2}{(b+c)^2} + \frac{1}{(b+c)^2 - 3bc} \le \frac{3}{2bc},$$

$$\frac{(b-c)^2(b^2 + c^2 + bc)}{2bc(b+c)^2(b^2 + c^2 - bc)} \ge 0.$$

□

P 1.138. *If a, b, c are nonnegative real numbers such that $a + b + c = 3$, then*

$$\frac{bc}{a^2 + a + 6} + \frac{ca}{b^2 + b + 6} + \frac{ab}{c^2 + c + 6} \le \frac{3}{8}.$$

(Vasile Cîrtoaje, 2009)

Solution. Write the inequality in the homogeneous form

$$\frac{bc}{3a^2+ap+2p^2}+\frac{ca}{3b^2+bp+2p^2}+\frac{ab}{3c^2+cp+2p^2}\le\frac{1}{8},\qquad p=a+b+c.$$

We need to prove that $f_6(a,b,c)\ge 0$, where

$$f_6(a,b,c)=\prod(3a^2+ap+2p^2)-8\sum bc(3b^2+bp+2p^2)(3c^2+cp+2p^2).$$

Clearly, $f_6(a,b,c)$ has the same highest coefficient as

$$27a^2b^2c^2-72\sum b^3c^3;$$

that is,

$$A=27-216<0.$$

By P 3.76-(a) in Volume 1, it suffices to prove the homogeneous inequality for $b=c=1$, and for $a=0$.

For $b=c=1$, the homogeneous inequality reduces to

$$\frac{1}{2(3a^2+5a+4)}+\frac{2a}{2a^2+9a+13}\le\frac{1}{8},$$

$$6a^4-11a^3+4a^2+a\ge 0,$$

$$a(6a+1)(a-1)^2\ge 0.$$

For $a=0$, the homogeneous inequality can be written as

$$\frac{bc}{2(b+c)^2}\le\frac{1}{8},$$

$$(b-c)^2\ge 0.$$

The equality holds for $a=b=c=1$, and for $a=0$ and $b=c=3/2$ (or any cyclic permutation).

□

P 1.139. *If a,b,c are nonnegative real numbers such that $ab+bc+ca=3$, then*

$$\frac{1}{8a^2-2bc+21}+\frac{1}{8b^2-2ca+21}+\frac{1}{8c^2-2ab+21}\ge\frac{1}{9}.$$

(Michael Rozenberg, 2013)

Solution. Write the inequality in the homogeneous form

$$\frac{1}{8a^2-2bc+7q}+\frac{1}{8b^2-2ca+7q}+\frac{1}{8c^2-2ab+7q}\ge\frac{1}{3q},\qquad q=ab+bc+ca.$$

We need to prove that $f_6(a,b,c)\ge 0$, where

$$f_6(a,b,c)=3q\sum(8b^2-2ca+7q)(8c^2-2ab+7q)-\prod(8a^2-2bc+7q).$$

Clearly, $f_6(a,b,c)$ has the same highest coefficient as $P_2(a,b,c)$, where

$$P_2(a,b,c)=-\prod(8a^2-2bc).$$

According to Remark 2 from the proof of P 2.75 in Volume 1, we have

$$A=P_2(1,1,1)=-6^3<0.$$

By P 3.76-(a) in Volume 1, it suffices to prove the homogeneous inequality for $b=c=1$, and for $a=0$.

For $b=c=1$, the homogeneous inequality reduces to

$$\frac{1}{8a^2+14a+5}+\frac{2}{12a+15}\ge\frac{1}{3(2a+1)},$$

$$\frac{1}{(4a+5)(2a+1)}+\frac{2}{3(4a+5)}\ge\frac{1}{3(2a+1)},$$

which is an identity.

For $a=0$, the homogeneous inequality can be written as

$$\frac{1}{b(8b+7c)}+\frac{1}{c(8c+7b)}\ge\frac{2}{15bc},$$

$$\frac{c}{8b+7c}+\frac{b}{8c+7b}\ge\frac{2}{15},$$

$$(b-c)^2\ge 0.$$

The equality holds when two of a,b,c are equal.

Remark. The following identity holds for $ab+bc+ca=3$:

$$\sum\frac{9}{8a^2-2bc+21}-1=\frac{8\prod(a-b)^2}{\prod(8a^2-2bc+21)}.$$

□

P 1.140. *Let a, b, c be real numbers, no two of which are zero. Prove that*

(a) $$\frac{a^2+bc}{b^2+c^2}+\frac{b^2+ca}{c^2+a^2}+\frac{c^2+ab}{a^2+b^2} \ge \frac{(a+b+c)^2}{a^2+b^2+c^2};$$

(b) $$\frac{a^2+3bc}{b^2+c^2}+\frac{b^2+3ca}{c^2+a^2}+\frac{c^2+3ab}{a^2+b^2} \ge \frac{6(ab+bc+ca)}{a^2+b^2+c^2}.$$

(Vasile Cîrtoaje, 2014)

Solution. (a) Using the known inequality

$$\sum \frac{a^2}{b^2+c^2} \ge \frac{3}{2}$$

and the Cauchy-Schwarz inequality yields

$$\sum \frac{a^2+bc}{b^2+c^2} = \sum \frac{a^2}{b^2+c^2} + \sum \frac{bc}{b^2+c^2} \ge \sum \left(\frac{1}{2}+\frac{bc}{b^2+c^2}\right)$$
$$= \sum \frac{(b+c)^2}{2(b^2+c^2)} \ge \frac{\left[\sum(b+c)\right]^2}{\sum 2(b^2+c^2)} = \frac{(a+b+c)^2}{a^2+b^2+c^2}.$$

The equality holds for $a = b = c$.

(b) We have

$$\sum \frac{a^2+3bc}{b^2+c^2} = \sum \frac{a^2}{b^2+c^2} + \sum \frac{3bc}{b^2+c^2} \ge \frac{3}{2} + \sum \frac{3bc}{b^2+c^2}$$
$$= -3 + 3\sum \left(\frac{1}{2}+\frac{bc}{b^2+c^2}\right) = -3 + 3\sum \frac{(b+c)^2}{2(b^2+c^2)}$$
$$\ge -3 + \frac{3\left[\sum(b+c)\right]^2}{\sum 2(b^2+c^2)} = -3 + \frac{3\left(\sum a\right)^2}{\sum a^2} = \frac{6(ab+bc+ca)}{a^2+b^2+c^2}.$$

The equality holds for $a = b = c$.

□

P 1.141. *Let a, b, c be real numbers such that $ab+bc+ca \ge 0$ and no two of which are zero. Prove that*

$$\frac{a(b+c)}{b^2+c^2}+\frac{b(c+a)}{c^2+a^2}+\frac{c(a+b)}{a^2+b^2} \ge \frac{3}{10}.$$

(Vasile Cîrtoaje, 2014)

Solution. Since the problem remains unchanged by replacing a, b, c with $-a, -b, -c$, it suffices to consider the cases $a, b, c \ge 0$ and $a < 0$, $b \ge 0$, $c \ge 0$.

Case 1: $a, b, c \geq 0$. We have

$$\sum \frac{a(b+c)}{b^2+c^2} \geq \sum \frac{a(b+c)}{(b+c)^2}$$
$$= \sum \frac{a}{b+c} \geq \frac{3}{2} > \frac{3}{10}.$$

Case 2: $a < 0$, $b \geq 0$, $c \geq 0$. Replacing a by $-a$, we need to show that

$$\frac{b(c-a)}{a^2+c^2} + \frac{c(b-a)}{a^2+b^2} - \frac{a(b+c)}{b^2+c^2} \geq \frac{3}{10},$$

where

$$a, b, c \geq 0, \qquad a \leq \frac{bc}{b+c}.$$

We show first that

$$\frac{b(c-a)}{a^2+c^2} \geq \frac{b(c-x)}{x^2+c^2},$$

where $x = \dfrac{bc}{b+c}$, $x \geq a$. This is equivalent to

$$b(x-a)[(c-a)x + ac + c^2] \geq 0,$$

which is true because

$$(c-a)x + ac + c^2 = \frac{c^2(a+2b+c)}{b+c} \geq 0.$$

Similarly, we can show that

$$\frac{c(b-a)}{a^2+b^2} \geq \frac{c(b-x)}{x^2+b^2}.$$

In addition, since

$$\frac{a(b+c)}{b^2+c^2} \leq \frac{x(b+c)}{b^2+c^2}.$$

it suffices to prove that

$$\frac{b(c-x)}{x^2+c^2} + \frac{c(b-x)}{x^2+b^2} - \frac{x(b+c)}{b^2+c^2} \geq \frac{3}{10}.$$

Denote

$$p = \frac{b}{b+c}, \qquad q = \frac{c}{b+c}, \qquad p+q = 1.$$

Since

$$\frac{b(c-x)}{x^2+c^2} = \frac{p}{1+p^2}, \qquad \frac{c(b-x)}{x^2+b^2} = \frac{q}{1+q^2},$$
$$\frac{x(b+c)}{b^2+c^2} = \frac{bc}{b^2+c^2} = \frac{pq}{1-2pq},$$

we need to show that

$$\frac{p}{1+p^2}+\frac{q}{1+q^2}-\frac{pq}{1-2pq} \geq \frac{3}{10}.$$

This inequality is equivalent to

$$\frac{1+pq}{2-2pq+p^2q^2}-\frac{pq}{1-2pq} \geq \frac{3}{10},$$

$$(pq+2)^2(1-4pq) \geq 0.$$

Since

$$1-4pq=(p+q)^2-4pq=(p-q)^2 \geq 0,$$

the proof is completed. The equality holds for $-2a=b=c$ (or any cyclic permutation).

□

P 1.142. *If a,b,c are positive real numbers such that $abc>1$, then*

$$\frac{1}{a+b+c-3}+\frac{1}{abc-1} \geq \frac{4}{ab+bc+ca-3}.$$

(Vasile Cîrtoaje, 2011)

Solution *(by Vo Quoc Ba Can).* By the AM-GM inequality, we have

$$a+b+c \geq 3\sqrt[3]{abc} > 3,$$

$$ab+bc+ca \geq \sqrt[3]{a^2b^2c^2} > 3.$$

Without loss of generality, assume that $a=\min\{a,b,c\}$. By the Cauchy-Schwarz inequality, we have

$$\left(\frac{1}{a+b+c-3}+\frac{1}{abc-1}\right)\left[a(a+b+c-3)+\frac{abc-1}{a}\right] \geq \left(\sqrt{a}+\frac{1}{\sqrt{a}}\right)^2.$$

Therefore, it suffices to prove that

$$\frac{(a+1)^2}{4a} \geq \frac{a(a+b+c-3)+\dfrac{abc-1}{a}}{ab+bc+ca-3}.$$

Since

$$a(a+b+c-3)+\frac{abc-1}{a}=ab+bc+ca-3+\frac{(a-1)^3}{a},$$

this inequality can be written as follows:

$$\frac{(a+1)^2}{4a}-1 \geq \frac{(a-1)^3}{a(ab+bc+ca-3)},$$

$$\frac{(a-1)^2}{4a} \ge \frac{(a-1)^3}{a(ab+bc+ca-3)},$$

$$(a-1)^2(ab+bc+ca+1-4a) \ge 0.$$

This is true since

$$bc \ge \sqrt[3]{(abc)^2} > 1,$$

hence

$$ab+bc+ca+1-4a > a^2+1+a^2+1-4a = 2(a-1)^2 \ge 0.$$

The equality holds for $a=b=1$ and $c>1$ (or any cyclic permutation).

Remark. Using this inequality, we can prove P 3.84 in Volume 1, which states that

$$(a+b+c-3)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-3\right)+abc+\frac{1}{abc} \ge 2$$

for any positive real numbers a,b,c. This inequality is clearly true for $abc=1$. In addition, it remains unchanged by substituting a, b, c with $1/a$, $1/b$, $1/c$, respectively. Therefore, it suffices to consider the case $abc>1$. Since $a+b+c \ge 3\sqrt[3]{abc} > 3$, we can write the required inequality as $E \ge 0$, where

$$E = ab+bc+ca-3abc+\frac{(abc-1)^2}{a+b+c-3}.$$

According to the inequality in P 1.142, we have

$$E \ge ab+bc+ca-3abc+(abc-1)^2\left(\frac{4}{ab+bc+ca-3}-\frac{1}{abc-1}\right)$$

$$= (ab+bc+ca-3)+\frac{4(abc-1)^2}{ab+bc+ca-3}-4(abc-1)$$

$$\ge 2\sqrt{(ab+bc+ca-3)\cdot\frac{4(abc-1)^2}{ab+bc+ca-3}}-4(abc-1) = 0.$$

□

P 1.143. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\sum \frac{(4b^2-ac)(4c^2-ab)}{b+c} \le \frac{27}{2}abc.$$

(Vasile Cîrtoaje, 2011)

Solution. Use the SOS method. Since

$$\begin{aligned}\sum \frac{(4b^2-ac)(4c^2-ab)}{b+c} &= \sum \frac{bc(16bc+a^2)}{b+c} - 4\sum \frac{a(b^3+c^3)}{b+c}\\ &= \sum \frac{bc(16bc+a^2)}{b+c} - 4\sum a(b^2+c^2) + 12abc\\ &= \sum bc\left[\frac{a^2}{b+c} + \frac{16bc}{b+c} - 4(b+c)\right] + 12abc\\ &= \sum bc\left[\frac{a^2}{b+c} - 4\frac{(b-c)^2}{b+c}\right] + 12abc\end{aligned}$$

we can write the inequality as follows:

$$\sum bc\left[\frac{a}{2} - \frac{a^2}{b+c} + \frac{4(b-c)^2}{b+c}\right] \ge 0,$$

$$8\sum \frac{bc(b-c)^2}{b+c} \ge abc\sum \frac{2a-b-c}{b+c}.$$

In addition, since

$$\begin{aligned}\sum \frac{2a-b-c}{b+c} &= \sum \frac{(a-b)+(a-c)}{b+c} = \sum \frac{a-b}{b+c} + \sum \frac{b-a}{c+a}\\ &= \sum \frac{(a-b)^2}{(b+c)(c+a)} = \sum \frac{(b-c)^2}{(c+a)(a+b)},\end{aligned}$$

the inequality can be restated as

$$8\sum \frac{bc(b-c)^2}{b+c} \ge abc\sum \frac{(b-c)^2}{(c+a)(a+b)},$$

$$\sum \frac{bc(b-c)^2(8a^2+8bc+7ab+7ac)}{(a+b)(b+c)(c+a)} \ge 0.$$

Since the last form is obvious, the proof is completed. The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.144. *Let a, b, c be nonnegative real numbers, no two of which are zero, such that*

$$a+b+c=2.$$

Prove that

$$\frac{a}{2a+bc} + \frac{b}{2b+ca} + \frac{c}{2c+ab} \ge 1.$$

Solution. Since

$$2a + bc = (a + b + c)a + bc = (a + b)(a + c),$$

we can write the inequality as follows:

$$a(b + c) + b(c + a) + c(a + b) \geq (a + b)(b + c)(c + a),$$

$$2(ab + bc + ca) \geq (a + b + c)(ab + bc + ca) - abc,$$

$$abc \geq 0.$$

The equality holds for $a = 0$, or $b = 0$, or $c = 0$.

□

P 1.145. *Let a, b, c be positive real numbers such that*

$$(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 10.$$

Prove that

$$\frac{19}{12} \leq \frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} \leq \frac{5}{3}.$$

(Vasile Cîrtoaje, 2012)

First Solution. Write the hypothesis

$$(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 10$$

as

$$\frac{b + c}{a} + \frac{c + a}{b} + \frac{a + b}{c} = 7$$

and

$$(a + b)(b + c)(c + a) = 9abc.$$

Using the substitution

$$x = \frac{b + c}{a}, \qquad y = \frac{c + a}{b}, \qquad z = \frac{a + b}{c},$$

we need to show that $x + y + z = 7$ and $xyz = 9$ involve

$$\frac{19}{12} \leq \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \leq \frac{5}{3},$$

or, equivalently,

$$\frac{19}{12} \leq \frac{1}{x} + \frac{x(7 - x)}{9} \leq \frac{5}{3}.$$

Clearly, $x, y, z \in (0, 7)$. The left inequality is equivalent to

$$(x-4)(2x-3)^2 \le 0,$$

while the right inequality is equivalent to

$$(x-1)(x-3)^2 \ge 0.$$

These inequalities are true if $1 \le x \le 4$. To show that $1 \le x \le 4$, from $(y+z)^2 \ge 4yz$, we get

$$(7-x)^2 \ge \frac{36}{x},$$

$$(x-1)(x-4)(x-9) \ge 0,$$

$$1 \le x \le 4.$$

Thus, the proof is completed. The left inequality is an equality for $2a = b = c$ (or any cyclic permutation), and the right inequality is an equality for $a/2 = b = c$ (or any cyclic permutation).

Second Solution. Due to homogeneity, assume that $b + c = 2$; this involves $bc \le 1$. From the hypothesis

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = 10,$$

we get

$$bc = \frac{2a(a+2)}{9a-2}.$$

Since

$$bc - 1 = \frac{(a-2)(2a-1)}{9a-2},$$

from the condition $bc \le 1$, we get

$$\frac{1}{2} \le a \le 2.$$

We have

$$\frac{b}{c+a} + \frac{c}{a+b} = \frac{a(b+c)+b^2+c^2}{a^2+(b+c)a+bc} = \frac{2a+4-2bc}{a^2+2a+bc}$$
$$= \frac{2(7a^2+12a-4)}{9a^2(a+2)} = \frac{2(7a-2)}{9a^2},$$

hence

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = \frac{a}{2} + \frac{2(7a-2)}{9a^2} = \frac{9a^3+28a-8}{18a^2}.$$

Thus, we need to show that

$$\frac{19}{12} \le \frac{9a^3+28a-8}{18a^2} \le \frac{5}{3}.$$

These inequalities are true, since the left inequality is equivalent to

$$(2a-1)(3a-4)^2 \ge 0,$$

and the right inequality is equivalent to

$$(a-2)(3a-2)^2 \le 0.$$

Remark. Similarly, we can prove the following generalization.

- *Let a, b, c be positive real numbers such that*

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = 9 + \frac{8k^2}{1-k^2},$$

where $k \in (0,1)$. Then,

$$\frac{k^2}{1+k} \le \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}-\frac{3}{2} \le \frac{k^2}{1-k}.$$

□

P 1.146. *Let a, b, c be nonnegative real numbers, no two of which are zero, such that $a+b+c = 3$. Prove that*

$$\frac{9}{10} < \frac{a}{2a+bc}+\frac{b}{2b+ca}+\frac{c}{2c+ab} \le 1.$$

(Vasile Cîrtoaje, 2012)

Solution. (a) Since

$$\frac{a}{2a+bc}-\frac{1}{2} = \frac{-bc}{2(2a+bc)},$$

we can write the right inequality as

$$\sum \frac{bc}{2a+bc} \ge 1.$$

According to the Cauchy-Schwarz inequality, we have

$$\sum \frac{bc}{2a+bc} \ge \frac{\left(\sum bc\right)^2}{\sum bc(2a+bc)} = \frac{\sum b^2c^2 + 2abc\sum a}{6abc+\sum b^2c^2} = 1.$$

The equality holds for $a = b = c = 1$, and also for $a = 0$, or $b = 0$, or $c = 0$.

(b) **First Solution.** For the nontrivial case $a, b, c > 0$, we can write the left inequality as

$$\sum \frac{1}{2+\dfrac{bc}{a}} > \frac{9}{10}.$$

Using the substitution

$$x = \sqrt{\frac{bc}{a}}, \quad y = \sqrt{\frac{ca}{b}}, \quad z = \sqrt{\frac{ab}{c}},$$

we need to show that

$$\sum \frac{1}{2+x^2} > \frac{9}{10}$$

for all positive real numbers x, y, z satisfying $xy+yz+zx = 3$. By expanding, the inequality becomes

$$4\sum x^2 + 48 > 9x^2y^2z^2 + 8\sum x^2y^2.$$

Since

$$\sum x^2y^2 = \left(\sum xy\right)^2 - 2xyz\sum x = 9 - 2xyz\sum x,$$

we can write the desired inequality as

$$4\sum x^2 + 16xyz\sum x > 9x^2y^2z^2 + 24,$$

which is equivalent to

$$4(p^2-12) + 16xyzp > 9x^2y^2z^2,$$

where $p = x+y+z$. Using Schur's inequality

$$p^3 + 9xyz \ge 4p(xy+yz+zx),$$

which is equivalent to

$$p(p^2-12) \ge -9xyz,$$

it suffices to prove that

$$-\frac{36xyz}{p} + 16xyzp > 9x^2y^2z^2.$$

This is true if

$$-\frac{36}{p} + 16p > 9xyz.$$

Since

$$x+y+z \ge \sqrt{3(xy+yz+zx)} = 3$$

and

$$1 = \frac{xy+yz+zx}{3} \ge \sqrt[3]{x^2y^2z^2},$$

we have

$$-\frac{36}{p} + 16p - 9xyz \ge -\frac{36}{3} + 48 - 9 > 0.$$

Second Solution. As it is shown at the first solution, it suffices to show that

$$\sum \frac{1}{2+x^2} > \frac{9}{10}$$

for all positive real numbers x, y, z satisfying $xy + yz + zx = 3$. Rewrite this inequality as

$$\sum \frac{x^2}{2 + x^2} < \frac{6}{5}.$$

Let p and q be two positive real numbers such that

$$p + q = \sqrt{3}.$$

By the Cauchy-Schwarz inequality, we have

$$\frac{x^2}{2 + x^2} = \frac{3x^2}{2(xy + yz + zx) + 3x^2} = \frac{(px + qx)^2}{2x(x + y + z) + (x^2 + 2yz)}$$

$$\leq \frac{p^2 x}{2(x + y + z)} + \frac{q^2 x^2}{x^2 + 2yz}.$$

Therefore,

$$\sum \frac{x^2}{2 + x^2} \leq \sum \frac{p^2 x}{2(x + y + z)} + \sum \frac{q^2 x^2}{x^2 + 2yz} = \frac{p^2}{2} + q^2 \sum \frac{x^2}{x^2 + 2yz}.$$

Thus, it suffices to prove that

$$\frac{p^2}{2} + q^2 \sum \frac{x^2}{x^2 + 2yz} < \frac{6}{5}.$$

We claim that

$$\sum \frac{x^2}{x^2 + 2yz} < 2.$$

Under this assumption, we only need to show that

$$\frac{p^2}{2} + 2q^2 \leq \frac{6}{5}.$$

Indeed, choosing $p = \frac{4\sqrt{3}}{5}$ and $q = \frac{\sqrt{3}}{5}$, we have $p + q = \sqrt{3}$ and $\frac{p^2}{2} + 2q^2 = \frac{6}{5}$. To complete the proof, we need to prove the homogeneous inequality $\sum \frac{x^2}{x^2 + 2yz} < 2$, which is equivalent to

$$\sum \frac{yz}{x^2 + 2yz} > \frac{1}{2}.$$

By the Cauchy-Schwarz inequality, we get

$$\sum \frac{yz}{x^2 + 2yz} \geq \frac{\left(\sum yz\right)^2}{\sum yz(x^2 + 2yz)} = \frac{\sum y^2 z^2 + 2xyz \sum x}{xyz \sum x + 2 \sum y^2 z^2} > \frac{1}{2}.$$

□

P 1.147. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{a^3}{2a^2+bc}+\frac{b^3}{2b^2+ca}+\frac{c^3}{2c^2+ab}\le\frac{a^3+b^3+c^3}{a^2+b^2+c^2}.$$

(Vasile Cîrtoaje, 2011)

Solution. Use the SOS method. Write the inequality as follows:

$$\sum\left[\frac{a^3}{a^2+b^2+c^2}-\frac{a^3}{2a^2+bc}\right]\ge 0,$$

$$\sum\frac{a^3(a^2+bc-b^2-c^2)}{2a^2+bc}\ge 0,$$

$$\sum\frac{a^3[a^2(b+c)-b^3-c^3]}{(b+c)(2a^2+bc)}\ge 0,$$

$$\sum\frac{a^3b(a^2-b^2)+a^3c(a^2-c^2)}{(b+c)(2a^2+bc)}\ge 0,$$

$$\sum\frac{a^3b(a^2-b^2)}{(b+c)(2a^2+bc)}+\sum\frac{b^3a(b^2-a^2)}{(c+a)(2b^2+ca)}\ge 0,$$

$$\sum\frac{ab(a+b)(a-b)^2[2a^2b^2+c(a^3+a^2b+ab^2+b^3)+c^2(a^2+ab+b^2)]}{(b+c)(c+a)(2a^2+bc)(2b^2+ca)}\ge 0.$$

The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.148. *If a, b, c are positive real numbers, then*

$$\frac{a^3}{4a^2+bc}+\frac{b^3}{4b^2+ca}+\frac{c^3}{4c^2+ab}\ge\frac{a+b+c}{5}.$$

(Vasile Cîrtoaje, 2011)

Solution. Use the SOS method. Write the inequality as follows:

$$\sum\left(\frac{a^3}{4a^2+bc}-\frac{a}{5}\right)\ge 0,$$

$$\sum\frac{a(a^2-bc)}{4a^2+bc}\ge 0,$$

$$\sum\frac{a[(a-b)(a+c)+(a-c)(a+b)]}{4a^2+bc}\ge 0,$$

$$\sum\frac{a(a-b)(a+c)}{4a^2+bc}+\sum\frac{b(b-a)(b+c)}{4b^2+ca}\ge 0,$$

$$\sum \frac{c(a-b)^2[(a-b)^2+bc+ca-ab]}{(4a^2+bc)(4b^2+ca)} \ge 0.$$

Clearly, it suffices to show that

$$\sum \frac{c(a-b)^2(bc+ca-ab)}{(4a^2+bc)(4b^2+ca)} \ge 0,$$

which can be written as

$$\sum (a-b)^2(bc+ca-ab)(4c^3+abc) \ge 0.$$

Assume that $a \ge b \ge c$. Since $ca+ab-bc > 0$, it is enough to prove that

$$(c-a)^2(ab+bc-ca)(4b^3+abc)+(a-b)^2(bc+ca-ab)(4c^3+abc) \ge 0,$$

which is equivalent to

$$(a-c)^2(ab+bc-ca)(4b^3+abc) \ge (a-b)^2(ab-bc-ca)(4c^3+abc).$$

This inequality is true since $ab+bc-ca > 0$ and

$$(a-c)^2 \ge (a-b)^2, \quad 4b^3+abc \ge 4c^3+abc, \quad ab+bc-ca \ge ab-bc-ca.$$

The equality holds for $a=b=c$.

□

P 1.149. *If a, b, c are positive real numbers, then*

$$\frac{1}{(2+a)^2}+\frac{1}{(2+b)^2}+\frac{1}{(2+c)^2} \ge \frac{3}{6+ab+bc+ca}.$$

(Vasile Cîrtoaje, 2013)

Solution. By the Cauchy-Schwarz inequality, we have

$$\sum \frac{1}{(2+a)^2} \ge \frac{4(a+b+c)^2}{\sum (2+a)^2(b+c)^2}.$$

Thus, it suffices to show that

$$4(a+b+c)^2(6+ab+bc+ca) \ge 3\sum (2+a)^2(b+c)^2.$$

This inequality is equivalent to

$$2p^2q-3q^2+3pr+12q \ge 6(pq+3r),$$

$$q(2p^2-3q-6p+12) \ge 3(6-p)r,$$

where

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.$$

For $p \geq 6$, the inequality is true since

$$2p^2 - 3q - 6p + 12 = (p^2 - 3q) + p(p - 6) + 12 > 0.$$

For $p \leq 6$, since $9r \leq pq$, it suffices to show that

$$3q(2p^2 - 3q - 6p + 12) \geq (6 - p)pq,$$

that is

$$q(7p^2 - 9q - 24p + 36) \geq 0,$$

$$q[3(p^2 - 3q) + 4(p - 3)^2] \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 1.150. *If a, b, c are positive real numbers, then*

$$\frac{1}{1+3a} + \frac{1}{1+3b} + \frac{1}{1+3c} \geq \frac{3}{3+abc}.$$

(Vasile Cîrtoaje, 2013)

Solution. Set

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = \sqrt[3]{abc},$$

and write the inequality as follows:

$$(3 + r^3)\sum(1 + 3b)(1 + 3c) \geq 3(1 + 3a)(1 + 3b)(1 + 3c),$$

$$(3 + r^3)(3 + 6p + 9q) \geq 3(1 + 3p + 9q + 27r^3),$$

$$r^3(2p + 3q) + 2 + 3p \geq 26r^3.$$

By virtue of the AM-GM inequality, we have

$$p \geq 3r, \quad q \geq 3r^2.$$

Therefore, it suffices to show that

$$r^3(6r + 9r^2) + 2 + 9r \geq 26r^3,$$

which is equivalent to the obvious inequality

$$(r - 1)^2(9r^3 + 24r^2 + 13r + 2) \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 1.151. *Let a, b, c be real numbers, no two of which are zero. If $1 < k \le 3$, then*

$$\left(k+\frac{2ab}{a^2+b^2}\right)\left(k+\frac{2bc}{b^2+c^2}\right)\left(k+\frac{2ca}{c^2+a^2}\right) \ge (k-1)(k^2-1).$$

(Vasile Cîrtoaje and Vo Quoc Ba Can, 2011)

Solution. If a, b, c have the same sign, then

$$\left(k+\frac{2ab}{a^2+b^2}\right)\left(k+\frac{2bc}{b^2+c^2}\right)\left(k+\frac{2ca}{c^2+a^2}\right) > k^3 > (k-1)(k^2-1).$$

Since the inequality remains unchanged by replacing a, b, c with $-a, -b, -c$, it suffices to consider further that $a \le 0$ and $b, c \ge 0$. Setting $-a$ for a, we need to show that

$$\left(k-\frac{2ab}{a^2+b^2}\right)\left(k+\frac{2bc}{b^2+c^2}\right)\left(k-\frac{2ca}{c^2+a^2}\right) \ge (k-1)(k^2-1)$$

for $a, b, c \ge 0$. Since

$$\left(k-\frac{2ab}{a^2+b^2}\right)\left(k-\frac{2ca}{c^2+a^2}\right) = \left[k-1+\frac{(a-b)^2}{a^2+b^2}\right]\left[k-1+\frac{(a-c)}{c^2+a^2}\right]$$

$$\ge (k-1)^2+(k-1)\left[\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2}{c^2+a^2}\right],$$

it suffices to prove that

$$\left[k-1+\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2}{c^2+a^2}\right]\left(k+\frac{2bc}{b^2+c^2}\right) \ge k^2-1.$$

According to Lemma below, we have

$$\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2}{c^2+a^2} \ge \frac{(b-c)^2}{(b+c)^2}.$$

Thus, it suffices to show that

$$\left[k-1+\frac{(b-c)^2}{(b+c)^2}\right]\left(k+\frac{2bc}{b^2+c^2}\right) \ge k^2-1,$$

which is equivalent to the obvious inequality

$$(b-c)^4+2(3-k)bc(b-c)^2 \ge 0.$$

The equality holds for $a = b = c$.

Lemma. *If $a, b, c \ge 0$, no two of which are zero, then*

$$\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2}{a^2+c^2} \ge \frac{(b-c)^2}{(b+c)^2}.$$

Proof. Consider two cases: $a^2 \le bc$ and $a^2 \ge bc$.

Case 1: $a^2 \le bc$. By the Cauchy-Schwarz inequality, we have

$$\frac{(a-b)^2}{a^2+b^2} + \frac{(a-c)^2)}{a^2+c^2} \ge \frac{[(b-a)+(a-c)]^2}{(a^2+b^2)+(a^2+c^2)} = \frac{(b-c)^2}{2a^2+b^2+c^2}.$$

Thus, it suffices to show that

$$\frac{1}{2a^2+b^2+c^2} \ge \frac{1}{(b+c)^2},$$

which is equivalent to $a^2 \le bc$.

Case 2: $a^2 \ge bc$. By the Cauchy-Schwarz inequality, we have

$$\frac{(a-b)^2}{a^2+b^2} + \frac{(a-c)^2}{a^2+c^2} \ge \frac{[c(b-a)+b(a-c)]^2}{c^2(a^2+b^2)+b^2(a^2+c^2)} = \frac{a^2(b-c)^2}{a^2(b^2+c^2)+2b^2c^2}.$$

Therefore, it suffices to prove that

$$\frac{a^2}{a^2(b^2+c^2)+2b^2c^2} \ge \frac{1}{(b+c)^2},$$

which reduces to $bc(a^2-bc) \ge 0$.

□

P 1.152. *If a, b, c are non-zero and distinct real numbers, then*

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+3\left[\frac{1}{(a-b)^2}+\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}\right] \ge 4\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right).$$

Solution. Write the inequality as

$$\left(\sum \frac{1}{a^2} - \sum \frac{1}{bc}\right) + 3\sum \frac{1}{(b-c)^2} \ge 3\sum \frac{1}{bc}.$$

In virtue of the AM-GM inequality, it suffices to prove that

$$2\sqrt{3\left(\sum \frac{1}{a^2} - \sum \frac{1}{bc}\right)\left[\sum \frac{1}{(b-c)^2}\right]} \ge 3\sum \frac{1}{bc},$$

which is true if

$$4\left(\sum \frac{1}{a^2} - \sum \frac{1}{bc}\right)\left[\sum \frac{1}{(b-c)^2}\right] \ge 3\left(\sum \frac{1}{bc}\right)^2.$$

Since

$$\sum \frac{1}{(b-c)^2} = \left(\sum \frac{1}{b-c}\right)^2 = \frac{\left(\sum a^2 - \sum ab\right)^2}{(a-b)^2(b-c)^2(c-a)^2},$$

we can rewrite this inequality as

$$4\left(\sum a^2b^2 - abc\sum a\right)\left(\sum a^2 - \sum ab\right)^2 \ge 3(a+b+c)^2(a-b)^2(b-c)^2(c-a)^2.$$

Using the notations

$$p = a+b+c, \quad q = ab+bc+ca, \quad r = abc,$$

and the identity

$$(a-b)^2(b-c)^2(c-a)^2 = -27r^2 - 2(2p^2-9q)pr + p^2q^2 - 4q^3,$$

the inequality can be written as

$$4(q^2-3pr)(p^2-3q)^2 \ge 3p^2[-27r^2 - 2(2p^2-9q)pr + p^2q^2 - 4q^3],$$

which is equivalent to

$$(9pr + p^2q - 6q^2)^2 \ge 0.$$

□

P 1.153. *Let a, b, c be positive real numbers, and let*

$$A = \frac{a}{b} + \frac{b}{a} + k, \quad B = \frac{b}{c} + \frac{c}{b} + k, \quad C = \frac{c}{a} + \frac{a}{b} + k,$$

where $-2 < k \le 4$. Prove that

$$\frac{1}{A} + \frac{1}{B} + \frac{1}{C} \le \frac{1}{k+2} + \frac{4}{A+B+C-k-2}.$$

(Vasile Cîrtoaje, 2009)

Solution. Let us denote

$$x = \frac{a}{b}, \quad y = \frac{b}{c}, \quad z = \frac{c}{a}.$$

We need to show that

$$\sum \frac{x}{x^2+kx+1} \le \frac{1}{k+2} + \frac{4}{\sum x + \sum xy + 2k - 2}$$

for all positive real numbers x, y, z satisfying $xyz = 1$. Write this inequality as follows:

$$\sum\left(\frac{1}{k+2} - \frac{x}{x^2+kx+1}\right) \ge \frac{2}{k+2} - \frac{4}{\sum x + \sum xy + 2k - 2},$$

$$\sum \frac{(x-1)^2}{x^2+kx+1} \ge \frac{2\sum yz(x-1)^2}{\sum x + \sum xy + 2k - 2},$$

$$\sum \frac{(x-1)^2[-x+y+z+x(y+z)-yz-2]}{x^2+kx+1} \geq 0.$$

Since

$$-x+y+z+x(y+z)-yz-2 = (x+1)(y+z)-(x+yz+2)$$
$$= (x+1)(y+z)-(x+1)(yz+1) = -(x+1)(y-1)(z-1),$$

the inequality is equivalent to

$$-(x-1)(y-1)(z-1)\sum \frac{x^2-1}{x^2+kx+1} \geq 0;$$

that is, $E \geq 0$, where

$$E = -(x-1)(y-1)(z-1)\sum (x^2-1)(y^2+ky+1)(z^2+kz+1).$$

We have

$$\sum (x^2-1)(y^2+ky+1)(z^2+kz+1) =$$
$$= k(k-2)\left(\sum x - \sum xy\right) + \left(\sum x^2y^2 - \sum x^2\right)$$
$$= k(k-2)(x-1)(y-1)(z-1) - (x^2-1)(y^2-1)(z^2-1)$$
$$= -(x-1)(y-1)(z-1)[(x+1)(y+1)(z+1)-k(k-2)],$$

hence

$$E = (x-1)^2(y-1)^2(z-1)^2[(x+1)(y+1)(z+1)-k(k-2)].$$

Since

$$(x+1)(y+1)(z+1)-k(k-2) \geq (2\sqrt{x})(2\sqrt{y})(2\sqrt{z})-k(k-2)$$
$$= (2+k)(4-k) \geq 0,$$

it follows that $E \geq 0$. The equality holds for $a=b$, or $b=c$, or $c=a$.

□

P 1.154. *If a,b,c are nonnegative real numbers, no two of which are zero, then*

$$\frac{1}{b^2+bc+c^2}+\frac{1}{c^2+ca+a^2}+\frac{1}{a^2+ab+b^2} \geq \frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}+\frac{1}{2c^2+ab}.$$

(Vasile Cîrtoaje, 2014)

Solution. Write the inequality as follows:

$$\sum\left(\frac{1}{b^2+bc+c^2}-\frac{1}{2a^2+bc}\right) \geq 0,$$

$$\sum \frac{(a^2-b^2)+(a^2-c^2)}{(b^2+bc+c^2)(2a^2+bc)} \ge 0,$$

$$\sum \frac{a^2-b^2}{(b^2+bc+c^2)(2a^2+bc)} + \sum \frac{b^2-a^2}{(c^2+ca+a^2)(2b^2+ca)} \ge 0,$$

$$(a^2+b^2+c^2-ab-bc-ca)\sum \frac{c(a^2-b^2)(a-b)}{(b^2+bc+c^2)(c^2+ca+a^2)(2a^2+bc)(2b^2+ca)} \ge 0.$$

Clearly, the last inequality is obvious. The equality holds for $a=b=c$.

□

P 1.155. *If a,b,c are nonnegative real numbers such that $a+b+c \le 3$, then*

(a)
$$\frac{1}{2a+1}+\frac{1}{2b+1}+\frac{1}{2c+1} \ge \frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2};$$

(b)
$$\frac{1}{2ab+1}+\frac{1}{2bc+1}+\frac{1}{2ca+1} \ge \frac{1}{a^2+2}+\frac{1}{b^2+2}+\frac{1}{c^2+2}.$$

(Vasile Cîrtoaje, 2014)

Solution. Denote

$$p=a+b+c, \qquad \sqrt{3q} \le p \le 3,$$

$$q=ab+bc+ca, \qquad 0 \le q \le 3.$$

(a) Use the SOS method. Write the inequality as follows

$$\sum \left(\frac{1}{2a+1} - \frac{1}{a+2} \right) \ge 0,$$

$$\sum \frac{1-a}{(2a+1)(a+2)} \ge 0,$$

$$\sum \frac{(a+b+c)-3a}{(2a+1)(a+2)} \ge 0,$$

$$\sum \frac{(b-a)+(c-a)}{(2a+1)(a+2)} \ge 0,$$

$$\sum \frac{b-a}{(2a+1)(a+2)} + \sum \frac{a-b}{(2b+1)(b+2)} \ge 0,$$

$$\sum (a-b)\left[\frac{1}{(2b+1)(b+2)} - \frac{1}{(2a+1)(a+2)}\right],$$

$$\sum (a-b)^2(2a+2b+5)(2c+1)(c+2) \ge 0.$$

The equality holds for $a=b=c=1$.

(b) Write the inequality as

$$\sum \frac{1}{2ab+1} \geq \sum \left(\frac{1}{a^2+2} - \frac{1}{2} \right) + \frac{3}{2},$$

$$\sum \frac{2}{2ab+1} + \sum \frac{a^2}{a^2+2} \geq 3.$$

By the AM-HM inequality, we have

$$\sum \frac{1}{2ab+1} \geq \frac{9}{\sum(2ab+1)} = \frac{9}{2q+3}$$

and

$$\sum \frac{a^2}{a^2+2} \geq \frac{(\sum a)^2}{\sum(a^2+2)} = \frac{p^2}{p^2-2q+6}$$

$$= 1 - \frac{2(3-q)}{p^2-2q+6} \geq 1 - \frac{2(3-q)}{q+6} = \frac{3q}{q+6}.$$

Therefore, it suffices to show that

$$\frac{18}{2q+3} + \frac{3q}{q+6} \geq 3,$$

which is equivalent to the obvious inequality $q \leq 3$. The equality holds for $a = b = c = 1$.

□

P 1.156. *If a, b, c are nonnegative real numbers such that $a + b + c = 4$, then*

$$\frac{1}{ab+2} + \frac{1}{bc+2} + \frac{1}{ca+2} \geq \frac{1}{a^2+2} + \frac{1}{b^2+2} + \frac{1}{c^2+2}.$$

(Vasile Cîrtoaje, 2014)

First Solution *(by Nguyen Van Quy).* Use the SOS method. Rewrite the inequality as follows:

$$\sum \left(\frac{2}{ab+2} - \frac{1}{a^2+2} - \frac{1}{b^2+2} \right) \geq 0,$$

$$\sum \left[\frac{a(a-b)}{(ab+2)(a^2+2)} + \frac{b(b-a)}{(ab+2)(b^2+2)} \right] \geq 0,$$

$$\sum \frac{(2-ab)(a-b)^2(c^2+2)}{ab+2} \geq 0.$$

Without loss of generality, assume that $a \geq b \geq c \geq 0$. Then,

$$bc \leq ac \leq \frac{a(b+c)}{2} \leq \frac{(a+b+c)^2}{8} = 2$$

and

$$\begin{aligned}\sum \frac{(2-ab)(a-b)^2(c^2+2)}{ab+2} &\ge \frac{(2-ab)(a-b)^2(c^2+2)}{ab+2} + \frac{(2-ac)(a-c)^2(b^2+2)}{ac+2}\\ &\ge \frac{(2-ab)(a-b)^2(c^2+2)}{ab+2} + \frac{(2-ac)(a-b)^2(c^2+2)}{ab+2}\\ &= \frac{(4-ab-ac)(a-b)^2(c^2+2)}{ab+2}\\ &= \frac{(a-b-c)^2(a-b)^2(c^2+2)}{4(ab+2)}\end{aligned}$$

The equality holds for $a=b=c=4/3$, and also for $a=2$ and $b=c=1$ (or any cyclic permutation).

Second Solution. Write the inequality as

$$\sum \frac{1}{bc+2} \ge \sum \left(\frac{1}{a^2+2} - \frac{1}{2}\right) + \frac{3}{2},$$

$$\sum \frac{1}{bc+2} + \sum \frac{a^2}{2(a^2+2)} \ge \frac{3}{2}.$$

Assume that $a \ge b \ge c$, and denote

$$s = \frac{b+c}{2}, \quad p = bc, \qquad 0 \le s \le \frac{4}{3}, \quad 0 \le p \le s^2.$$

By the Cauchy-Schwarz inequality, we have

$$\frac{b^2}{2(b^2+2)} + \frac{c^2}{2(c^2+2)} \ge \frac{(b+c)^2}{2(b^2+2)+2(c^2+2)+4} = \frac{s^2}{2s^2-p+2}.$$

In addition,

$$\frac{1}{ca+2} + \frac{1}{ab+2} = \frac{a(b+c)+4}{(ab+2)(ac+2)} = \frac{2as+4}{a^2p+4as+4}.$$

Therefore, it suffices to show that $E(a,b,c) \ge 0$, where

$$E(a,b,c) = \frac{1}{p+2} + \frac{s^2}{2s^2-p+2} + \frac{2(as+2)}{a^2p+4as+4} + \frac{a^2}{2(a^2+2)} - \frac{3}{2}.$$

Use the mixing variables method. We will prove that

$$E(a,b,c) \ge E(a,s,s) \ge 0.$$

We have

$$\begin{aligned}E(a,b,c)-E(a,s,s)=&\left(\frac{1}{p+2}-\frac{1}{s^2+2}\right)+s^2\left(\frac{1}{2s^2-p+2}-\frac{1}{s^2+2}\right)\\&+2(as+2)\left(\frac{1}{a^2p+4as+4}-\frac{1}{a^2s^2+4as+4}\right)\\=&\frac{s^2-p}{(p+2)(s^2+2)}-\frac{s^2(s^2-p)}{(s^2+2)(2s^2-p+2)}\\&+\frac{2a^2(s^2-p)}{(a^2p+4as+4)(as+2)}.\end{aligned}$$

Since $s^2-p\geq 0$, we need to show that

$$\frac{1}{(p+2)(s^2+2)}+\frac{2a^2}{(a^2p+4as+4)(as+2)}\geq\frac{s^2}{(s^2+2)(2s^2-p+2)},$$

which is equivalent to

$$\frac{2a^2}{(a^2p+4as+4)(as+2)}\geq\frac{p(s^2+1)-2}{(p+2)(s^2+2)(2s^2-p+2)}.$$

Since

$$a^2p+4as+4\leq a^2s^2+4as+4=(as+2)^2$$

and

$$2s^2-p+2\geq s^2+2,$$

it is enough to prove that

$$\frac{2a^2}{(as+2)^3}\geq\frac{p(s^2+1)-2}{(p+2)(s^2+2)^2}.$$

In addition, since

$$as+2=(4-2s)s+2\leq 4$$

and

$$\frac{p(s^2+1)-2}{p+2}=s^2+1-\frac{2(s^2+2)}{p+2}\leq s^2+1-\frac{2(s^2+2)}{s^2+2}=s^2-1,$$

it suffices to show that

$$\frac{a^2}{32}\geq\frac{s^2-1}{(s^2+2)^2},$$

which is equivalent to

$$(2-s)^2(2+s^2)^2\geq 8(s^2-1).$$

Indeed, for the nontrivial case $1 < s \le \frac{4}{3}$, we have

$$\begin{aligned}(2-s)^2(2+s^2)^2 - 8(s^2-1) &\ge \left(2-\frac{4}{3}\right)^2(2+s^2)^2 - 8(s^2-1)\\ &= \frac{4}{9}(s^4-14s^2+22) = \frac{4}{9}\left[(7-s^2)^2-27\right]\\ &\ge \frac{4}{9}\left[\left(7-\frac{16}{9}\right)^2-27\right] = \frac{88}{729} > 0.\end{aligned}$$

To end the proof, we need to show that $E(a,s,s) \ge 0$. We have

$$\begin{aligned}E(a,s,s) &= \frac{1}{s^2+2} + \frac{s^2}{s^2+2} + \frac{2}{as+2} + \frac{a^2}{2(a^2+2)} - \frac{3}{2}\\ &= \frac{(s-1)^2(3s-4)^2}{2(s^2+2)(1+2s-s^2)(2s^2-8s+9)} \ge 0.\end{aligned}$$

□

P 1.157. *If a, b, c are nonnegative real numbers, no two of which are zero, then*

(a) $$\frac{ab+bc+ca}{a^2+b^2+c^2} + \frac{(a-b)^2(b-c)^2(c-a)^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)} \le 1;$$

(b) $$\frac{ab+bc+ca}{a^2+b^2+c^2} + \frac{(a-b)^2(b-c)^2(c-a)^2}{(a^2-ab+b^2)(b^2-bc+c^2)(c^2-ca+a^2)} \le 1.$$

(Vasile Cîrtoaje, 2014)

Solution. (a) ***First Solution.*** Consider the nontrivial case where a, b, c are distinct and write the inequality as follows:

$$\frac{(a-b)^2(b-c)^2(c-a)^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)} \le \frac{(a-b)^2+(b-c)^2+(c-a)^2}{2(a^2+b^2+c^2)},$$

$$\frac{(a^2+b^2)+(b^2+c^2)+(c^2+a^2)}{(a^2+b^2)(b^2+c^2)(c^2+a^2)} \le \frac{(a-b)^2+(b-c)^2+(c-a)^2}{(a-b)^2(b-c)^2(c-a)^2},$$

$$\sum \frac{1}{(b^2+c^2)(c^2+a^2)} \le \sum \frac{1}{(b-c)^2(c-a)^2}.$$

Since

$$a^2+b^2 \ge (a-b)^2, \quad b^2+c^2 \ge (b-c)^2, \quad c^2+a^2 \ge (c-a)^2,$$

the conclusion follows. The equality holds for $a = b = c$.

Second Solution. Assume that $a \ge b \ge c$. We have

$$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{(a-b)^2(b-c)^2(c-a)^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)} \le \frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{(a-b)^2(a-c)^2}{(a^2+b^2)(a^2+c^2)}$$
$$\le \frac{2ab+c^2}{a^2+b^2+c^2}+\frac{(a-b)^2a^2}{a^2(a^2+b^2+c^2)}$$
$$= \frac{2ab+c^2+(a-b)^2}{a^2+b^2+c^2}=1.$$

(b) Consider the nontrivial case where a, b, c are distinct and write the inequality as follows:

$$\frac{(a-b)^2(b-c)^2(c-a)^2}{(a^2-ab+b^2)(b^2-bc+c^2)(c^2-ca+a^2)} \le \frac{(a-b)^2+(b-c)^2+(c-a)^2}{2(a^2+b^2+c^2)},$$

$$\frac{2(a^2+b^2+c^2)}{(a^2-ab+b^2)(b^2-bc+c^2)(c^2-ca+a^2)} \le \frac{(a-b)^2+(b-c)^2+(c-a)^2}{(a-b)^2(b-c)^2(c-a)^2},$$

$$\sum \frac{1}{(a-b)^2(a-c)^2} \ge \frac{2(a^2+b^2+c^2)}{(a^2-ab+b^2)(b^2-bc+c^2)(c^2-ca+a^2)}.$$

Assume that $a = \min\{a, b, c\}$, and use the substitution

$$b = a + x, \quad c = a + y, \quad x, y \ge 0.$$

The inequality can be written as

$$\frac{1}{x^2y^2}+\frac{1}{x^2(x-y)^2}+\frac{1}{y^2(x-y)^2} \ge 2f(a),$$

where

$$f(a) = \frac{3a^2+2(x+y)a+x^2+y^2}{(a^2+xa+x^2)(a^2+ya+y^2)[a^2+(x+y)a+x^2-xy+y^2]}.$$

We will show that

$$\frac{1}{x^2y^2}+\frac{1}{x^2(x-y)^2}+\frac{1}{y^2(x-y)^2} \ge 2f(0) \ge 2f(a).$$

The left inequality is equivalent to

$$\frac{x^2+y^2-xy}{x^2y^2(x-y)^2} \ge \frac{x^2+y^2}{x^2y^2(x^2-xy+y^2)}.$$

Indeed,

$$\frac{x^2+y^2-xy}{x^2y^2(x-y)^2}-\frac{x^2+y^2}{x^2y^2(x^2-xy+y^2)} = \frac{1}{(x-y)^2(x^2-xy+y^2)} \ge 0.$$

Also, since

$$(a^2+xa+x^2)(a^2+ya+y^2) \ge (x^2+y^2)a^2+xy(x+y)a+x^2y^2$$

and

$$a^2 + (x+y)a + x^2 - xy + y^2 \geq x^2 - xy + y^2,$$

we get $f(a) \leq g(a)$, where

$$g(a) = \frac{3a^2 + 2(x+y)a + x^2 + y^2}{[(x^2+y^2)a^2 + xy(x+y)a + x^2y^2](x^2 - xy + y^2)}.$$

Therefore,

$$\begin{aligned} f(0) - f(a) &\geq \frac{x^2 + y^2}{x^2y^2(x^2 - xy + y^2)} - g(a) \\ &= \frac{(x^4 - x^2y^2 + y^4)a^2 + xy(x+y)(x-y)^2 a}{x^2y^2(x^2 - xy + y^2)[(x^2+y^2)a^2 + xy(x+y)a + x^2y^2]} \geq 0. \end{aligned}$$

Thus, the proof is completed. The equality holds for $a = b = c$.

□

P 1.158. *If a, b, c are nonnegative real numbers, no two of which are zero, then*

$$\frac{a^2+b^2+c^2}{ab+bc+ca} \geq 1 + \frac{9(a-b)^2(b-c)^2(c-a)^2}{(a+b)^2(b+c)^2(c+a)^2}.$$

(Vasile Cîrtoaje, 2014)

Solution. Consider the nontrivial case where

$$0 \leq a < b < c,$$

and write the inequality as follows:

$$\frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{2(ab+bc+ca)} \geq \frac{9(a-b)^2(b-c)^2(c-a)^2}{(a+b)^2(b+c)^2(c+a)^2},$$

$$\frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{(a-b)^2(b-c)^2(c-a)^2} \geq \frac{18(ab+bc+ca)}{(a+b)^2(b+c)^2(c+a)^2},$$

$$\sum \frac{1}{(b-a)^2(c-a)^2} \geq \frac{18(ab+bc+ca)}{(a+b)^2(a+c)^2(b+c)^2}.$$

Since

$$\sum \frac{1}{(b-a)^2(c-a)^2} \geq \frac{1}{b^2c^2} + \frac{1}{b^2(b-c)^2} + \frac{1}{c^2(b-c)^2} = \frac{2(b^2+c^2-bc)}{b^2c^2(b-c)^2}$$

and

$$\frac{ab+bc+ca}{(a+b)^2(a+c)^2(b+c)^2} \leq \frac{ab+bc+ca}{(ab+bc+ca)^2(b+c)^2} \leq \frac{1}{bc(b+c)^2},$$

it suffices to show that

$$\frac{b^2+c^2-bc}{b^2c^2(b-c)^2} \geq \frac{9}{bc(b+c)^2}.$$

Write this inequality as follows:

$$\frac{(b+c)^2-3bc}{bc} \geq \frac{9(b+c)^2-36bc}{(b+c)^2},$$

$$\frac{(b+c)^2}{bc} - 12 + \frac{36bc}{(b+c)^2} \geq 0,$$

$$(b+c)^4 - 12bc(b+c)^2 + 36b^2c^2 \geq 0,$$

$$[(b+c)^2 - 6bc]^2 \geq 0.$$

Thus, the proof is completed. The equality holds for $a = b = c$, and also for $a = 0$ and $b/c + c/b = 4$ (or any cyclic permutation).

□

P 1.159. *If a, b, c are nonnegative real numbers, no two of which are zero, then*

$$\frac{a^2+b^2+c^2}{ab+bc+ca} \geq 1 + (1+\sqrt{2})^2 \frac{(a-b)^2(b-c)^2(c-a)^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}.$$

(Vasile Cîrtoaje, 2014)

Solution. Consider the nontrivial case where a, b, c are distinct and denote $k = 1+\sqrt{2}$. Write the inequality as follows:

$$\frac{(a-b)^2+(b-c)^2+(c-a)^2}{2(ab+bc+ca)} \geq \frac{k^2(a-b)^2(b-c)^2(c-a)^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)},$$

$$\frac{(a-b)^2+(b-c)^2+(c-a)^2}{(a-b)^2(b-c)^2(c-a)^2} \geq \frac{2k^2(ab+bc+ca)}{(a^2+b^2)(b^2+c^2)(c^2+a^2)},$$

$$\sum \frac{1}{(b-a)^2(c-a)^2} \geq \frac{2k^2(ab+bc+ca)}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}.$$

Assume that $a = \min\{a, b, c\}$, and use the substitution

$$b = a + x, \quad c = a + y, \quad x, y \geq 0.$$

The inequality becomes

$$\frac{1}{x^2y^2} + \frac{1}{x^2(x-y)^2} + \frac{1}{y^2(x-y)^2} \geq 2k^2 f(a),$$

where

$$f(a) = \frac{3a^2+2(x+y)a+xy}{(2a^2+2xa+x^2)(2a^2+2ya+y^2)[2a^2+2(x+y)a+x^2+y^2]}.$$

We will show that

$$\frac{1}{x^2y^2}+\frac{1}{x^2(x-y)^2}+\frac{1}{y^2(x-y)^2}\ge 2k^2f(0)\ge 2k^2f(a).$$

We have

$$\frac{1}{x^2y^2}+\frac{1}{x^2(x-y)^2}+\frac{1}{y^2(x-y)^2}-2k^2f(0)=\frac{2(x^2+y^2-xy)}{x^2y^2(x-y)^2}-\frac{2k^2xy}{x^2y^2(x^2+y^2)}$$
$$=\frac{2[x^2+y^2-(2+\sqrt{2}\ )xy]^2}{x^2y^2(x-y)^2(x^2-xy+y^2)}\ge 0.$$

Also, since

$$(2a^2+2xa+x^2)(2a^2+2ya+y^2)\ge 2(x^2+y^2)a^2+2xy(x+y)a+x^2y^2$$

and

$$2a^2+2(x+y)a+x^2+y^2\ge x^2+y^2,$$

we get $f(a)\le g(a)$, where

$$g(a)=\frac{3a^2+2(x+y)a+xy}{[2(x^2+y^2)a^2+2xy(x+y)a+x^2y^2](x^2+y^2)}.$$

Therefore,

$$f(0)-f(a)\ge\frac{1}{xy(x^2+y^2)}-g(a)$$
$$=\frac{(2x^2+2y^2-3xy)a^2}{xy(x^2+y^2)[2(x^2+y^2)a^2+2xy(x+y)a+x^2y^2]}\ge 0.$$

Thus, the proof is completed. The equality holds for $a=b=c$, and also for $a=0$ and $b/c+c/b=2+\sqrt{2}$ (or any cyclic permutation).

□

P 1.160. *If a,b,c are nonnegative real numbers, no two of which are zero, then*

$$\frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{c+a}\ge\frac{5}{3a+b+c}+\frac{5}{3b+c+a}+\frac{5}{3c+a+b}.$$

Solution. Use the SOS method. Write the inequality as follows:

$$\sum\left(\frac{2}{b+c}-\frac{5}{3a+b+c}\right)\ge 0,$$

$$\sum\frac{2a-b-c}{(b+c)(3a+b+c)}\ge 0,$$

$$\sum \frac{a-b}{(b+c)(3a+b+c)} + \sum \frac{a-c}{(b+c)(3a+b+c)} \ge 0,$$

$$\sum \frac{a-b}{(b+c)(3a+b+c)} + \sum \frac{b-a}{(c+a)(3b+c+a)} \ge 0,$$

$$\sum \frac{(a-b)^2(a+b-c)}{(b+c)(c+a)(3a+b+c)(3b+c+a)},$$

$$\sum (b-c)^2 S_a \ge 0,$$

where

$$S_a = (b+c-a)(b+c)(3a+b+c).$$

Assume that $a \ge b \ge c$. Since $S_c > 0$, it suffices to show that

$$(b-c)^2 S_a + (a-c)^2 S_b \ge 0.$$

Since $S_b \ge 0$, we have

$$(b-c)^2 S_a + (a-c)^2 S_b \ge (b-c)^2 S_a + (b-c)^2 S_b = (b-c)^2 (S_a + S_b).$$

Thus, it is enough to prove that $S_a + S_b \ge 0$, which is equivalent to

$$(c+a-b)(c+a)(3b+c+a) \ge (b+c-a)(b+c)(3a+b+c).$$

For the nontrivial case $b+c-a > 0$, since $c+a-b \ge b+c-a$, we only need to show that

$$(c+a)(3b+c+a) \ge (b+c)(3a+b+c).$$

Indeed,

$$(c+a)(3b+c+a) - (b+c)(3a+b+c) = (a-b)(a+b-c) \ge 0.$$

The equality holds for $a = b = c$, and also for $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 1.161. *If a, b, c are real numbers, no two of which are zero, then*

(a) $$\frac{8a^2+3bc}{b^2+bc+c^2} + \frac{8b^2+3ca}{c^2+ca+a^2} + \frac{8c^2+3ab}{a^2+ab+b^2} \ge 11;$$

(b) $$\frac{8a^2-5bc}{b^2-bc+c^2} + \frac{8b^2-5ca}{c^2-ca+a^2} + \frac{8c^2-5ab}{a^2-ab+b^2} \ge 9.$$

(Vasile Cîrtoaje, 2011)

Solution. Consider the more general inequality

$$\frac{a^2+mbc}{b^2+kbc+c^2}+\frac{b^2+mca}{c^2+kca+a^2}+\frac{c^2+mab}{a^2+kab+b^2}\geq\frac{3(m+1)}{k+2},$$

which can be written as $f_6(a,b,c)\geq 0$, where

$$f_6(a,b,c)=(k+2)\sum(a^2+mbc)(a^2+kab+b^2)(a^2+kac+c^2)$$

$$-3(m+1)\prod(b^2+kbc+c^2).$$

Let

$$p=a+b+c,\qquad q=ab+bc+ca.$$

From

$$f_6(a,b,c)=(k+2)\sum(a^2+mbc)(kab-c^2+p^2-2q)(kac-b^2+p^2-2q)$$

$$-3(m+1)\prod(kbc-a^2+p^2-2q).$$

it follows that $f_6(a,b,c)$ has the same highest coefficient A as

$$(k+2)P_2(a,b,c)-3(m+1)P_3(a,b,c),$$

where

$$P_2(a,b,c)=\sum(a^2+mbc)(kab-c^2)(kac-b^2),$$

$$P_3(a,b,c)=\prod(kbc-a^2).$$

According to Remark 2 from the proof of P 2.75 in Volume 1,

$$\begin{aligned}A&=(k+2)P_2(1,1,1)-3(m+1)P_3(1,1,1)\\&=3(k+2)(m+1)(k-1)^2-3(m+1)(k-1)^3=9(m+1)(k-1)^2.\end{aligned}$$

Also, we have

$$f_6(a,1,1)=(k+2)(a^2+ka+1)(a-1)^2[a^2+(k+2)a+1+2k-2m].$$

(a) For our particular case $m=3/8$ and $k=1$, we have $A=0$. Therefore, according to P 2.75 in Volume 1, it suffices to prove that $f_6(a,1,1)\geq 0$ for all real a. Indeed,

$$f_6(a,1,1)=3(a^2+a+1)(a-1)^2\left(a+\frac{3}{2}\right)^2\geq 0.$$

Thus, the proof is completed. The equality holds for $a=b=c$, and also for $-2a/3=b=c$ (or any cyclic permutation).

(b) For $m=-5/8$ and $k=-1$, we have

$$A=\frac{27}{2}$$

and

$$f_6(a,1,1) = \frac{1}{4}(a^2 - a + 1)(a-1)^2(2a+1)^2.$$

Since $A > 0$, we will use the *highest coefficient cancellation method*. Consider the homogeneous polynomial

$$P(a,b,c) = abc + Bp^3 + Cpq,$$

where B and C are real constants. Since the desired inequality becomes an equality for $a = b = c = 1$, and also for $a = -1/2$ and $b = c = 1$, we will determine B and C such that $P(1,1,1) = P(-1/2,1,1) = 0$; that is,

$$B = \frac{4}{27}, \quad C = \frac{-5}{9},$$

when

$$P(a,b,c) = abc + \frac{4p^3}{27} - \frac{5pq}{9},$$

$$P(a,1,1) = \frac{2}{27}(a-1)^2(2a+1).$$

We will show that

$$f_6(a,b,c) \geq \frac{27}{2}P^2(a,b,c).$$

Let us denote

$$g_6(a,b,c) = f_6(a,b,c) - \frac{27}{2}P^2(a,b,c).$$

Since $g_6(a,b,c)$ has the highest coefficient equal to zero, it suffices to prove that $g_6(a,1,1) \geq 0$ for all real a (see P 2.75 in Volume 1). Indeed,

$$g_6(a,1,1) = f_6(a,1,1) - \frac{27}{2}P^2(a,1,1) = \frac{1}{108}(a-1)^2(2a+1)^2(19a^2 - 11a + 19) \geq 0.$$

The equality holds for $a = b = c$, and also for $-2a = b = c$ (or any cyclic permutation).

□

P 1.162. *If a, b, c are real numbers, no two of which are zero, then*

$$\frac{4a^2 + bc}{4b^2 + 7bc + 4c^2} + \frac{4b^2 + ca}{4c^2 + 7ca + 4a^2} + \frac{4c^2 + ab}{4a^2 + 7ab + 4b^2} \geq 1.$$

(Vasile Cîrtoaje, 2011)

Solution. Write the inequality as $f_6(a,b,c) \geq 0$, where

$$f_6(a,b,c) = \sum (4a^2 + bc)(4a^2 + 7ab + 4b^2)(4a^2 + 7ac + 4c^2) - \prod (4b^2 + 7bc + 4c^2).$$

Let

$$p = a + b + c, \quad q = ab + bc + ca.$$

From

$$f_6(a,b,c) = \sum(4a^2+bc)(7ab-4c^2+4p^2-8q)(7ac-4b^2+4p^2-8q)$$
$$-\prod(7bc-4a^2+4p^2-8q),$$

it follows that $f_6(a,b,c)$ has the same highest coefficient A as

$$P_2(a,b,c) - P_3(a,b,c),$$

where

$$P_2(a,b,c) = \sum(4a^2+bc)(7ab-4c^2)(7ac-4b^2),$$
$$P_3(a,b,c) = \prod(7bc-4a^2).$$

According to Remark 2 from the proof of P 2.75 in Volume 1,

$$A = P_2(1,1,1) - P_3(1,1,1) = 135 - 27 = 108.$$

Since $A > 0$, we will apply the *highest coefficient cancellation method*. Consider the homogeneous polynomial

$$P(a,b,c) = abc + Bp^3 + Cpq,$$

where B and C are real constants. We will show that there are two real numbers B and C such that the following sharper inequality holds

$$f_6(a,b,c) \geq 108P^2(a,b,c).$$

Let us denote

$$g_6(a,b,c) = f_6(a,b,c) - 108P^2(a,b,c).$$

Clearly, $g_6(a,b,c)$ has the highest coefficient equal to zero. Then, by P 2.75 in Volume 1, it suffices to prove that there exist B and C such that $g_6(a,1,1) \geq 0$ for all real a.

We have

$$g_6(a,1,1) = f_6(a,1,1) - 108P^2(a,1,1),$$

where

$$f_6(a,1,1) = 4(4a^2+7a+4)(a-1)^2(4a^2+15a+16),$$
$$P(a,1,1) = a + B(a+2)^3 + C(a+2)(2a+1).$$

Let us denote $g(a) = f_6(a,1,1)$. Since

$$g(-2) = 0,$$

the condition

$$g'(-2) = 0,$$

which involves $C = -5/9$, is necessary to have $g(a) \geq 0$ in the vicinity of $a = -2$. On the other hand, from $g(1) = 0$, we get $B = 4/27$. For these values of B and C, we get

$$P(a,1,1) = \frac{2(a-1)^2(2a+1)}{27},$$

$$g_6(a,1,1) = \frac{4}{27}(a-1)^2(a+2)^2(416a^2+728a+431) \ge 0.$$

The proof is completed. The equality holds for $a = b = c$, and for $a = 0$ and $b + c = 0$ (or any cyclic permutation).

□

P 1.163. *If a, b, c are real numbers, no two of which are equal, then*

$$\frac{1}{(a-b)^2} + \frac{1}{(b-c)^2} + \frac{1}{(c-a)^2} \ge \frac{27}{4(a^2+b^2+c^2-ab-bc-ca)}.$$

First Solution. Write the inequality as follows:

$$\left[(a-b)^2+(b-c)^2+(a-c)^2\right]\left[\frac{1}{(a-b)^2}+\frac{1}{(b-c)^2}+\frac{1}{(a-c)^2}\right] \ge \frac{27}{2},$$

$$\left[\frac{(a-b)^2}{(a-c)^2}+\frac{(b-c)^2}{(a-c)^2}+1\right]\left[\frac{(a-c)^2}{(a-b)^2}+\frac{(a-c)^2}{(b-c)^2}+1\right] \ge \frac{27}{2},$$

$$(x^2+y^2+1)\left(\frac{1}{x^2}+\frac{1}{y^2}+1\right) \ge \frac{27}{2},$$

where

$$x = \frac{a-b}{a-c}, \qquad y = \frac{b-c}{a-c}, \qquad x + y = 1.$$

We have

$$(x^2+y^2+1)\left(\frac{1}{x^2}+\frac{1}{y^2}+1\right) - \frac{27}{2} = \frac{(x+1)^2(x-2)^2(2x-1)^2}{2x^2(1-x)^2} \ge 0.$$

The proof is completed. The equality holds for $2a = b + c$ (or any cyclic permutation).

Second Solution. Assume that $a > b > c$. We have

$$\frac{1}{(a-b)^2} + \frac{1}{(b-c)^2} \ge \frac{2}{(a-b)(b-c)} \ge \frac{8}{[(a-b)+(b-c)]^2} = \frac{8}{(a-c)^2}.$$

Therefore, it suffices to show that

$$\frac{9}{(a-c)^2} \ge \frac{27}{4(a^2+b^2+c^2-ab-bc-ca)},$$

which is equivalent to

$$(a-2b+c)^2 \ge 0.$$

Third Solution. Write the inequality as $f_6(a,b,c) \ge 0$, where

$$f_6(a,b,c) = 4(a^2+b^2+c^2-ab-bc-ca)\sum(a-b)^2(a-c)^2 - 27(a-b)^2(b-c)^2(c-a)^2.$$

Clearly, $f_6(a,b,c)$ has the same highest coefficient A as

$$-27(a-b)^2(b-c)^2(c-a)^2;$$

that is,

$$A = -27(-27) = 729.$$

Since $A > 0$, we will use the *highest coefficient cancellation method*. Define the homogeneous polynomial

$$P(a,b,c) = abc + B(a+b+c)^3 - \left(3B + \frac{1}{9}\right)(a+b+c)(ab+bc+ca)$$

which satisfies $P(1,1,1) = 0$. We will show that there is a real value of B such that the following sharper inequality holds

$$f_6(a,b,c) \ge 729P^2(a,b,c).$$

Let us denote

$$g_6(a,b,c) = f_6(a,b,c) - 729P^2(a,b,c).$$

Clearly, $g_6(a,b,c)$ has the highest coefficient equal to zero. Then, by P 2.75 in Volume 1, it suffices to prove that $g_6(a,1,1) \ge 0$ for all real a. We have

$$f_6(a,1,1) = 4(a-1)^6$$

and

$$P(a,1,1) = \frac{1}{9}(a-1)^2[9B(a+2)+2],$$

hence

$$\begin{aligned} g_6(a,1,1) &= f_6(a,1,1) - 729P^2(a,1,1) \\ &= (27B+2)(a+2)(a-1)^4[(2-27B)a - 54B - 8]. \end{aligned}$$

Choosing $B = -2/27$, we get $g_6(a,1,1) = 0$ for all real a.

Remark. The inequality is equivalent to

$$(a-2b+c)^2(b-2c+a)^2(c-2a+b)^2 \ge 0.$$

□

P 1.164. *If a, b, c are real numbers, no two of which are zero, then*

$$\frac{1}{a^2-ab+b^2}+\frac{1}{b^2-bc+c^2}+\frac{1}{c^2-ca+a^2}\ge\frac{14}{3(a^2+b^2+c^2)}.$$

(Vasile Cîrtoaje and BJSL, 2014)

Solution. Write the inequality as $f_6(a,b,c)\ge 0$, where

$$\begin{aligned}f_6(a,b,c)=&3(a^2+b^2+c^2)\sum(a^2-ab+b^2)(a^2-ac+c^2)\\&-14(a^2-ab+b^2)(b^2-bc+c^2)(c^2-ca+a^2).\end{aligned}$$

Clearly, $f_6(a,b,c)$ has the same highest coefficient A as

$$-14(a^2-ab+b^2)(b^2-bc+c^2)(c^2-ca+a^2);$$

that is, according to Remark 2 from the proof of P 2.75 in Volume 1,

$$A=-14(-1-1)^3=112.$$

Since $A>0$, we apply the *highest coefficient cancellation method*. Consider the homogeneous polynomial

$$P(a,b,c)=abc+B(a+b+c)^3+C(a+b+c)(ab+bc+ca).$$

We will show that there are two real numbers B and C such that the following sharper inequality holds

$$f_6(a,b,c)\ge 112P^2(a,b,c).$$

Let us denote

$$g_6(a,b,c)=f_6(a,b,c)-112P^2(a,b,c).$$

Clearly, $g_6(a,b,c)$ has the highest coefficient equal to zero. By P 2.75 in Volume 1, it suffices to prove that $g_6(a,1,1)\ge 0$ for all real a. We have

$$g_6(a,1,1)=f_6(a,1,1)-112P^2(a,1,1),$$

where

$$f_6(a,1,1)=(a^2-a+1)(3a^4-3a^3+a^2+8a+4),$$

$$P(a,1,1)=a+B(a+2)^3+C(a+2)(2a+1).$$

Let us denote $g(a)=g_6(a,1,1)$. Since

$$g(-2)=0,$$

the condition

$$g'(-2)=0,$$

which involves $C = -4/7$, is necessary to have $g(a) \ge 0$ in the vicinity of $a = -2$. In addition, setting $B = 9/56$, we get

$$P(a,1,1) = \frac{1}{56}(9a^3 - 10a^2 + 4a + 8),$$

$$\begin{aligned} g_6(a,1,1) &= \frac{3}{28}(a^6 + 4a^5 + 8a^4 + 16a^3 + 20a^2 + 16a + 16) \\ &= \frac{3(a+2)^2(a^2+2)^2}{28} \ge 0. \end{aligned}$$

The proof is completed. The equality holds for $a = 0$ and $b + c = 0$ (or any cyclic permutation).

□

P 1.165. *If a, b, c are real numbers, then*

$$\frac{a^2+bc}{2a^2+b^2+c^2} + \frac{b^2+ca}{a^2+2b^2+c^2} + \frac{c^2+ab}{a^2+b^2+2c^2} \ge \frac{1}{6}.$$

(Vasile Cîrtoaje, 2010)

Solution. Write the inequality as $f_6(a,b,c) \ge 0$, where

$$\begin{aligned} f_6(a,b,c) =& 6\sum(a^2+bc)(a^2+2b^2+c^2)(a^2+b^2+2c^2) \\ & - (2a^2+b^2+c^2)(a^2+2b^2+c^2)(a^2+b^2+2c^2). \end{aligned}$$

Clearly, $f_6(a,b,c)$ has the same highest coefficient A as $f(a,b,c)$, where

$$f(a,b,c) = 6\sum(a^2+bc)b^2c^2 - a^2b^2c^2 = 17a^2b^2c^2 + 6(a^3b^3 + b^3c^3 + c^3a^3);$$

that is,

$$A = 17 + 6 \cdot 3 = 35.$$

Since $A > 0$, we apply the *highest coefficient cancellation method*. Consider the homogeneous polynomial

$$P(a,b,c) = abc + B(a+b+c)^3 + C(a+b+c)(ab+bc+ca)$$

and show that there are two real numbers B and C such that the following sharper inequality holds

$$f_6(a,b,c) \ge 35P^2(a,b,c).$$

Let us denote

$$g_6(a,b,c) = f_6(a,b,c) - 35P^2(a,b,c).$$

Clearly, $g_6(a,b,c)$ has the highest coefficient equal to zero. By P 2.75 in Volume 1, it suffices to prove that $g_6(a,1,1) \ge 0$ for all real a. We have

$$g_6(a,1,1) = f_6(a,1,1) - 35P^2(a,1,1),$$

where

$$f_6(a,1,1) = 4(a^2+1)(a^2+3)(a+3)^2,$$
$$P(a,1,1) = a + B(a+2)^3 + C(a+2)(2a+1).$$

Let

$$g(a) = g_6(a,1,1).$$

Since $g(-2) = 0$, we can have $g(a) \geq 0$ in the vicinity of $a = -2$ only if $g'(-2) = 0$, which involves $C = 19/35$. Since $f_6(-3,1,1) = 0$, we enforce $P(-3,1,1) = 0$, which provides $B = -2/7$. Thus,

$$P(a,1,1) = a - \frac{2}{7}(a+1)^3 + \frac{19}{35}(a+2)(2a+1) = \frac{-2(a+3)(5a^2-4a+7)}{35}$$

and

$$\begin{aligned} g_6(a,1,1) &= 4(a^2+1)(a^2+3)(a+3)^2 - \frac{4}{35}(a+3)^2(5a^2-4a+7)^2 \\ &= \frac{8}{35}(a+3)^2(a+2)^2(5a^2+7) \geq 0. \end{aligned}$$

The proof is completed. The equality holds for $a = 0$ and $b + c = 0$ (or any cyclic permutation), and also for $-a/3 = b = c$ (or any cyclic permutation).

□

P 1.166. *If a, b, c are real numbers, then*

$$\frac{2b^2+2c^2+3bc}{(a+3b+3c)^2} + \frac{2c^2+2a^2+3ca}{(b+3c+3a)^2} + \frac{2a^2+2b^2+3ab}{(c+3a+3b)^2} \geq \frac{3}{7}.$$

(Vasile Cîrtoaje, 2010)

Solution. Write the inequality as $f_6(a,b,c) \geq 0$, where

$$f_6(a,b,c) = 7\sum(2b^2+2c^2+3bc)(b+3c+3a)^2(c+3a+3b)^2 - 3\prod(a+3b+3c)^2.$$

We have

$$f_6(a,1,1) = (a-1)^2(a-8)^2(3a+4)^2.$$

Let

$$p = a+b+c, \quad q = ab+bc+ca, \quad r = abc.$$

From

$$f_6(a,b,c) = 7\sum(2p^2-4q+3bc-2a^2)(3p-2b)^2(3p-2c)^2 - 3\prod(3p-2a)^2,$$

it follows that $f(a,b,c)$ has the same highest coefficient A as $g(a,b,c)$, where

$$g(a,b,c) = 7\sum(3bc-2a^2)(-2b)^2(-2c)^2 - 3\prod(-2a)^2 = 48\left(7\sum b^3c^3 - 18a^2b^2c^2\right);$$

that is,

$$A = 48(21 - 18) = 144.$$

Since the highest coefficient A is positive, we will use the *highest coefficient cancellation method*. There are two cases to consider: $p^2 + q \geq 0$ and $p^2 + q < 0$.

Case 1: $p^2 + q \geq 0$. Since

$$f_6(1, 1, 1) = f_6(8, 1, 1) = 0,$$

define the homogeneous function

$$P(a, b, c) = r + Bp^3 + Cpq$$

such that $P(1, 1, 1) = P(8, 1, 1) = 0$; that is,

$$P(a, b, c) = r + \frac{1}{45}p^3 - \frac{8}{45}pq,$$

which leads to

$$P(a, 1, 1) = \frac{45a + (a+2)^3 - 8(a+2)(2a+1)}{45} = \frac{(a-1)^2(a-8)}{45}.$$

We will show that the following sharper inequality holds for $p^2 + q \geq 0$:

$$f_6(a, b, c) \geq 144P^2(a, b, c).$$

Let us denote

$$g_6(a, b, c) = f_6(a, b, c) - 144P^2(a, b, c).$$

Since the highest coefficient of $g_6(a, b, c)$ is zero, it suffices to prove that $g_6(a, 1, 1) \geq 0$ for all real a such that $(a+2)^2 + 2a + 1 \geq 0$, that is

$$a \in (-\infty, -5] \cup [-1, \infty)$$

(see Remark 3 from the proof of P 2.75 in Volume 1). We have

$$\begin{aligned} g_6(a, 1, 1) &= f_6(a, 1, 1) - 144P^2(a, 1, 1) \\ &= \frac{1}{225}(a-1)^2(a-8)^2[225(3a+4)^2 - 16(a-1)^2] \\ &= \frac{7}{225}(a-1)^2(a-8)^2(41a+64)(7a+8) \geq 0. \end{aligned}$$

Case 2: $p^2 + q < 0$. Since

$$f_6(1, 1, 1) = f_6(-4/3, 1, 1) = 0,$$

define the homogeneous function

$$P(a, b, c) = r + Bp^3 + Cpq$$

such that $P(1,1,1) = P(-4/3,1,1) = 0$; that is,

$$P(a,b,c) = r + \frac{1}{3}p^3 - \frac{10}{9}pq,$$

which leads to

$$P(a,1,1) = \frac{9a + 3(a+2)^3 - 10(a+2)(2a+1)}{9} = \frac{(a-1)^2(3a+4)}{9}.$$

We will show that the following sharper inequality holds for $p^2 + q < 0$:

$$f_6(a,b,c) \geq 144P^2(a,b,c).$$

Let us denote

$$g_6(a,b,c) = f_6(a,b,c) - 144P^2(a,b,c).$$

Since the highest coefficient of $g_6(a,b,c)$ is zero, it suffices to prove that $g_6(a,1,1) \geq 0$ for all real a such that $(a+2)^2 + 2a + 1 < 0$, that is

$$a \in (-5,-1)$$

(see Remark 3 from the proof of P 2.75 in Volume 1). We have

$$\begin{aligned} g_6(a,1,1) &= f_6(a,1,1) - 144P^2(a,1,1) \\ &= \frac{1}{9}(a-1)^2(3a+4)^2[9(a-8)^2 - 16(a-1)^2] \\ &= \frac{7}{9}(a-1)^2(3a+4)^2(20+a)(4-a) \geq 0. \end{aligned}$$

The proof is completed. The equality holds for $a = b = c$, for $a/8 = b = c$ (or any cyclic permutation), and also for $-3a/4 = b = c$ (or any cyclic permutation).

□

P 1.167. *If a,b,c are nonnegative real numbers, then*

$$\frac{6b^2+6c^2+13bc}{(a+2b+2c)^2} + \frac{6c^2+6a^2+13ca}{(b+2c+2a)^2} + \frac{6a^2+6b^2+13ab}{(c+2a+2b)^2} \leq 3.$$

(Vasile Cîrtoaje, 2010)

Solution. Write the inequality as $f_6(a,b,c) \geq 0$, where

$$f_6(a,b,c) = 3\prod(a+2b+2c)^2 - \sum(6b^2+6c^2+13bc)(b+2c+2a)^2(c+2a+2b)^2.$$

Let

$$p = a+b+c, \quad q = ab+bc+ca.$$

From

$$f_6(a,b,c) = 3\prod(2p-a)^2 - \sum(6p^2-12q+13bc-6a^2)(2p-b)^2(2p-c)^2,$$

it follows that $f(a,b,c)$ has the same highest coefficient A as $g(a,b,c)$, where

$$g(a,b,c) = 3\prod(-a)^2 - \sum(13bc-6a^2)(-b)^2(-c)^2 = 21a^2b^2c^2 - 13\sum b^3c^3;$$

that is,

$$A = 21 - 39 = -18.$$

Since the highest coefficient A is negative, it suffices to prove the desired inequality for $b=c=1$, and for $a=0$ (see P 3.76-(a) in Volume 1).

For $b=c=1$, the inequality becomes

$$\frac{25}{(a+4)^2} + \frac{2(6a^2+13a+6)}{(2a+3)^2} \le 3,$$

$$\frac{2(6a^2+13a+6)}{(2a+3)^2} \le \frac{3a^2+24a+23}{(a+4)^2},$$

$$\frac{5(2a+3)(a-1)^2}{(2a+3)^2(a+4)^2} \ge 0.$$

For $a=0$, the inequality turns into

$$\frac{6b^2+6c^2+13bc}{4(b+c)^2} + \frac{6c^2}{(b+2c)^2} + \frac{6b^2}{(2b+c)^2} \le 3,$$

$$\frac{6b^2+6c^2+13bc}{4(b+c)^2} + \frac{6[(b^2+c^2)^2+4bc(b^2+c^2)+6b^2c^2]}{(2b^2+2c^2+5bc)^2} \le 3.$$

If $bc=0$, then the inequality is an identity. For $bc \ne 0$, we may consider $bc=1$ (due to homogeneity). Denoting

$$x = b^2+c^2, \qquad x \ge 2,$$

the inequality becomes

$$\frac{6x+13}{4(x+2)} + \frac{6(x^2+4x+6)}{(2x+5)^2} \le 3,$$

which reduces to the obvious inequality

$$20x^2+34x-13 \ge 0.$$

The equality holds for $a=b=c$, and also for $a=b=0$ (or any cyclic permutation).

□

P 1.168. *If a, b, c are nonnegative real numbers such that $a+b+c=3$, then*

$$\frac{3a^2+8bc}{9+b^2+c^2}+\frac{3b^2+8ca}{9+c^2+a^2}+\frac{3c^2+8ab}{9+a^2+b^2}\le 3.$$

(Vasile Cîrtoaje, 2010)

Solution. Let

$$p=a+b+c, \quad q=ab+bc+ca.$$

Write the inequality in the homogeneous form

$$\frac{3a^2+8bc}{p^2+b^2+c^2}+\frac{3b^2+8ca}{p^2+c^2+a^2}+\frac{3c^2+8ab}{p^2+a^2+b^2}\le 3,$$

which is equivalent to $f_6(a,b,c)\ge 0$, where

$$f_6(a,b,c)=3\prod(p^2+b^2+c^2)-\sum(3a^2+8bc)(p^2+c^2+a^2)(p^2+a^2+b^2).$$

From

$$f_6(a,b,c)=3\prod(2p^2-2q-a^2)-\sum(3a^2+8bc)(2p^2-2q-b^2)(2p^2-2q-c^2),$$

it follows that $f(a,b,c)$ has the same highest coefficient A as $g(a,b,c)$, where

$$g(a,b,c)=3\prod(-a)^2-\sum(3a^2+8bc)(-b^2)(-c^2)=-12a^2b^2c^2-8\sum b^3c^3;$$

that is,

$$A=-12-24=-36.$$

Since the highest coefficient A is negative, it suffices to prove the homogeneous inequality for $b=c=1$ and for $a=0$ (see P 3.76-(a) in Volume 1).

For $b=c=1$, we need to show that

$$\frac{3a^2+8}{(a+2)^2+2}+\frac{2(3+8a)}{(a+2)^2+a^2+1}\le 3,$$

which is equivalent to

$$\frac{3a^2+8}{a^2+4a+6}+\frac{2(8a+3)}{2a^2+4a+5}\le 3,$$

$$\frac{8a+3}{2a^2+4a+5}\le\frac{6a+5}{a^2+4a+6},$$

$$4a^3-a^2-10a+7\ge 0,$$

$$(a-1)^2(4a+7)\ge 0.$$

For $a=0$, we need to show that

$$\frac{8bc}{(b+c)^2+b^2+c^2}+\frac{3b^2}{(b+c)^2+c^2}+\frac{3c^2}{(b+c)^2+b^2}\le 3.$$

Clearly, it suffices to show that

$$\frac{8bc}{(b+c)^2+b^2+c^2}+\frac{3(b^2+c^2)}{(b+c)^2}\le 3,$$

which is equivalent to

$$\frac{4bc}{b^2+c^2+bc}\le\frac{6bc}{(b+c)^2},$$

$$bc(b-c)^2\ge 0.$$

The equality holds for $a=b=c=1$, and also for $a=b=0$ and $c=3$ (or any cyclic permutation).

□

P 1.169. *If a,b,c are nonnegative real numbers such that $a+b+c=3$, then*

$$\frac{5a^2+6bc}{9+b^2+c^2}+\frac{5b^2+6ca}{9+c^2+a^2}+\frac{5c^2+6ab}{9+a^2+b^2}\ge 3.$$

(Vasile Cîrtoaje, 2010)

Solution. We use the *highest coefficient method.* Let

$$p=a+b+c,\qquad q=ab+bc+ca.$$

Write the inequality in the homogeneous form $f_6(a,b,c)\ge 0$, where

$$f_6(a,b,c)=\sum(5a^2+6bc)(p^2+c^2+a^2)(p^2+a^2+b^2)-3\prod(p^2+b^2+c^2).$$

From

$$f_6(a,b,c)=\sum(5a^2+6bc)(2p^2-2q-b^2)(2p^2-2q-c^2)-3\prod(2p^2-2q-a^2),$$

it follows that $f_6(a,b,c)$ has the same highest coefficient A as

$$f(a,b,c)=\sum(5a^2+6bc)(-b^2)(-c^2)-3(-a^2)(-b^2)(-c^2)=18a^2b^2c^2+6\sum b^3c^3;$$

therefore,

$$A=18+18=36.$$

On the other hand,

$$f_6(a,1,1)=4a(2a^2+4a+5)(a+1)(a-1)^2\ge 0$$

and

$$\begin{aligned}f_6(0,b,c)&=6bcBC+5b^2AB+5c^2AC-3ABC\\&=-3(A-2bc)BC+5A(b^2B+c^2C),\end{aligned}$$

where

$$A = (b+c)^2 + b^2 + c^2, \qquad B = (b+c)^2 + b^2, \qquad C = (b+c)^2 + c^2.$$

Substituting

$$(b+c)^2 = 4x, \qquad bc = y, \qquad x \ge y,$$

we have

$$A = 2(4x - y), \qquad B = 4x + b^2, \qquad C = 4x + c^2,$$

$$A - 2bc = 4(2x - y),$$

$$BC = 16x^2 + 4x(b^2 + c^2) + b^2c^2 = 16x^2 + 4x(4x - 2y) + y^2 = 32x^2 - 8xy + y^2,$$

$$b^2B + c^2C = 4x(b^2 + c^2) + b^4 + c^4 = 2(16x^2 - 12xy + y^2),$$

therefore

$$\begin{aligned} f_6(0,b,c) &= -12(2x-y)(32x^2 - 8xy + y^2) + 20(4x - y)(16x^2 - 12xy + y^2) \\ &= 8(64x^3 - 88x^2y + 25xy^2 - y^3) = 8(x-y)(64x^2 - 24xy + y^2). \end{aligned}$$

Since

$$f_6(1,1,1) = f_6(0,1,1) = 0,$$

define the homogeneous function

$$P(a,b,c) = abc + B(a+b+c)^3 + C(a+b+c)(ab+bc+ca)$$

such that $P(1,1,1) = P(0,1,1) = 0$; that is,

$$P(a,b,c) = abc + \frac{1}{9}(a+b+c)^3 - \frac{4}{9}(a+b+c)(ab+bc+ca).$$

We have

$$P(a,1,1) = \frac{a(a-1)^2}{9}, \qquad P^2(a,1,1) = \frac{a^2(a-1)^4}{81},$$

$$P(0,b,c) = \frac{(b+c)(b-c)^2}{9}, \qquad P^2(0,b,c) = \frac{64x(x-y)^2}{81}.$$

We will prove the sharper inequality $g_6(a,b,c) \ge 0$, where

$$g_6(a,b,c) = f_6(a,b,c) - 36P^2(a,b,c).$$

Clearly, $g_6(a,b,c)$ has the highest coefficient $A = 0$. Then, according to P 3.76-(a) in Volume 1, it suffices to prove that $g_6(a,1,1) \ge 0$ and $g_6(0,b,c) \ge 0$ for $a,b,c \ge 0$.

We have

$$g_6(a,1,1) = f_6(a,1,1) - 36P^2(a,1,1) = \frac{4a(a-1)^2h(a)}{9},$$

where

$$\begin{aligned} h(a) &= 9(2a^2 + 4a + 5)(a+1) - a(a-1)^2 \\ &> (a-1)^2(a+1) - a(a-1)^2 = (a-1)^2 \ge 0. \end{aligned}$$

Also, we have

$$g_6(0,b,c) = f_6(0,b,c) - 36P^2(0,b,c) = \frac{8(x-y)g(x,y)}{9},$$

where

$$\begin{aligned} g(x,y) &= 9(64x^2 - 24xy + y^2) - 32x(x-y) \\ &> (64x^2 - 24xy + y^2) - 32x(x-y) = 32x^2 + 8xy + y^2 > 0. \end{aligned}$$

The equality holds for $a = b = c = 1$, and also for $a = 0$ and $b = c = 3/2$ (or any cyclic permutation).

□

P 1.170. *If a, b, c are nonnegative real numbers such that $a + b + c = 3$, then*

$$\frac{1}{a^2 + bc + 12} + \frac{1}{b^2 + ca + 12} + \frac{1}{c^2 + ab + 12} \le \frac{3}{14}.$$

(Vasile Cîrtoaje, 2010)

Solution. Write the inequality in the homogeneous form

$$\frac{1}{3(a^2 + bc) + 4p^2} + \frac{1}{3(b^2 + ca) + 4p^2} + \frac{1}{3(c^2 + ab) + 4p^2} \le \frac{9}{14p^2},$$

where

$$p = a + b + c.$$

The inequality is equivalent to $f_6(a,b,c) \ge 0$, where

$$f_6(a,b,c) = 9\prod(3a^2 + 3bc + 4p^2) - 14p^2\sum(3b^2 + 3ca + 4p^2)(3c^2 + 3ab + 4p^2).$$

Clearly, $f_6(a,b,c)$ has the same highest coefficient A as

$$g(a,b,c) = 243\prod(a^2 + bc).$$

According to Remark 2 from the proof of P 2.75 in Volume 1, we have

$$A = g(1,1,1) = 243 \cdot 8 = 1944.$$

Since the highest coefficient A is positive, we will apply the *highest coefficient cancellation method*. We have

$$\begin{aligned} f_6(a,1,1) =& 9[3a^2 + 3 + 4(a+2)^2][3a + 3 + 4(a+2)^2]^2 \\ &- 14(a+2)^2[3a + 3 + 4(a+2)^2]^2 \\ &- 28(a+2)^2[3a + 3 + 4(a+2)^2][3a^2 + 3 + 4(a+2)^2] \\ =& 9(7a^2 + 16a + 19)(4a^2 + 19a + 19)^2 - 14(a+2)^2(4a^2 + 19a + 19)^2 \\ &- 28(a+2)^2(4a^2 + 19a + 19)(7a^2 + 16a + 19) \\ =& 3(4a^2 + 19a + 19)f(a), \end{aligned}$$

where

$$f(a) = 3(7a^2 + 16a + 19)(4a^2 + 19a + 19) - 14(a + 2)^2(6a^2 + 17a + 19)$$
$$= 17a^3 - 15a^2 - 21a + 19 = (a - 1)^2(17a + 19);$$

therefore,

$$f_6(a, 1, 1) = 3(4a^2 + 19a + 19)(a - 1)^2(17a + 19).$$

Since

$$f_6(1, 1, 1) = f_6(1, 0, 0) = 0,$$

define the homogeneous function

$$P(a, b, c) = abc + B(a + b + c)^3 + C(a + b + c)(ab + bc + ca)$$

such that $P(1, 1, 1) = P(1, 0, 0) = 0$; that is,

$$P(a, b, c) = abc - \frac{1}{9}(a + b + c)(ab + bc + ca).$$

We will prove the sharper inequality $g_6(a, b, c) \geq 0$, where

$$g_6(a, b, c) = f_6(a, b, c) - 1944P^2(a, b, c).$$

Clearly, $g_6(a, b, c)$ has the highest coefficient $A = 0$. Then, it suffices to prove that $g_6(a, 1, 1) \geq 0$ and $g_6(0, b, c) \geq 0$ for $a, b, c \geq 0$ (see P 3.76-(a) in Volume 1).

To show that $g_6(a, 1, 1) \geq 0$, which can be written as

$$f_6(a, 1, 1) - 1944P^2(a, 1, 1) \geq 0,$$

we see that

$$P(a, 1, 1) = a - \frac{(a + 2)(2a + 1)}{9} = \frac{-2(a - 1)^2}{9},$$
$$P^2(a, 1, 1) = \frac{4(a - 1)^4}{81},$$

hence

$$g_6(a, 1, 1) = 3(4a^2 + 19a + 19)(a - 1)^2(17a + 19) - 96(a - 1)^4$$
$$= 3(a - 1)^2h(a),$$

where

$$h(a) = (4a^2 + 19a + 19)(17a + 19) - 32(a - 1)^2.$$

We need to show that $h(a) \geq 0$ for $a \geq 0$. Indeed, since

$$(4a^2 + 19a + 19)(17a + 19) > (19a + 19)(17a + 17) > 32(a + 1)^2,$$

we get

$$h(a) > 32[(a + 1)^2 - (a - 1)^2] = 128a \geq 0.$$

To show that $g_6(0,b,c) \ge 0$, denote

$$x = (b+c)^2, \qquad y = bc.$$

We have

$$f_6(0,b,c) = 9ABC - 14x[BC + A(B+C)] = (9A-14x)BC - 14xA(B+C),$$

where

$$A = 4x+3y, \quad B = 4x+3b^2, \quad C = 4x+3c^2.$$

Since

$$9A - 14x = 22x+27y, \quad B+C = 8x+3(x-2y) = 11x-6y,$$
$$BC = 16x^2 + 12x(x-2y) + 9y^2 = 28x^2 - 24xy + 9y^2,$$

we get

$$\begin{aligned} f_6(0,b,c) &= (22x+27y)(28x^2-24xy+9y^2) - 14x(4x+3y)(11x-6y) \\ &= 3y(34x^2 - 66xy + 81y^2). \end{aligned}$$

Also,

$$P(0,b,c) = \frac{-bc(b+c)}{9}, \qquad P^2(0,b,c) = \frac{xy^2}{81}.$$

Hence

$$\begin{aligned} g_6(0,b,c) &= f_6(0,b,c) - 1944P^2(0,b,c) = 3y(34x^2 - 74xy + 81y^2) \\ &\ge 3y(25x^2 - 90xy + 81y^2) = 3y(5x-9y)^2 \ge 0. \end{aligned}$$

The equality holds for $a = b = c$, and also for $a = b = 0$ (or any cyclic permutation).

□

P 1.171. *If a,b,c are nonnegative real numbers, no two of which are zero, then*

$$\frac{1}{a^2+b^2} + \frac{1}{b^2+c^2} + \frac{1}{c^2+a^2} \ge \frac{45}{8(a^2+b^2+c^2)+2(ab+bc+ca)}.$$

(Vasile Cîrtoaje, 2014)

First Solution *(by Nguyen Van Quy).* Multiplying by $a^2+b^2+c^2$, the inequality becomes

$$\sum \frac{a^2}{b^2+c^2} + 3 \ge \frac{45(a^2+b^2+c^2)}{8(a^2+b^2+c^2)+2(ab+bc+ca)}.$$

Applying the Cauchy-Schwarz inequality, we have

$$\sum \frac{a^2}{b^2+c^2} \ge \frac{\left(\sum a^2\right)^2}{\sum a^2(b^2+c^2)} = \frac{(a^2+b^2+c^2)^2}{2(a^2b^2+b^2c^2+c^2a^2)}.$$

Therefore, it suffices to show that

$$\frac{(a^2+b^2+c^2)^2}{2(a^2b^2+b^2c^2+c^2a^2)}+3 \ge \frac{45(a^2+b^2+c^2)}{8(a^2+b^2+c^2)+2(ab+bc+ca)},$$

which is equivalent to

$$\frac{(a^2+b^2+c^2)^2}{a^2b^2+b^2c^2+c^2a^2}-3 \ge \frac{45(a^2+b^2+c^2)}{4(a^2+b^2+c^2)+ab+bc+ca}-9,$$

$$\frac{a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2}{a^2b^2+b^2c^2+c^2a^2} \ge \frac{9(a^2+b^2+c^2-ab-bc-ca)}{4(a^2+b^2+c^2)+ab+bc+ca}.$$

By Schur's inequality of degree four, we have

$$a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2 \ge (a^2+b^2+c^2-ab-bc-ca)(ab+bc+ca).$$

Therefore, it suffices to show that

$$[4(a^2+b^2+c^2)+ab+bc+ca](ab+bc+ca) \ge 9(a^2b^2+b^2c^2+c^2a^2).$$

Since

$$(ab+bc+ca)^2 \ge a^2b^2+b^2c^2+c^2a^2,$$

this inequality is true if

$$4(a^2+b^2+c^2)(ab+bc+ca) \ge 8(a^2b^2+b^2c^2+c^2a^2),$$

which is equivalent to the obvious inequality

$$ab(a-b)^2+bc(b-c)^2+ca(c-a)^2+abc(a+b+c) \ge 0.$$

The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

Second Solution. Write the inequality as $f_6(a,b,c) \ge 0$, where

$$f_6(a,b,c) = \left[8(a^2+b^2+c^2)+2(ab+bc+ca)\right]\sum(a^2+b^2)(a^2+c^2) - 45\prod(b^2+c^2).$$

Clearly, $f_6(a,b,c)$ has the same highest coefficient A as

$$f(a,b,c) = -45\prod(b^2+c^2) = -45\prod(p^2-2q-a^2),$$

where $p=a+b+c$ and $q=ab+bc+ca$; that is,

$$A = 45.$$

Since $A>0$, we will apply the *highest coefficient cancellation method.* We have

$$f_6(a,1,1) = 4a(2a+5)(a^2+1)(a-1)^2,$$

$$f_6(0,b,c) = (b-c)^2[8(b^4+c^4)+18bc(b^2+c^2)+15b^2c^2].$$

Since

$$f_6(1,1,1) = f_6(0,1,1) = 0,$$

define the homogeneous function

$$P(a,b,c) = abc + B(a+b+c)^3 + C(a+b+c)(ab+bc+ca)$$

such that $P(1,1,1) = P(0,1,1) = 0$; that is,

$$P(a,b,c) = abc + \frac{1}{9}(a+b+c)^3 - \frac{4}{9}(a+b+c)(ab+bc+ca).$$

We will show that the following sharper inequality holds

$$f_6(a,b,c) \geq 45P^2(a,b,c).$$

Let us denote

$$g_6(a,b,c) = f_6(a,b,c) - 45P^2(a,b,c).$$

Clearly, $g_6(a,b,c)$ has the highest coefficient equal to zero. By P 3.76-(a) in Volume 1, it suffices to prove that $g_6(a,1,1) \geq 0$ and $g_6(0,b,c) \geq 0$ for all $a,b,c \geq 0$. We have

$$P(a,1,1) = \frac{a(a-1)^2}{9},$$

hence

$$g_6(a,1,1) = f_6(a,1,1) - 45P^2(a,1,1) = \frac{a(a-1)^2(67a^3+190a^2+67a+180)}{9} \geq 0.$$

Also, we have

$$P(0,b,c) = \frac{(b+c)(b-c)^2}{9},$$

hence

$$\begin{aligned} g_6(0,b,c) &= f_6(0,b,c) - 45P^2(0,b,c) \\ &= \frac{(b-c)^2[67(b^4+c^4)+162bc(b^2+c^2)+145b^2c^2]}{9} \geq 0. \end{aligned}$$

□

P 1.172. *If a,b,c are real numbers, no two of which are zero, then*

$$\frac{a^2-7bc}{b^2+c^2} + \frac{b^2-7ca}{a^2+b^2} + \frac{c^2-7ab}{a^2+b^2} + \frac{9(ab+bc+ca)}{a^2+b^2+c^2} \geq 0.$$

(Vasile Cîrtoaje, 2014)

Solution. Let

$$p = a + b + c, \qquad q = ab + bc + ca, \qquad r = abc.$$

Write the inequality as $f_8(a, b, c) \geq 0$, where

$$\begin{aligned} f_8(a, b, c) =& (a^2 + b^2 + c^2) \sum (a^2 - 7bc)(a^2 + b^2)(a^2 + c^2) \\ &+ 9(ab + bc + ca) \prod (b^2 + c^2) \end{aligned}$$

is a symmetric homogeneous polynomial of degree eight. Notice that any symmetric homogeneous polynomial of degree eight $f_8(a, b, c)$ can be written in the form

$$f_8(a, b, c) = A(p, q)r^2 + B(p, q)r + C(p, q),$$

where the *highest polynomial* $A(p, q)$ has the form

$$A(p, q) = \alpha p^2 + \beta q.$$

Since

$$\begin{aligned} f_8(a, b, c) =& (p^2 - 2q) \sum (a^2 - 7bc)(p^2 - 2q - c^2)(p^2 - 2q - b^2) \\ &+ 9q \prod (p^2 - 2q - a^2), \end{aligned}$$

$f_8(a, b, c)$ has the same highest polynomial as

$$\begin{aligned} g_8(a, b, c) =& (p^2 - 2q) \sum (a^2 - 7bc)(-c^2)(-b^2) + 9q(-a^2)(-b^2)(-c^2) \\ =& (p^2 - 2q) \left(3r^2 - 7 \sum b^3 c^3\right) - 9qr^2; \end{aligned}$$

that is,

$$A(p, q) = (p^2 - 2q)(3 - 21) - 9q = -9(p^2 - 3q).$$

Since $A(p, q) \leq 0$ for all real a, b, c, it suffices to prove the original inequality for $b = c = 1$ (see Lemma below). We need to show that

$$\frac{a^2 - 7}{2} - \frac{2(7a - 1)}{a^2 + 1} + \frac{9(2a + 1)}{a^2 + 2} \geq 0,$$

which is equivalent to

$$(a - 1)^2(a + 2)^2(a^2 - 2a + 3) \geq 0.$$

The equality holds for $a = b = c$, and also for $-a/2 = b = c$ (or any cyclic permutation).

Lemma. *Let*

$$p = a + b + c, \qquad q = ab + bc + ca, \qquad r = abc,$$

and let $f_8(a, b, c)$ be a symmetric homogeneous polynomial of degree eight written in the form

$$f_8(a, b, c) = A(p, q)r^2 + B(p, q)r + C(p, q),$$

where $A(p,q) \le 0$ for all real a,b,c. The inequality $f_8(a,b,c) \ge 0$ holds for all real numbers a,b,c if and only if $f_8(a,1,1) \ge 0$ for all real a.

Proof. For fixed p and q,

$$h_8(r) = A(p,q)r^2 + B(p,q)r + C(p,q)$$

is a concave quadratic function of r which is minimum when r is minimum or maximum; that is, according to P 2.53 in Volume 1, when two of a,b,c are equal. Thus, the inequality $f_8(a,b,c) \ge 0$ holds for all real numbers a,b,c if and only if $f_8(a,1,1) \ge 0$ and $f_8(a,0,0) \ge 0$ for all real a. The last condition is not necessary because it follows from the first condition as follows:

$$f_8(a,0,0) = \lim_{t\to 0} f_8(a,t,t) = \lim_{t\to 0} t^8 f_8(a/t,1,1) \ge 0.$$

Notice that $A(p,q)$ is called the *highest polynomial* of $f_8(a,b,c)$.

Remark. This Lemma can be extended for the case where the highest polynomial $A(p,q$ is not nonnegative for all real a,b,c.

- *The inequality $f_8(a,b,c) \ge 0$ in the previous Lemma holds for all real numbers a,b,c satisfying*

$$A(p,q) \le 0$$

if and only if $f_8(a,1,1) \ge 0$ for all real a satisfying $A(a+2,2a+1) \le 0$.

□

P 1.173. *If a,b,c are nonnegative real numbers, no two of which are zero, then*

$$\frac{a^2-4bc}{b^2+c^2} + \frac{b^2-4ca}{c^2+a^2} + \frac{c^2-4ab}{a^2+b^2} + \frac{9(ab+bc+ca)}{a^2+b^2+c^2} \ge \frac{9}{2}.$$

(Vasile Cîrtoaje, 2014)

Solution. Let

$$p = a+b+c, \quad q = ab+bc+ca, \quad r = abc.$$

Write the inequality as $f_8(a,b,c) \ge 0$, where

$$\begin{aligned} f_8(a,b,c) =& 2(a^2+b^2+c^2)\sum(a^2-4bc)(a^2+b^2)(a^2+c^2) \\ &+ 9(2ab+2bc+2ca-a^2-b^2-c^2)\prod(b^2+c^2) \end{aligned}$$

is a symmetric homogeneous polynomial of degree eight. Any symmetric homogeneous polynomial of degree eight can be written in the form

$$f_8(a,b,c) = A(p,q)r^2 + B(p,q)r + C(p,q),$$

where $A(p,q) = \alpha p^2 + \beta q$ is called the *highest polynomial* of $f_8(a,b,c)$. From

$$\begin{aligned} f_8(a,b,c) =& 2(p^2-2q)\sum(a^2-4bc)(p^2-2q-c^2)(p^2-2q-b^2) \\ &+ 9(4q-p^2\prod(p^2-2q-a^2), \end{aligned}$$

it follows that $f_8(a,b,c)$ has the same highest polynomial as

$$\begin{aligned} g_8(a,b,c) =& 2(p^2-2q)\sum(a^2-4bc)b^2c^2 + 9(4q-p^2)(-a^2b^2c^2) \\ =& 2(p^2-2q)\left(3r^2 - 4\sum b^3c^3\right) - 9(4q-p^2)r^2; \end{aligned}$$

that is,

$$A(p,q) = 2(p^2-2q)(3-12) - 9(4q-p^2) = -9p^2.$$

Since $A(p,q) \le 0$ for all $a,b,c \ge 0$, it suffices to prove the original inequality for $b = c = 1$, and for $a = 0$ (see Lemma below).

For $b = c = 1$, the original inequality becomes

$$\frac{a^2-4}{2} - \frac{2(4a-1)}{a^2+1} + \frac{9(2a+1)}{a^2+2} \ge \frac{9}{2},$$

which is equivalent to

$$a(a+4)(a-1)^4 \ge 0.$$

For $a = 0$, the original inequality turns into

$$\frac{b^2}{c^2} + \frac{c^2}{b^2} + \frac{5bc}{b^2+c^2} \ge \frac{9}{2}.$$

Substituting

$$x = \frac{b}{c} + \frac{c}{b}, \qquad x \ge 2,$$

the inequality becomes

$$(x^2-2) + \frac{5}{x} \ge \frac{9}{2},$$

$$(x-2)(2x^2+4x-5) \ge 0.$$

Thus, the proof is completed. The equality holds for $a = b = c$, and also for $a = 0$ and $b = c$ (or any cyclic permutation).

Lemma. *Let*

$$p = a+b+c, \qquad q = ab+bc+ca, \qquad r = abc,$$

and let $f_8(a,b,c)$ be a symmetric homogeneous polynomial of degree eight written in the form

$$f_8(a,b,c) = A(p,q)r^2 + B(p,q)r + C(p,q),$$

where $A(p,q) \le 0$ for all $a,b,c \ge 0$. The inequality $f_8(a,b,c) \ge 0$ holds for all $a,b,c \ge 0$ if and only if the inequalities $f_8(a,1,1) \ge 0$ and $f_8(0,b,c) \ge 0$ hold for all $a,b,c \ge 0$.

Proof. For fixed p and q,

$$h_8(r) = A(p,q)r^2 + B(p,q)r + C(p,q)$$

is a concave quadratic function of r. Therefore, $h_8(r)$ is minimum when r is minimum or maximum; that is, according to P 3.57 in Volume 1, when $b = c$ or $a = 0$. Thus, the conclusion follows. Notice that $A(p,q)$ is called the *highest polynomial* of $f_8(a,b,c)$.

Remark. This Lemma can be extended for the case where the highest polynomial $A(p,q$ is not nonnegative for all $a,b,c \ge 0$.

- *The inequality $f_8(a,b,c) \ge 0$ in the previous Lemma holds for all $a,b,c \ge 0$ satisfying $A(p,q) \le 0$ if and only if the inequalities $f_8(a,1,1) \ge 0$ and $f_8(0,b,c) \ge 0$ hold for all $a,b,c \ge 0$ satisfying $A(a+2,2a+1) \le 0$ and $A(b+c,bc) \le 0$.*

□

P 1.174. *If a,b,c are real numbers such that $abc \ne 0$, then*

$$\frac{(b+c)^2}{a^2} + \frac{(c+a)^2}{b^2} + \frac{(a+b)^2}{c^2} \ge 2 + \frac{10(a+b+c)^2}{3(a^2+b^2+c^2)}.$$

(Vasile Cîrtoaje and Michael Rozenberg, 2014)

Solution. Let

$$p = a+b+c, \qquad q = ab+bc+ca, \qquad r = abc.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{(b+c)^2}{a^2} \ge \frac{\left[\sum(b+c)^2\right]^2}{\sum a^2(b+c)^2} = \frac{2\left(\sum a^2 + \sum ab\right)^2}{\sum a^2b^2 + abc\sum a} = \frac{2(p^2-q)^2}{q^2-pr}.$$

Therefore, it suffices to show that

$$\frac{2(p^2-q)^2}{q^2-pr} \ge 2 + \frac{10p^2}{3(p^2-2q)},$$

which is equivalent to

$$\frac{3(p^2-q)^2}{q^2-pr} \ge \frac{8p^2-6q}{p^2-2q}.$$

Using Schur's inequality

$$p^3 + 9r \ge 4pq,$$

we get

$$q^2 - pr \le q^2 - p \cdot \frac{4pq-p^3}{9} = \frac{p^4-4p^2q+9q^2}{9}.$$

Thus, it suffices to prove that

$$\frac{27(p^2-q)^2}{p^4-4p^2q+9q^2} \ge \frac{8p^2-6q}{p^2-2q},$$

which is equivalent to the obvious inequality

$$p^2(p^2 - 3q)(19p^2 - 13q) \geq 0.$$

The equality holds for $a = b = c$.

□

P 1.175. *Let a, b, c be real numbers such that $ab+bc+ca \geq 0$ and no two of which are zero. Prove that*

(a)
$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2};$$

(b) if $ab \leq 0$, then
$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq 2.$$

(Vasile Cîrtoaje, 2014)

Solution. Let as show first that $b + c \neq 0$, $c + a \neq 0$ and $a + b \neq 0$. Indeed, if $b + c = 0$, then $ab + bc + ca \geq 0$ yields $bc \geq 0$, hence $b = c = 0$, which is not possible (because, by hypothesis, at most one of a, b, c can be zero).

(a) Use the SOS method. Write the inequality as follows:

$$\sum \left(\frac{a}{b+c} + 1\right) \geq \frac{9}{2},$$

$$\left[\sum (b+c)\right]\left(\sum \frac{1}{b+c}\right) \geq 9,$$

$$\sum \left(\frac{a+b}{a+c} + \frac{a+c}{a+b} - 2\right) \geq 0,$$

$$\sum \frac{(b-c)^2}{(a+b)(a+c)} \geq 0,$$

$$\sum \frac{(b-c)^2}{a^2 + (ab+bc+ca)} \geq 0.$$

Clearly, the last inequality is true. The equality holds for $a = b = c \neq 0$.

(b) From $ab+bc+ca \geq 0$, it follows that if one of a, b, c is zero, then the others are the same sign. In this case, the desired inequality is trivial. Consider further that $abc \neq 0$. Since the problem remains unchanged by replacing a, b, c with $-a, -b, -c$, it suffices to consider

$$a < 0 < b, c.$$

First Solution. We will show that

$$F(a,b,c) > F(0,b,c) \ge 2,$$

where

$$F(a,b,c) = \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}.$$

The right inequality is true because

$$F(0,b,c) = \frac{b}{c} + \frac{c}{b} \ge 2.$$

Since

$$F(a,b,c) - F(0,b,c) = a\left[\frac{1}{b+c} - \frac{b}{c(c+a)} - \frac{c}{b(a+b)}\right],$$

the left inequality is true if

$$\frac{b}{c(c+a)} + \frac{c}{b(a+b)} > \frac{1}{b+c}.$$

From $ab+bc+ca \ge 0$, we get

$$c+a \ge \frac{-ca}{b} > 0, \qquad a+b \ge \frac{-ab}{c} > 0,$$

hence

$$\frac{b}{c(c+a)} > \frac{b}{c^2}, \qquad \frac{c}{b(a+b)} > \frac{c}{b^2}.$$

Therefore, it suffices to prove that

$$\frac{b}{c^2} + \frac{c}{b^2} \ge \frac{1}{b+c}.$$

Indeed, by virtue of the AM-GM inequality, we have

$$\frac{b}{c^2} + \frac{c}{b^2} - \frac{1}{b+c} \ge \frac{2}{\sqrt{bc}} - \frac{1}{2\sqrt{bc}} > 0.$$

This completes the proof. The equality holds for $a=0$ and $b=c$, or $b=0$ and $a=c$.

Second Solution. From $b+c>0$ and

$$(b+c)(a+b) = b^2 + (ab+bc+ca) > 0,$$

$$(b+c)(c+a) = c^2 + (ab+bc+ca) > 0,$$

it follows that

$$a+b > 0, \qquad c+a > 0.$$

By virtue of the Cauchy-Schwarz inequality and AM-GM inequality, we have

$$\begin{aligned}\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} &\geq \frac{a}{b+c}+\frac{(b+c)^2}{b(c+a)+c(a+b)}\\ &= \frac{a}{b+c}+\frac{(b+c)^2}{2bc+a(b+c)}\\ &> \frac{a}{2a+b+c}+\frac{(b+c)^2}{\frac{(b+c)^2}{2}+a(b+c)}\\ &> \frac{4a}{2a+b+c}+\frac{2(b+c)}{2a+b+c}=2.\end{aligned}$$

□

P 1.176. *If a, b, c are nonnegative real numbers, then*

$$\frac{a}{7a+b+c}+\frac{b}{7b+c+a}+\frac{c}{7c+a+b} \geq \frac{ab+bc+ca}{(a+b+c)^2}.$$

(Vasile Cîrtoaje, 2014)

First Solution. Use the SOS method. Write the inequality as follows:

$$\sum\left[\frac{2a}{7a+b+c}-\frac{a(b+c)}{(a+b+c)^2}\right] \geq 0,$$

$$\sum\frac{a[(a-b)+(a-c)](a-b-c)}{7a+b+c} \geq 0,$$

$$\sum\frac{a(a-b)(a-b-c)}{7a+b+c}+\sum\frac{a(a-c)(a-b-c)}{7a+b+c} \geq 0,$$

$$\sum\frac{a(a-b)(a-b-c)}{7a+b+c}+\sum\frac{b(b-a)(b-c-a)}{7b+c+a} \geq 0,$$

$$\sum(a-b)\left[\frac{a(a-b-c)}{7a+b+c}-\frac{b(b-c-a)}{7b+c+a}\right] \geq 0,$$

$$\sum(a-b)^2(a^2+b^2-c^2+14ab)(a+b+7c) \geq 0.$$

Since

$$a^2+b^2-c^2+14ab \geq (a+b)^2-c^2=(a+b+c)(a+b-c),$$

it suffices to show that

$$\sum(a-b)^2(a+b-c)(a+b+7c) \geq 0.$$

Assume that $a \ge b \ge c$. It is enough to show that

$$(a-c)^2(a-b+c)(a+7b+c)+(b-c)^2(-a+b+c)(7a+b+c) \ge 0.$$

For the nontrivial case $b>0$, we have

$$(a-c)^2 \ge \frac{a^2}{b^2}(b-c)^2 \ge \frac{a}{b}(b-c)^2.$$

Thus, it suffices to prove that

$$a(a-b+c)(a+7b+c)+b(-a+b+c)(7a+b+c) \ge 0.$$

Since

$$a(a+7b+c) \ge b(7a+b+c),$$

we have

$$a(a-b+c)(a+7b+c)+b(-a+b+c)(7a+b+c) \ge$$
$$\ge b(a-b+c)(7a+b+c)+b(-a+b+c)(7a+b+c)$$
$$= 2bc(7a+b+c) \ge 0.$$

This completes the proof. The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

Second Solution. Assume that

$$a \le b \le c, \qquad a+b+c=3,$$

and use the substitution

$$x=\frac{2a+1}{3}, \quad y=\frac{2b+1}{3}, \quad z=\frac{2c+1}{3}.$$

We have $b+c \ge 2$ and

$$\frac{1}{3} \le x \le y \le z, \qquad x+y+z=3, \quad x \le 1, \quad y+z \ge 2.$$

The inequality can be written as follows:

$$\frac{a}{2a+1}+\frac{b}{2b+1}+\frac{c}{2c+1} \ge \frac{9-a^2-b^2-c^2}{6},$$

$$\frac{a^2+b^2+c^2}{3} \ge \frac{1}{2a+1}+\frac{1}{2b+1}+\frac{1}{2c+1},$$

$$\frac{(2a+1)^2+(2b+1)^2+(2c+1)^2-15}{12} \ge \frac{1}{2a+1}+\frac{1}{2b+1}+\frac{1}{2c+1},$$

$$9(x^2+y^2+z^2) \ge 4\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)+15.$$

We will use the mixing variables method. More precisely, we will show that

$$E(x,y,z) \ge E(x,t,t) \ge 0,$$

where

$$t = (y+z)/2 = (3-x)/2,$$

$$E(x,y,z) = 9(x^2+y^2+z^2) - 4\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) - 15.$$

We have

$$E(x,y,z) - E(x,t,t) = 9(y^2+z^2-2t^2) - 4\left(\frac{1}{y}+\frac{1}{z}-\frac{2}{t}\right)$$
$$= \frac{(y-z)^2[9yz(y+z)-8]}{2yz(y+z)} \ge 0$$

since

$$9yz = (2b+1)(2c+1) \ge 2(b+c)+1 \ge 5, \qquad y+z \ge 2.$$

Also,

$$E(x,t,t) = 9x^2 + 2t^2 - 15 - \frac{4}{x} - \frac{8}{t} = \frac{(x-1)^2(3x-1)(8-3x)}{2x(3-x)} \ge 0.$$

Third Solution. Write the inequality as $f_5(a,b,c) \ge 0$, where $f_5(a,b,c)$ is a symmetric homogeneous inequality of degree five. According to P 3.68-(a) in Volume 1, it suffices to prove the inequality for $a=0$ and for $b=c=1$, when the inequality is equivalent to

$$(b-c)^2(b^2+c^2+11bc) \ge 0$$

and

$$a(a-1)^2(a+14) \ge 0,$$

respectively.

□

P 1.177. *If a,b,c are positive real numbers such that $abc=1$, then*

$$\frac{a+b+c}{30} + \frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} \ge \frac{8}{5}.$$

(Vasile Cîrtoaje, 2018)

Solution. Assume that $a \ge b \ge c$, which involves $ab \ge 1$. Since $a+b \ge 2\sqrt{ab}$ and

$$\frac{1}{a+1} + \frac{1}{b+1} - \frac{2}{\sqrt{ab}+1} = \frac{(\sqrt{a}-\sqrt{b})^2(\sqrt{ab}-1)}{(a+1)(b+1)(\sqrt{ab}+1)} \ge 0,$$

it suffices to show that

$$\frac{2\sqrt{ab}+c}{30}+\frac{2}{\sqrt{ab}+1}+\frac{1}{c+1}\ge \frac{8}{5}.$$

Substituting $\sqrt{ab}=1/t$, which implies $c=t^2$, the inequality becomes

$$\frac{t^3+2}{30t}+\frac{2t}{t+1}+\frac{1}{t^2+1}\ge \frac{8}{5},$$

$$t^6+t^5+13t^4-45t^3+44t^2-16t+2\ge 0,$$

$$(t-1)^2[t^4+3t^3+2(3t-1)^2]\ge 0.$$

The equality holds for $a=b=c=1$.

□

P 1.178. *Let a,b,c be positive real numbers such that at most one of them is larger that 1 and $abc=1$. Prove that*

$$\frac{11(b^2+c^2)-10a^2}{b+c}+\frac{11(c^2+a^2)-10b^2}{c+a}+\frac{11(a^2+b^2)-10c^2}{a+b}\le 18.$$

(Vasile Cîrtoaje, RMM, 38, 2025)

Solution. Assume that $a\ge 1\ge b\ge c$ and denote $p=a+b+c$ and $q=ab+bc+ca$. By the AM-GM inequality, we have

$$p\ge 3(abc)^{1/3}=3.$$

In addition, from $(a-1)(b-1)(c-1)\ge 0$, we get

$$q\le p.$$

Write the inequality as follows:

$$\sum_{cyc}\frac{11(a^2+b^2+c^2)-21a^2}{b+c}\le 18,$$

$$11(a^2+b^2+c^2)\sum_{cyc}\frac{1}{b+c}-21\sum_{cyc}\frac{a^2}{b+c}\le 18,$$

$$11(a^2+b^2+c^2)\sum_{cyc}(a+b)(a+c)-21\sum_{cyc}a^2(a+b)(a+c)\le 18(a+b)(b+c)(c+a),$$

$$11(p^2-2q)(p^2+q)-21(p^4-3p^2q+4p)\le 18(pq-1).$$

So, for fixed p, we need to show that $f(q)\ge 0$, where

$$f(q)=11q^2-p(26p-9)q+5p^4+42p-9.$$

Since

$$f'(q) = 22q - p(26p - 9) < 8p^2 - p(26p - 9) = -9p(2p - 1) < 0,$$

$f(q)$ is decreasing, therefore

$$f(q) \ge f(p) = 5p^4 - 26p^3 + 20p^2 + 42p - 9 = (p - 3)^2(5p^2 + 4p - 1) \ge 0.$$

The equality occurs for $a = b = c = 1$.

□

P 1.179. *Let $a, b, c \le 8$ be real numbers such that $a + b + c = 3$. Prove that*

$$\frac{13a - 1}{a^2 + 23} + \frac{13b - 1}{b^2 + 23} + \frac{13c - 1}{c^2 + 23} \le \frac{3}{2}.$$

(Vasile Cîrtoaje, 2008)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) \ge \frac{-3}{2},$$

where

$$f(u) = \frac{1 - 13u}{u^2 + 23}.$$

Assume that $a \le b \le c$, and denote

$$s = \frac{b + c}{2}.$$

We have

$$s = \frac{3 - a}{2}, \quad 1 \le s \le 8.$$

We claim that

$$f(b) + f(c) \ge 2f(s).$$

To show this, according to Lemma below, it suffices to show that

$$h(b, c) \ge 0,$$

where

$$h(b, c) = \frac{g(b) - g(c)}{b - c}, \qquad g(u) = \frac{f(u) - f(s)}{u - s}.$$

We have

$$g(u) = \frac{(13s - 1)u - s - 299}{(s^2 + 23)(u^2 + 23)},$$

$$h(b, c) = \frac{(1 - 13s)bc + (s + 299)(b + c) + 23(13s - 1)}{(s^2 + 23)(b^2 + 23)(c^2 + 23)}.$$

Since $1-13s<0$ and $bc \le s^2$, we get

$$\begin{aligned} h(b,c) &\ge \frac{(1-13s)s^2+(s+299)(2s)+23(13s-1)}{(s^2+23)(b^2+23)(c^2+23)} \\ &= \frac{-13s^3+3s^2+897s-23}{(s^2+23)(b^2+23)(c^2+23)} \\ &> \frac{-13s^3+3s^2+897s-712}{(s^2+23)(b^2+23)(c^2+23)} \\ &= \frac{(8-s)(13s^2+101s-89)}{(s^2+23)(b^2+23)(c^2+23)} \ge 0. \end{aligned}$$

Therefore,

$$\begin{aligned} f(a)+f(b)+f(c)+\frac{3}{2} \ge f(a)+2f(s)+\frac{3}{2} &= f(a)+2f\left(\frac{3-a}{2}\right)+\frac{3}{2} \\ &= \frac{1-13a}{a^2+23}+\frac{4(13a-37)}{a^2-6a+101}+\frac{3}{2} \\ &= \frac{3(a-1)^2(a+11)^2}{2(a^2+23)(a^2-6a+101)} \ge 0. \end{aligned}$$

The equality holds for $a=b=c=1$, and also for $a=-11$ and $b=c=7$ (or any cyclic permutation).

Lemma. *Let f be a real function defined on an interval $\mathbb{I}$, and let $x,y \in \mathbb{I}$. If $s=\dfrac{x+y}{2}$, then the inequality*

$$f(x)+f(y) \ge 2f(s)$$

holds if and only if

$$h(x,y) \ge 0,$$

where

$$h(x,y)=\frac{g(x)-g(y)}{x-y}, \qquad g(u)=\frac{f(u)-f(s)}{u-s}.$$

In addition, the property is valid for the case where f is defined on $\mathbb{I}\setminus\{u_0\}$ and $x,y,s \ne u_0$.

Proof. From

$$\begin{aligned} f(x)+f(y)-2f(s) &= [f(x)-f(s)]+[f(y)-f(s)] \\ &= (x-s)g(x)+(y-s)g(y) \\ &= \frac{(x-y)[g(x)-g(y)]}{2} \\ &= \frac{(x-y)^2h(x,y)}{2}, \end{aligned}$$

the conclusion follows.

□

P 1.180. *Let $a, b, c \neq \frac{3}{4}$ be nonnegative real numbers such that $a+b+c=3$. Prove that*

$$\frac{1-a}{(4a-3)^2}+\frac{1-b}{(4b-3)^2}+\frac{1-c}{(4c-3)^2} \geq 0.$$

(Vasile Cîrtoaje, 2006)

Solution. Write the inequality as

$$f(a)+f(b)+f(c) \geq 0,$$

where

$$f(u)=\frac{1-u}{(4u-3)^2}.$$

Assume that $a \leq b \leq c$, and denote

$$s=\frac{b+c}{2}.$$

We have

$$s=\frac{3-a}{2}, \quad 1 \leq s \leq \frac{3}{2}.$$

We claim that

$$f(b)+f(c) \geq 2f(s).$$

According to Lemma from P 1.179, it suffices to show that

$$h(b,c) \geq 0,$$

where

$$h(b,c)=\frac{g(b)-g(c)}{b-c}, \qquad g(u)=\frac{f(u)-f(s)}{u-s}.$$

We have

$$g(u)=\frac{16(s-1)u-16s+15}{(4s-3)^2(4u-3)^2},$$

$$\frac{1}{8}h(b,c)=\frac{-32(s-1)bc+64s^2-90s+27}{(4s-3)^2(4b-3)^2(4c-3)^2}.$$

Since $s-1 \geq 0$ and $bc \leq s^2$, we get

$$\frac{1}{8}h(b,c) \geq \frac{-32(s-1)s^2+64s^2-90s+27}{(4s-3)^2(4b-3)^2(4c-3)^2}$$
$$=\frac{-32s^3+96s^2-90s+27}{(4s-3)^2(4b-3)^2(4c-3)^2}$$
$$=\frac{(3-2s)(3-4s)^2}{(4s-3)^2(4b-3)^2(4c-3)^2} \geq 0.$$

Therefore,

$$f(a)+f(b)+f(c) \ge f(a)+2f(s) = f(a)+2f\left(\frac{3-a}{2}\right)$$
$$= \frac{1-a}{(4a-3)^2} + \frac{a-1}{(3-2a)^2} = \frac{12a(a-1)^2}{(4a-3)^2(3-2a)^2} \ge 0.$$

The equality holds for $a=b=c=1$, and also for $a=0$ and $b=c=3/2$ (or any cyclic permutation).

□

P 1.181. *If a,b,c are the lengths of the sides of a triangle, then*

$$\frac{a^2}{4a^2+5bc} + \frac{b^2}{4b^2+5ca} + \frac{c^2}{4c^2+5ab} \ge \frac{1}{3}.$$

(Vasile Cîrtoaje, 2009)

Solution. Use the highest coefficient method. Write the inequality as $f_6(a,b,c) \ge 0$, where

$$f_6(a,b,c) = 3\sum a^2(4b^2+5ca)(4c^2+5ab) - \prod(4a^2+5bc)$$
$$= -45a^2b^2c^2 - 25abc\sum a^3 + 40\sum a^3b^3.$$

Since $f_6(a,b,c)$ has the highest coefficient

$$A = -45 - 75 + 120 = 0,$$

according to P 3.76-(b) in Volume 1, it suffices to prove the original inequality for $b=c=1$ and $0 \le a \le 2$, and for $a=b+c$.

Case 1: $b=c=1$, $0 \le a \le 2$. The original inequality becomes

$$\frac{a^2}{4a^2+5} + \frac{2}{5a+4} \ge \frac{1}{3},$$

$$(2-a)(a-1)^2 \ge 0.$$

Case 2: $a=b+c$. Using the Cauchy-Schwarz inequality

$$\frac{b^2}{4b^2+5ca} + \frac{c^2}{4c^2+5ab} \ge \frac{(b+c)^2}{4(b^2+c^2)+5a(b+c)},$$

it suffices to show that

$$\frac{a^2}{4a^2+5bc} + \frac{(b+c)^2}{4(b^2+c^2)+5a(b+c)} \ge \frac{1}{3},$$

which is equivalent to

$$\frac{1}{4(b^2+c^2)+13bc}+\frac{1}{9(b^2+c^2)+10bc}\ge\frac{1}{3(b^2+c^2+2bc)}.$$

Using the substitution

$$x=\frac{b}{c}+\frac{c}{b},\qquad x\ge 2,$$

the inequality becomes

$$\frac{1}{4x+13}+\frac{1}{9x+10}\ge\frac{1}{3(x+2)},$$

$$(x-2)(3x-4)\ge 0.$$

The equality holds for an equilateral triangle, and for a degenerate triangle with $a/2=b=c$ (or any cyclic permutation).

□

P 1.182. *If a,b,c are the lengths of the sides of a triangle, then*

$$\frac{1}{7a^2+b^2+c^2}+\frac{1}{7b^2+c^2+a^2}+\frac{1}{7c^2+a^2+b^2}\ge\frac{3}{(a+b+c)^2}.$$

(Vo Quoc Ba Can, 2010)

Solution. Use the highest coefficient method. Denote

$$p=a+b+c,\qquad q=ab+bc+ca,$$

and write the inequality as $f_6(a,b,c)\ge 0$, where

$$\begin{aligned}f_6(a,b,c)&=p^2\sum(7b^2+c^2+a^2)(7c^2+a^2+b^2)-3\prod(7a^2+b^2+c^2)\\&=p^2\sum(6b^2+p^2-2q)(6c^2+p^2-2q)-3\prod(6a^2+p^2-2q).\end{aligned}$$

Since $f_6(a,b,c)$ has the highest coefficient

$$A=-3\cdot 6^3<0,$$

according to P 3.76-(b) in Volume 1, it suffices to prove the original inequality for $b=c=1$ and $0\le a\le 2$, and for $a=b+c$.

Case 1: $b=c=1$, $0\le a\le 2$. The original inequality reduces to

$$\frac{1}{7a^2+2}+\frac{2}{a^2+8}\ge\frac{3}{(a+2)^2},$$

$$a(8-a)(a-1)^2\ge 0.$$

Case 2: $a = b + c$. Write the inequality as

$$\frac{1}{4(b^2+c^2)+7bc} + \frac{1}{4b^2+c^2+bc} + \frac{1}{4c^2+b^2+bc} \ge \frac{3}{2(b+c)^2}.$$

Since

$$\frac{3}{2(b+c)^2} - \frac{1}{4(b^2+c^2)+7bc} \le \frac{3}{2(b+c)^2} - \frac{1}{4(b^2+c^2)+8bc} = \frac{5}{4(b+c)^2},$$

it suffices to show that

$$\frac{1}{4b^2+c^2+bc} + \frac{1}{4c^2+b^2+bc} \ge \frac{5}{4(b+c)^2},$$

which is equivalent to

$$4[5(b^2+c^2)+2bc][(b^2+c^2)+2bc] \ge 5(4b^2+c^2+bc)(4c^2+b^2+bc),$$

$$4[5(b^2+c^2)^2+12bc(b^2+c^2)+4b^2c^2] \ge 5[4(b^2+c^2)^2+5bc(b^2+c^2)+10b^2c^2],$$

$$bc[23(b-c)^2+12bc] \ge 0.$$

The equality holds for an equilateral triangle, and for a degenerate triangle with $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 1.183. *Let a, b, c be the lengths of the sides of a triangle. If $k > -2$, then*

$$\sum \frac{a(b+c)+(k+1)bc}{b^2+kbc+c^2} \le \frac{3(k+3)}{k+2}.$$

(Vasile Cîrtoaje, 2009)

Solution. Use the highest coefficient method. Let

$$p = a+b+c, \qquad q = ab+bc+ca.$$

Write the inequality as $f_6(a,b,c) \ge 0$, where

$$f_6(a,b,c) = 3(k+3)\prod(b^2+kbc+c^2)$$

$$-(k+2)\sum[a(b+c)+(k+1)bc](c^2+kca+a^2)(a^2+kab+b^2).$$

From

$$f_6(a,b,c) = 3(k+3)\prod(p^2-2q+kbc-a^2)$$

$$-(k+2)\sum(q+kbc)(p^2-2q+kca-b^2)(p^2-2q+kab-c^2),$$

it follows that $f_6(a,b,c)$ has the same highest coefficient A as $f(a,b,c)$, where

$$f(a,b,c) = 3(k+3)P_3(a,b,c) - k(k+2)P_2(a,b,c),$$

$$P_3(a,b,c) = \prod(kbc - a^2), \qquad P_2(a,b,c) = \sum bc(kca - b^2)(kab - c^2).$$

According to Remark 2 from the proof of P 2.75 in Volume 1, we have

$$\begin{aligned} A &= 3(k+3)P_3(1,1,1) - k(k+2)P_2(1,1,1) \\ &= 3(k+3)(k-1)^3 - 3k(k+2)(k-1)^2 = -9(k-1)^2 \le 0. \end{aligned}$$

Taking into account P 3.76-(b) in Volume 1, it suffices to prove the original inequality for $b = c = 1$ and $0 \le a \le 2$, and for $a = b + c$.

Case 1: $b = c = 1$, $0 \le a \le 2$. The original inequality reduces to

$$\frac{2a+k+1}{k+2} + \frac{2(k+2)a+2}{a^2+ka+1} \le \frac{3(k+3)}{k+2},$$

$$\frac{a-k-4}{k+2} + \frac{(k+2)a+1}{a^2+ka+1} \le 0,$$

$$(2-a)(a-1)^2 \ge 0.$$

Case 2: $a = b + c$. Write the inequality as follows:

$$\sum \left[\frac{a(b+c)+(k+1)bc}{b^2+kbc+c^2} - 1\right] \le \frac{3}{k+2},$$

$$\sum \frac{ab+bc+ca-b^2-c^2}{b^2+kbc+c^2} \le \frac{3}{k+2},$$

$$\frac{3bc}{b^2+kbc+c^2} + \frac{bc-c^2}{b^2+(k+2)(bc+c^2)} + \frac{bc-b^2}{c^2+(k+2)(bc+b^2)} \le \frac{3}{k+2}.$$

Since

$$\frac{3bc}{b^2+kbc+c^2} \le \frac{3}{k+2},$$

it suffices to prove that

$$\frac{bc-c^2}{b^2+(k+2)(bc+c^2)} + \frac{bc-b^2}{c^2+(k+2)(bc+b^2)} \le 0.$$

This reduces to the obvious inequality

$$(b-c)^2(b^2+bc+c^2) \ge 0.$$

The equality holds for an equilateral triangle, and for a degenerate triangle with $a/2 = b = c$ (or any cyclic permutation).

□

P 1.184. *Let a, b, c be the lengths of the sides of a triangle. If $k > -2$, then*

$$\sum \frac{2a^2 + (4k+9)bc}{b^2 + kbc + c^2} \leq \frac{3(4k+11)}{k+2}.$$

(Vasile Cîrtoaje, 2009)

Solution. Use the highest coefficient method. Let

$$p = a + b + c, \qquad q = ab + bc + ca.$$

Write the inequality as $f_6(a, b, c) \geq 0$, where

$$f_6(a, b, c) = 3(4k+11)\prod(b^2 + kbc + c^2)$$

$$-(k+2)\sum[2a^2 + (4k+9)bc](c^2 + kca + a^2)(a^2 + kab + b^2).$$

From

$$f_6(a, b, c) = 3(4k+11)\prod(p^2 - 2q + kbc - a^2)$$

$$-(k+2)\sum[2a^2 + (4k+9)bc](p^2 - 2q + kca - b^2)(p^2 - 2q + kab - c^2),$$

it follows that $f_6(a, b, c)$ has the same highest coefficient A as $f(a, b, c)$, where

$$f(a, b, c) = 3(4k+11)P_3(a, b, c) - (k+2)P_2(a, b, c),$$

$$P_3(a, b, c) = \prod(kbc - a^2),$$

$$P_2(a, b, c) = \sum[2a^2 + (4k+9)bc](kca - b^2)(kab - c^2).$$

According to Remark 2 from the proof of P 2.75 in Volume 1, we have

$$\begin{aligned} A &= 3(4k+11)P_3(1,1,1) - (k+2)P_2(1,1,1) \\ &= 3(4k+11)(k-1)^3 - 3(k+2)(4k+11)(k-1)^2 \\ &= -9(4k+11)(k-1)^2 \leq 0. \end{aligned}$$

Taking into account P 3.76-(b) in Volume 1, it suffices to prove the original inequality for $b = c = 1$ and $0 \leq a \leq 2$, and for $a = b + c$.

Case 1: $b = c = 1$, $0 \leq a \leq 2$. The original inequality reduces to

$$\frac{2a^2 + 4k + 9}{k+2} + \frac{2(4k+9)a + 4}{a^2 + ka + 1} \leq \frac{3(4k+11)}{k+2},$$

$$\frac{a^2 - 4k - 12}{k+2} + \frac{(4k+9)a + 2}{a^2 + ka + 1} \leq 0,$$

$$(2-a)(a-1)^2 \geq 0,$$

Case 2: $a = b + c$. Write the inequality as follows:

$$\sum \left[\frac{2a^2 + (4k+9)bc}{b^2 + kbc + c^2} - 4\right] \le \frac{9}{k+2},$$

$$\sum \frac{2a^2 - 4b^2 - 4c^2 + 9bc}{b^2 + kbc + c^2} \le \frac{9}{k+2},$$

$$\frac{13bc - 2b^2 - 2c^2}{b^2 + kbc + c^2} + \frac{bc - 2b^2 + c^2}{b^2 + (k+2)(bc + c^2)} + \frac{bc - 2c^2 + b^2}{c^2 + (k+2)(bc + b^2)} \le \frac{9}{k+2}.$$

Since

$$\frac{9}{k+2} - \frac{13bc - 2b^2 - 2c^2}{b^2 + kbc + c^2} = \frac{(2k+13)(b-c)^2}{(k+2)(b^2 + kbc + c^2)}$$

and

$$\frac{bc - 2b^2 + c^2}{b^2 + (k+2)(bc + c^2)} + \frac{bc - 2c^2 + b^2}{c^2 + (k+2)(bc + b^2)} =$$

$$= \frac{(b-c)^2(b^2 + c^2 + 3bc) - 2(k+2)(b^2 - c^2)^2}{[b^2 + (k+2)(bc + c^2)][c^2 + (k+2)(bc + b^2]},$$

we only need to show that

$$\frac{2k+13}{(k+2)(b^2 + kbc + c^2)} + \frac{2(k+2)(b+c)^2 - b^2 - c^2 - 3bc}{[b^2 + (k+2)(bc + c^2)][c^2 + (k+2)(bc + b^2]} \ge 0.$$

Using the substitution

$$x = \frac{b}{c} + \frac{c}{b}, \qquad x \ge 2,$$

the inequality can be written as

$$\frac{2k+13}{(k+2)(x+k)} + \frac{(2k+3)x + 4k + 5}{(k+2)x^2 + (k+2)(k+3)x + 2k^2 + 6k + 5} \ge 0,$$

which is equivalent to

$$4(k+2)(k+4)x^2 + 2(k+2)Bx + C \ge 0,$$

where

$$B = 2k^2 + 13k + 22, \quad C = 8k^3 + 51k^2 + 98k + 65.$$

Since

$$B = 2(k+2)^2 + 5(k+2) + 4 > 0,$$

$$C = 8(k+2)^3 + 2k^2 + (k+1)^2 > 0,$$

the conclusion follows. The equality holds for an equilateral triangle, and for a degenerate triangle with $a/2 = b = c$ (or any cyclic permutation).

□

P 1.185. *If a,b,c are nnonnegative numbers such that $abc=1$, then*

$$\frac{1}{(a+1)^2}+\frac{1}{(b+1)^2}+\frac{1}{(c+1)^2}+\frac{1}{2(a+b+c-1)}\ge 1.$$

(Vasile Cîrtoaje, 2018)

Solution. Let

$$p=a+b+c, \qquad q=ab+bc+ca.$$

By the Cauchy-Schwarz inequality, we have

$$\sum\frac{1}{(a+1)^2}=\sum\frac{b^2c^2}{(1+bc)^2}\ge\frac{(\sum bc)^2}{\sum(1+bc)^2}$$

$$=\frac{q^2}{q^2+2q-2p+3}.$$

Thus we only need to show that

$$\frac{q^2}{q^2+2q-2p+3}+\frac{1}{2(p-1)}\ge 1,$$

which is equivalent to

$$(q-2p+3)^2\ge 0.$$

The equality occurs for $a=b=c=1$.

□

P 1.186. *If a,b,c are positive real numbers such that*

$$a\le b\le c, \qquad a^2bc\ge 1,$$

then

$$\frac{1}{1+a^3}+\frac{1}{1+b^3}+\frac{1}{1+c^3}\ge\frac{3}{1+abc}.$$

(Vasile Cîrtoaje, 2008)

Solution. Since

$$\frac{1}{1+x^2}+\frac{1}{1+y^2}-\frac{2}{1+xy}=\frac{(x-y)^2(xy-1)}{(1+x^2)(1+y^2)(1+xy)},$$

we have

$$\frac{1}{1+b^3}+\frac{1}{1+c^3}\ge\frac{2}{1+t^3},$$

where

$$t=\sqrt{bc}, \quad at\ge 1, \quad t\ge 1, \quad t\ge a.$$

So, we only need to show that

$$\frac{1}{1+a^3}+\frac{2}{1+t^3}\ge\frac{3}{1+at^2}\ ,$$

which is equivalent to

$$\frac{a(t^2-a^2)}{1+a^3}\ge\frac{2t^2(t-a)}{1+t^3}\ ,$$

$$(t-a)^2[at^2(2a+t)-a-2t]\ge 0.$$

This is true since

$$at^2(2a+t)-a-2t\ge t(2a+t)-a-2t=(t-1)^2+(at-1)+a(t-1)\ge 0.$$

The equality occurs for $a=b=c\ge 1$.

Remark 1. The inequality is true for the weaker condition

$$a^{8/5}bc\ge 1,$$

that is $a^4t^5\ge 1$. Since $bc\ge 1$, it suffices to show that $at^2(2a+t)-a-2t\ge 0$. This is true if the following homogeneous inequality is true:

$$\frac{at^2}{(a^4t^5)^{1/3}}(2a+t)\ge a+2t,$$

that is

$$t^{1/3}(2a+t)\ge a^{1/3}(a+2t).$$

Setting $a=1$ and $t=z^3\ge 1$, the inequality becomes as follows:

$$z(2+z^3)\ge 1+2z^3,$$

$$z^4-1\ge 2z(z^2-1),$$

$$(z^2-1)(z-1)^2\ge 0.$$

Remark 2. The inequality is also true for the condition

$$a^4b^5\ge 1.$$

Indeed, if $a^4b^5\ge 1$, then $b\ge 1$, $bc\ge b^2\ge 1$ and

$$a^4(bc)^{5/2}\ge 1,$$

which is equivalent to to the condition $a^{8/5}bc\ge 1$ from Remark 1.

Remark 3. From P 1.186, the following statement follows (*Vasile Cîrtoaje* and *Vasile Vornicu*):

- *If a,b,c,d are positive real numbers such that*

$$a \ge b \ge c \ge d, \qquad abcd \ge 1,$$

then

$$\frac{1}{1+a^3}+\frac{1}{1+b^3}+\frac{1}{1+c^3} \ge \frac{3}{1+abc}.$$

This is valid because $c \le b \le a$ and $c^2ba \ge 1$.

□

P 1.187. *If a,b,c are positive real numbers such that*

$$a \le b \le c, \qquad a^2c \ge 1,$$

then

$$\frac{1}{1+a^3}+\frac{1}{1+b^3}+\frac{1}{1+c^3} \ge \frac{3}{1+abc}.$$

(Vasile Cîrtoaje, 2021)

Solution. Denote

$$d = \sqrt{ac}, \qquad d \ge 1.$$

If $d = 1$, then $ac = 1$ and $a^2c \ge 1$ yield $a = b = c = 1$, and the required inequality is an equality. Next assume $d > 1$. For fixed a and c, write the inequality as $f(b) \ge 0$, where

$$f(b) = \frac{1}{1+a^3}+\frac{1}{1+b^3}+\frac{1}{1+c^3} - \frac{3}{1+abc}, \qquad b \in [a,c],$$

and calculate the derivative

$$\frac{1}{3}f'(b) = \frac{d^2}{(1+d^2b)^2} - \frac{b^2}{(1+b^3)^2}$$

$$= \frac{(db^2-1)(b-d)[d(1+b^3)+b(d^2b+1)]}{(1+d^2b)^2(1+b^3)^2}.$$

If $a \le \dfrac{1}{\sqrt{d}}$, then $f'(b) \le 0$ for $b \in [1/\sqrt{d}, d]$ and $f'(b) \ge 0$ for $b \in [a, 1/\sqrt{d}] \cup [d,c]$, hence $f(b)$ is decreasing on $[1/\sqrt{d}, d]$ and increasing on $[a, 1/\sqrt{d}] \cup [d,c]$. Thus, it suffices to show that $f(a) \ge 0$ and $f(d) \ge 0$. If $a \ge \dfrac{1}{\sqrt{d}}$, then $f'(b) \le 0$ for $b \in [a,d]$ and $f'(b) \ge 0$ for $b \in [d,c]$, $f(b)$ is decreasing on $[a,d]$ and increasing on $[d,c]$, hence it suffices to show that $f(d) \ge 0$. In conclusion, we only need to show that $f(a) \ge 0$ and $f(d) \ge 0$. Write the inequality $f(a) \ge 0$ as follows:

$$\frac{2}{1+a^3}+\frac{1}{1+c^3} \ge \frac{3}{1+a^2c},$$

$$\frac{2a^2(c-a)}{1+a^3} \geq \frac{c(c^2-a^2)}{1+c^3}\ ,$$

$$(c-a)^2[a^2c(a+2c)-2a-c] \geq 0.$$

This is true because

$$a^2c(a+2c)-2a-c \geq (a+2c)-2a-c = c-a \geq 0.$$

Write now the inequality $f(d) \geq 0$ as

$$\frac{1}{1+a^3}+\frac{1}{1+c^3} \ \geq\ \frac{2}{1+(ac)^{3/2}}.$$

Since

$$\frac{1}{1+x^2}+\frac{1}{1+y^2}-\frac{2}{1+xy} = \frac{(x-y)^2(xy-1)}{(1+x^2)(1+y^2)(1+xy)}\ ,$$

the inequality is equivalent to

$$\left(a^{3/2}-c^{3/2}\right)^2\left[(ac)^{3/2}-1\right] \geq 0.$$

This is true because

$$(ac)^3 \geq (a^2c)^2 \geq 1.$$

The equality occurs for $a=b=c \geq 1$.

□

P 1.188. *If a,b,c are positive real numbers such that*

$$a \leq b \leq c, \qquad 2a+c \geq 3,$$

then

$$\frac{1}{3+a^2}+\frac{1}{3+b^2}+\frac{1}{3+c^2} \geq \frac{3}{3+\left(\frac{a+b+c}{3}\right)^2}.$$

(Vasile Cîrtoaje, 2021)

Solution. Denote

$$s=\frac{a+b+c}{3}, \qquad s \geq 1.$$

For fixed a and c, write the inequality as $f(b) \geq 0$, where

$$f(b)=\frac{1}{3+a^2}+\frac{1}{3+b^2}+\frac{1}{3+c^2} - \frac{3}{3+s^2}, \qquad b \in [a,c],$$

and calculate the derivative

$$\frac{1}{2}f'(b) = \frac{s}{(3+s^2)^2}-\frac{b}{(3+b^2)^2} = \frac{(b-s)g(b)}{(3+s^2)^2(3+b^2)^2}\ ,$$

where

$$g(b) = bs(b^2 + bs + s^2 + 6) - 9.$$

Denote

$$d = \frac{a+c}{2}, \qquad d \geq 1.$$

If $d = 1$, then $a + c = 2$ and $2a + c \geq 3$ yield $a = b = c = 1$, and the required inequality is an equality. Next assume $d > 1$. Since

$$s = \frac{b+2d}{3},$$

we have

$$b - s = \frac{2(b-d)}{3},$$

$$g(b) = \frac{b(b+2d)}{3}\left[b^2 + \frac{b(b+2d)}{3} + \frac{(b+2d)^2}{9} + 6\right] - 9.$$

Since $g(b)$ is strictly increasing, $g(0) = -9$ and

$$g(d) = 3(d^4 + 2d^2 - 3) > 0,$$

there is an unique $d_1 \in (0, d)$ such that $g(d_1) = 0$, $g(b) \leq 0$ for $b \leq d_1$ and $g(b) \geq 0$ for $b \geq d_1$. If $a \leq d_1$, then $f'(b) \leq 0$ for $b \in [d_1, d]$ and $f'(b) \geq 0$ for $b \in [a, d_1] \cup [d, c]$, hence $f(b)$ is decreasing on $[d_1, d]$ and increasing on $[a, d_1] \cup [d, c]$. Thus, it suffices to show that $f(a) \geq 0$ and $f(d) \geq 0$. If $a \geq d_1$, then $f'(b) \leq 0$ for $b \in [a, d]$ and $f'(b) \geq 0$ for $b \in [d, c]$, $f(b)$ is decreasing on $[a, d]$ and increasing on $[d, c]$, hence it suffices to show that $f(d) \geq 0$. In conclusion, we only need to show that $f(a) \geq 0$ and $f(d) \geq 0$. Denoting

$$p = \frac{2a+c}{3},$$

we may write the inequality $f(a) \geq 0$ as follows:

$$\frac{2}{3+a^2} + \frac{1}{3+c^2} \geq \frac{3}{3+p^2},$$

$$\frac{2(p^2 - a^2)}{3+a^2} \geq \frac{c^2 - p^2}{3+c^2},$$

$$(a-c)^2[(a+c)p + ac - 3] \geq 0,$$

$$(a-c)^2(2a^2 + 6ac + c^2 - 9) \geq 0.$$

This is true because

$$2a^2 + 6ac + c^2 - 9 = (2a+c)^2 - 9 + 2a(c-a) \geq 0.$$

Write now the inequality $f(d) \geq 0$ as follows:

$$\frac{1}{3+a^2} + \frac{1}{3+c^2} \geq \frac{2}{3+d^2},$$

$$\frac{d^2-a^2}{3+a^2} \ge \frac{c^2-d^2}{3+c^2},$$

$$(a-c)^2[(a+c)d+ac)-3] \ge 0,$$

$$(a-c)^2(a^2+4ac+c^2-6) \ge 0.$$

This is true because

$$3(a^2+4ac+c^2)-18 \ge 3(a^2+4ac+c^2)-2(2a+c)^2 = (c-a)(c+5a) \ge 0.$$

The equality occurs for $a=b=c\ge 1$, and also for $a=b=0$ and $c=3$.

□

P 1.189. *If a,b,c are positive real numbers such that*

$$a \le b \le c, \qquad 9a+8b \ge 17,$$

then

$$\frac{1}{3+a^2}+\frac{1}{3+b^2}+\frac{1}{3+c^2} \ge \frac{3}{3+\left(\frac{a+b+c}{3}\right)^2}.$$

(Vasile Cîrtoaje, 2021)

Solution. From $a\le b\le c$ and $9a+8b\ge 17$, it follows that

$$1 \le b \le c, \qquad a+b+c \ge 3.$$

As in the previous P 1.188, denote

$$s=\frac{a+b+c}{3}, \qquad 1\le s\le c,$$

and, for fixed a and b, write the inequality as $f(c)\ge 0$, where

$$f(c)=\frac{1}{3+a^2}+\frac{1}{3+b^2}+\frac{1}{3+c^2} - \frac{3}{3+s^2}, \qquad c \ge b.$$

We show that

$$f(c) \ge f(b) \ge 0.$$

Since

$$\frac{1}{2}f'(c)=\frac{s}{(3+s^2)^2}-\frac{c}{(3+c^2)^2}=\frac{(c-s)[cs(c^2+cs+s^2+6)-9]}{(3+s^2)^2(3+c^2)^2} \ge 0,$$

$f(c)$ is increasing, therefore $f(c)\ge f(b)$. Denote

$$p=\frac{a+2b}{3},$$

Write now the inequality $f(b) \geq 0$ as follows:

$$\frac{1}{3+a^2} + \frac{2}{3+b^2} \geq \frac{3}{3+p^2} ,$$

$$\frac{p^2-a^2}{3+a^2} \geq \frac{2(b^2-p^2)}{3+b^2} ,$$

$$(a-b)^2[(a+b)p + ab - 3] \geq 0 ,$$

$$(a-b)^2(a^2+6ab+2b^2-9) \geq 0 .$$

This is true if

$$16(a^2+6ab+2b^2) \geq (7a+5b)^2,$$

which is equivalent to

$$(b-a)(b+220a) \geq 0.$$

The equality occurs for $a = b = c \geq 1$.

Remark. Actually, the inequality is valid for the weaker condition

$$ka + b \geq k+1, \qquad k = \frac{3}{\sqrt{2}} - 1,$$

when the inequality

$$(k+1)^2(a^2+6ab+2b^2) \geq 9(ka+b)^2,$$

reduces to the form

$$a(b-a) \geq 0.$$

The equality occurs for $a = b = c \geq 1$, and also for $a = 0$ and $b = c = \dfrac{3}{\sqrt{2}}$.

□

P 1.190. *Let a, b, c, d be positive real numbers such that $abcd = 1$. Prove that*

$$\sum \frac{1}{1+ab+bc+ca} \leq 1.$$

Solution. From

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}} + \frac{1}{\sqrt{ab}} = \frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{\sqrt{abc}},$$

we get

$$ab+bc+ca \geq \sqrt{abc}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right) = \frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{\sqrt{d}}.$$

Therefore,

$$\sum \frac{1}{1+ab+bc+ca} \leq \sum \frac{\sqrt{d}}{\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}} = 1,$$

which is just the required inequality. The equality occurs for $a = b = c = d = 1$.

□

P 1.191. *Let a, b, c, d be positive real numbers such that $abcd = 1$. Prove that*

$$\frac{1}{(1+a)^2}+\frac{1}{(1+b)^2}+\frac{1}{(1+c)^2}+\frac{1}{(1+d)^2}\ge 1.$$

(Vasile Cîrtoaje, Crux Mathematicorum, 6, 2004)

First Solution. The inequality follows by summing the following inequalities (see P 1.1):

$$\frac{1}{(1+a)^2}+\frac{1}{(1+b)^2}\ge\frac{1}{1+ab},$$

$$\frac{1}{(1+c)^2}+\frac{1}{(1+d)^2}\ge\frac{1}{1+cd}=\frac{ab}{1+ab}.$$

The equality occurs for $a = b = c = d = 1$.

Second Solution. Using the substitution

$$a=\frac{1}{x^4},\quad b=\frac{1}{y^4},\quad c=\frac{1}{z^4},\quad d=\frac{1}{t^4},$$

where x, y, z, t are positive real numbers such that $xyzt = 1$, the inequality becomes

$$\frac{x^6}{\left(x^3+\frac{1}{x}\right)^2}+\frac{y^6}{\left(y^3+\frac{1}{y}\right)^2}+\frac{z^6}{\left(z^3+\frac{1}{z}\right)^2}+\frac{t^6}{\left(t^3+\frac{1}{t}\right)^2}\ge 1.$$

By the Cauchy-Schwarz inequality, we get

$$\sum\frac{x^6}{\left(x^3+\frac{1}{x}\right)^2}\ge\frac{\left(\sum x^3\right)^2}{\sum\left(x^3+\frac{1}{x}\right)^2}=\frac{\left(\sum x^3\right)^2}{\sum x^6+2\sum x^2+\sum x^2y^2z^2}.$$

Thus, it suffices to prove the homogeneous inequality

$$2(x^3y^3+x^3z^3+x^3t^3+y^3z^3+y^3t^3+z^3t^3)\ge 2xyzt\sum x^2+\sum x^2y^2z^2.$$

We can get it by summing the inequalities

$$4(x^3y^3+x^3z^3+x^3t^3+y^3z^3+y^3t^3+z^3t^3)\ge 6xyzt\sum x^2$$

and

$$2(x^3y^3+x^3z^3+x^3t^3+y^3z^3+y^3t^3+z^3t^3)\ge 3\sum x^2y^2z^2,$$

Write these inequalities as

$$\sum x^3(y^3+z^3+t^3-3yzt)\ge 0$$

and

$$\sum(x^3y^3 + y^3z^3 + z^3x^3 - 3x^2y^2z^2) \ge 0,$$

respectively. By the AM-GM inequality, we have

$$y^3 + z^3 + t^3 \ge 3yzt, \qquad x^3y^3 + y^3z^3 + z^3x^3 \ge 3x^2y^2z^2.$$

Thus the conclusion follows.

Third Solution. Using the substitution

$$a = \frac{yz}{x^2}, \quad b = \frac{zt}{y^2}, \quad c = \frac{tx}{z^2}, \quad d = \frac{xy}{t^2},$$

where x, y, z, t are positive real numbers, the inequality becomes

$$\frac{x^4}{(x^2+yz)^2} + \frac{y^4}{(y^2+zt)^2} + \frac{z^4}{(z^2+tx)^2} + \frac{t^4}{(t^2+xy)^2} \ge 1.$$

Using the Cauchy-Schwarz inequality two times, we deduce

$$\frac{x^4}{(x^2+yz)^2} + \frac{z^4}{(z^2+tx)^2} \ge \frac{x^4}{(x^2+y^2)(x^2+z^2)} + \frac{z^4}{(z^2+t^2)(z^2+x^2)}$$

$$= \frac{1}{x^2+z^2}\left(\frac{x^4}{x^2+y^2} + \frac{z^4}{z^2+t^2}\right) \ge \frac{x^2+z^2}{x^2+y^2+z^2+t^2},$$

hence

$$\frac{x^4}{(x^2+yz)^2} + \frac{z^4}{(z^2+tx)^2} \ge \frac{x^2+z^2}{x^2+y^2+z^2+t^2}.$$

Adding this to the similar inequality

$$\frac{y^4}{(y^2+zt)^2} + \frac{t^4}{(t^2+xy)^2} \ge \frac{y^2+t^2}{x^2+y^2+z^2+t^2},$$

we get the required inequality.

Fourth Solution. Using the substitution

$$a = \frac{x}{y}, \quad b = \frac{y}{z}, \quad c = \frac{z}{t}, \quad d = \frac{t}{x},$$

where x, y, z, t are positive real numbers, the inequality can be written as

$$\frac{y^2}{(x+y)^2} + \frac{z^2}{(y+z)^2} + \frac{t^2}{(z+t)^2} + \frac{x^2}{(t+x)^2} \ge 1.$$

By the Cauchy-Schwarz inequality and the AM-GM inequality, we get

$$\sum \frac{y^2}{(x+y)^2} \ge \frac{[\sum y(y+z)]^2}{\sum (x+y)^2(y+z)^2}$$

$$= \frac{[(x+y)^2+(y+z)^2+(z+t)^2+(t+x)^2]^2}{4[(x+y)^2+(z+t)^2][(y+z)^2+(t+x)^2]} \geq 1.$$

Remark. The following generalization holds true (*Vasile Cîrtoaje*, 2005):

- *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1a_2\cdots a_n = 1$. If $k \geq \sqrt{n}-1$, then*

$$\frac{1}{(1+ka_1)^2} + \frac{1}{(1+ka_2)^2} + \cdots + \frac{1}{(1+ka_n)^2} \geq \frac{n}{(1+k)^2}.$$

□

P 1.192. *Let $a, b, c, d \neq \frac{1}{3}$ be positive real numbers such that $abcd = 1$. Prove that*

$$\frac{1}{(3a-1)^2} + \frac{1}{(3b-1)^2} + \frac{1}{(3c-1)^2} + \frac{1}{(3d-1)^2} \geq 1.$$

(Vasile Cîrtoaje, 2006)

First Solution. It suffices to show that

$$\frac{1}{(3a-1)^2} \geq \frac{a^{-3}}{a^{-3}+b^{-3}+c^{-3}+d^{-3}}.$$

This inequality is equivalent to

$$6a^{-2} + b^{-3} + c^{-3} + d^{-3} \geq 9a^{-1},$$

which follows by the AM-GM inequality, as follows:

$$6a^{-2} + b^{-3} + c^{-3} + d^{-3} \geq 9\sqrt[9]{a^{-12}b^{-3}c^{-3}d^{-3}} = 9a^{-1}.$$

The equality occurs for $a = b = c = d = 1$.

Second Solution. Let $a \leq b \leq c \leq d$. If $a \leq 2/3$, then

$$\frac{1}{(3a-1)^2} \geq 1,$$

and the desired inequality is clearly true. Otherwise, if $2/3 < a \leq b \leq c \leq d$, we have

$$4a^3 - (3a-1)^2 = (a-1)^2(4a-1) \geq 0.$$

Using this result and the AM-GM inequality, we get

$$\sum \frac{1}{(3a-1)^2} \geq \frac{1}{4}\sum \frac{1}{a^3} \geq \sqrt[4]{\frac{1}{a^3b^3c^3d^3}} = 1.$$

Third Solution. We have

$$\frac{1}{(3a-1)^2}-\frac{1}{(a^3+1)^2}=\frac{a(a-1)^2(a+2)(a^2+3)}{(3a-1)^2(a^3+1)^2}\geq 0;$$

therefore,

$$\sum\frac{1}{(3a-1)^2}\geq\sum\frac{1}{(a^3+1)^2}.$$

Thus, it suffices to prove that

$$\sum\frac{1}{(a^3+1)^2}\geq 1,$$

which is an immediate consequence of the inequality in P 1.191.

□

P 1.193. *Let a,b,c,d be positive real numbers such that $abcd=1$. Prove that*

$$\frac{1}{1+a+a^2+a^3}+\frac{1}{1+b+b^2+b^3}+\frac{1}{1+c+c^2+c^3}+\frac{1}{1+d+d^2+d^3}\geq 1.$$

(Vasile Cîrtoaje, 1999)

First Solution. We get the desired inequality by summing the inequalities

$$\frac{1}{1+a+a^2+a^3}+\frac{1}{1+b+b^2+b^3}\geq\frac{1}{1+(ab)^{3/2}},$$

$$\frac{1}{1+c+c^2+c^3}+\frac{1}{1+d+d^2+d^3}\geq\frac{1}{1+(cd)^{3/2}}.$$

Thus, it suffices to show that

$$\frac{1}{1+x^2+x^4+x^6}+\frac{1}{1+y^2+y^4+y^6}\geq\frac{1}{1+x^3y^3},$$

where x and y are positive real numbers. Putting $p=xy$ and $s=x^2+xy+y^2$, this inequality becomes

$$p^3(x^6+y^6)+p^2(p-1)(x^4+y^4)-p^2(p^2-p+1)(x^2+y^2)-p^6-p^4+2p^3-p^2+1\geq 0,$$

$$p^3(x^3-y^3)^2+p^2(p-1)(x^2-y^2)^2-p^2(p^2-p+1)(x-y)^2+p^6-p^4-p^2+1\geq 0,$$

$$p^3s^2(x-y)^2+p^2(p-1)(s+p)^2(x-y)^2-p^2(p^2-p+1)(x-y)^2+p^6-p^4-p^2+1\geq 0,$$

$$p^2(s+1)(ps-1)(x-y)^2+(p^2-1)(p^4-1)\geq 0.$$

If $ps-1\geq 0$, then the inequality is clearly true. Consider further that $ps<1$. From $ps<1$ and $s\geq 3p$, we get $p^2<1/3$. Write the desired inequality in the form

$$p^2(1+s)(1-ps)(x-y)^2\leq(1-p^2)(1-p^4).$$

Since

$$p(x-y)^2 = p(s-3p) < 1-3p^2 < 1-p^2,$$

it suffices to show that

$$p(1+s)(1-ps) \le 1-p^4.$$

Indeed,

$$4p(1+s)(1-ps) \le [p(1+s)+(1-ps)]^2 = (1+p)^2 < 2(1+p^2) < 4(1-p^4).$$

The equality occurs for $a=b=c=d=1$.

Second Solution. Assume that $a \ge b \ge c \ge d$, and write the inequality as

$$\sum \frac{1}{(1+a)(1+a^2)} \ge 1.$$

Since

$$\frac{1}{1+a} \le \frac{1}{1+b} \le \frac{1}{1+c}, \quad \frac{1}{1+a^2} \le \frac{1}{1+b^2} \le \frac{1}{1+c^2},$$

by Chebyshev's inequality, it suffices to prove that

$$\frac{1}{3}\left(\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}\right)\left(\frac{1}{1+a^2}+\frac{1}{1+b^2}+\frac{1}{1+c^2}\right)+\frac{1}{(1+d)(1+d^2)} \ge 1.$$

On the other hand, from Remark 3 of P 1.186, we have

$$\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c} \ge \frac{3}{1+\sqrt[3]{abc}} = \frac{3\sqrt[3]{d}}{\sqrt[3]{d}+1}$$

and

$$\frac{1}{1+a^2}+\frac{1}{1+b^2}+\frac{1}{1+c^2} \ge \frac{3}{1+\sqrt[3]{a^2b^2c^2}} = \frac{3\sqrt[3]{d^2}}{\sqrt[3]{d^2}+1}.$$

Thus, it suffices to prove that

$$\frac{3d}{(1+\sqrt[3]{d})(1+\sqrt[3]{d^2}\ )}+\frac{1}{(1+d)(1+d^2)} \ge 1.$$

Putting $x=\sqrt[3]{d}$, this inequality becomes as follows:

$$\frac{3x^3}{(1+x)(1+x^2)}+\frac{1}{(1+x^3)(1+x^6)} \ge 1,$$

$$3x^3(1-x+x^2)(1-x^2+x^4)+1 \ge (1+x^3)(1+x^6),$$

$$x^3(2-3x+2x^3-3x^5+2x^6) \ge 0,$$

$$x^3(1-x)^2(2+x+x^3+2x^4) \ge 0.$$

Remark. The following generalization holds true (*Vasile Cîrtoaje*, 2004):

- *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1a_2\cdots a_n = 1$, then*

$$\frac{1}{1+a_1+\cdots+a_1^{n-1}} + \frac{1}{1+a_2+\cdots+a_2^{n-1}} + \cdots + \frac{1}{1+a_n+\cdots+a_n^{n-1}} \ge 1.$$

Gabriel Dospinescu gave a nice proof by induction in 2005. So, using the induction hypothesis for $n-1$, we have for each $k \in \{1, 2, \ldots, n\}$:

$$\sum_{i\neq k} x_i^{n-1} + (n-1)(n-2)\prod_{i\neq k} x_i \ge \prod_{i\neq k} x_i \left(\sum_{i\neq k} x_i\right)\left(\sum_{i\neq k}\frac{1}{x_i}\right).$$

Multiplying by x_k, we obtain

$$x_k\sum_{i\neq k} x_i^{n-1} + (n-1)(n-2)x_1x_2\cdots x_n \ge x_1x_2\cdots x_n\left(\sum_{i\neq k} x_i\right)\left(\sum_{i\neq k}\frac{1}{x_i}\right),$$

that is

$$x_k\sum_{i=1}^n x_i^{n-1} - x_k^n + (n-1)(n-2)x_1x_2\cdots x_n \ge$$

$$\ge x_1x_2\cdots x_n\left[\left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n\frac{1}{x_i}\right) - x_k\sum_{i=1}^n\frac{1}{x_i} - \frac{1}{x_k}\sum_{i=1}^n x_i + 1\right].$$

Summing over k, we find

$$\left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n x_i^{n-1}\right) - \sum_{i=1}^n x_i^n + n(n-1)(n-2)x_1x_2\cdots x_n \ge$$

$$\ge x_1x_2\cdots x_n\left[(n-2)\left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n\frac{1}{x_i}\right) + n\right].$$

Using Surany's inequality,

$$\left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n x_i^{n-1}\right) \le (n-1)\sum_{i=1}^n x_i^n + nx_1x_2\cdots x_n,$$

we get

$$(n-2)\sum_{i=1}^n x_i^n + nx_1x_2\cdots x_n + n(n-1)(n-2)x_1x_2\cdots x_n \ge$$

$$\ge x_1x_2\cdots x_n\left[(n-2)\left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n\frac{1}{x_i}\right) + n\right],$$

which simplifies to the desired result.

□

P 1.194. *Let a, b, c, d be positive real numbers such that $abcd = 1$. Prove that*

$$\frac{1}{1+a+2a^2}+\frac{1}{1+b+2b^2}+\frac{1}{1+c+2c^2}+\frac{1}{1+d+2d^2} \ge 1.$$

(Vasile Cîrtoaje, 2006)

Solution. We will show that

$$\frac{1}{1+a+2a^2} \ge \frac{1}{1+a^k+a^{2k}+a^{3k}},$$

where $k = 5/6$. Then, it suffices to show that

$$\sum \frac{1}{1+a^k+a^{2k}+a^{3k}} \ge 1,$$

which immediately follows from the inequality in P 1.193. Setting $a = x^6$, $x > 0$, the claimed inequality can be written as

$$\frac{1}{1+x^6+2x^{12}} \ge \frac{1}{1+x^5+x^{10}+x^{15}},$$

which is equivalent to

$$x^{10}+x^5+1 \ge 2x^7+x.$$

We can prove it by summing the AM-GM inequalities

$$x^5+4 \ge 5x$$

and

$$5x^{10}+4x^5+1 \ge 10x^7.$$

This completes the proof. The equality occurs for $a = b = c = d = 1$.

Remark. The inequalities in P 1.191, P 1.193 and P 1.194 are particular cases of the following more general inequality (*Vasile Cîrtoaje*, 2009):

- *Let $a_1, a_2, \ldots, a_n$ $(n \ge 4)$ be positive real numbers such that $a_1a_2\cdots a_n = 1$. If p, q, r are nonnegative real numbers satisfying $p+q+r = n-1$, then*

$$\sum_{i=1}^{i=n} \frac{1}{1+pa_i+qa_i^2+ra_i^3} \ge 1.$$

□

P 1.195. *Let a, b, c, d be positive real numbers such that $abcd = 1$. Prove that*

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{9}{a+b+c+d} \ge \frac{25}{4}.$$

Solution *(by Vo Quoc Ba Can).* Replacing a, b, c, d by a^4, b^4, c^4, d^4, respectively, the inequality becomes as follows:

$$\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}+\frac{1}{d^4}+\frac{9}{a^4+b^4+c^4+d^4} \ge \frac{25}{4abcd},$$

$$\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}+\frac{1}{d^4}-\frac{4}{abcd} \ge \frac{9}{4abcd}-\frac{9}{a^4+b^4+c^4+d^4},$$

$$\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}+\frac{1}{d^4}-\frac{4}{abcd} \ge \frac{9(a^4+b^4+c^4+d^4-4abcd)}{4abcd(a^4+b^4+c^4+d^4)}.$$

Using the identities

$$a^4+b^4+c^4+d^4-4abcd = (a^2-b^2)^2+(c^2-d^2)^2+2(ab-cd)^2,$$

$$\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}+\frac{1}{d^4}-\frac{4}{abcd} = \frac{(a^2-b^2)^2}{a^4b^4}+\frac{(c^2-d^2)^2}{c^4d^4}+\frac{2(ab-cd)^2}{a^2b^2c^2d^2},$$

the inequality can be written as

$$\frac{(a^2-b^2)^2}{a^4b^4}+\frac{(c^2-d^2)^2}{c^4d^4}+\frac{2(ab-cd)^2}{a^2b^2c^2d^2} \ge \frac{9[(a^2-b^2)^2+(c^2-d^2)^2+2(ab-cd)^2]}{4abcd(a^4+b^4+c^4+d^4)},$$

$$(a^2-b^2)^2\left[\frac{4cd(a^4+b^4+c^4+d^4)}{a^3b^3}-9\right]+(c^2-d^2)^2\left[\frac{4ab(a^4+b^4+c^4+d^4)}{c^3d^3}-9\right]$$
$$+2(ab-cd)^2\left[\frac{4(a^4+b^4+c^4+d^4)}{abcd}-9\right] \ge 0.$$

By the AM-GM inequality, we have

$$a^4+b^4+c^4+d^4 \ge 4abcd.$$

Therefore, it suffices to show that

$$(a^2-b^2)^2\left[\frac{4cd(a^4+b^4+c^4+d^4)}{a^3b^3}-9\right]+(c^2-d^2)^2\left[\frac{4ab(a^4+b^4+c^4+d^4)}{c^3d^3}-9\right] \ge 0.$$

Without loss of generality, assume that $a \ge c \ge d \ge b$. Since

$$(a^2-b^2)^2 \ge (c^2-d^2)^2$$

and

$$\frac{4cd(a^4+b^4+c^4+d^4)}{a^3b^3} \ge \frac{4(a^4+b^4+c^4+d^4)}{a^3b} \ge \frac{4(a^4+3b^4)}{a^3b} > 9,$$

it is enough to prove that

$$\left[\frac{4cd(a^4+b^4+c^4+d^4)}{a^3b^3}-9\right]+\left[\frac{4ab(a^4+b^4+c^4+d^4)}{c^3d^3}-9\right] \ge 0,$$

which is equivalent to

$$2(a^4+b^4+c^4+d^4)\left(\frac{cd}{a^3b^3}+\frac{ab}{c^3d^3}\right)\ge 9.$$

Indeed, by the AM-GM inequality,

$$2(a^4+b^4+c^4+d^4)\left(\frac{cd}{a^3b^3}+\frac{ab}{c^3d^3}\right)\ge 8abcd\left(\frac{2}{abcd}\right)=16>9.$$

The equality occurs for $a=b=c=d=1$.

□

P 1.196. *If a,b,c,d are real numbers such that $a+b+c+d=0$, then*

$$\frac{(a-1)^2}{3a^2+1}+\frac{(b-1)^2}{3b^2+1}+\frac{(c-1)^2}{3c^2+1}+\frac{(d-1)^2}{3d^2+1}\le 4.$$

Solution. Since

$$4-\frac{3(a-1)^2}{3a^2+1}=\frac{(3a+1)^2}{3a^2+1},$$

we can write the inequality as

$$\sum\frac{(3a+1)^2}{3a^2+1}\ge 4.$$

On the other hand, since

$$4a^2=3a^2+(b+c+d)^2\le 3a^2+3(b^2+c^2+d^2)=3(a^2+b^2+c^2+d^2),$$

$$3a^2+1\le\frac{9}{4}(a^2+b^2+c^2+d^2)+1=\frac{9(a^2+b^2+c^2+d^2)+4}{4},$$

we have

$$\sum\frac{(3a+1)^2}{3a^2+1}\ge\frac{4\sum(3a+1)^2}{9(a^2+b^2+c^2+d^2)+4}=4.$$

The equality holds for $a=b=c=d=0$, and also for $a=1$ and $b=c=d=-1/3$ (or any cyclic permutation).

Remark. The following generalization is also true.

- *If $a_1,a_2,\ldots,a_n$ are real numbers such that $a_1+a_2+\cdots+a_n=0$, then*

$$\frac{(a_1-1)^2}{(n-1)a_1^2+1}+\frac{(a_2-1)^2}{(n-1)a_2^2+1}+\cdots+\frac{(a_n-1)^2}{(n-1)a_n^2+1}\le n,$$

with equality for $a_1=a_2=\cdots=a_n=0$, and also for $a_1=1$ and $a_2=a_3=\cdots=a_n=-1/(n-1)$ (or any cyclic permutation).

□

P 1.197. *If $a, b, c, d \ge -5$ such that $a+b+c+d=4$, then*

$$\frac{1-a}{(1+a)^2}+\frac{1-b}{(1+b)^2}+\frac{1-c}{(1+c)^2}+\frac{1-d}{(1+d)^2} \ge 0.$$

Solution. Assume that $a \le b \le c \le d$. We show first that $x \in \mathbb{R} \setminus \{-1\}$ involves

$$\frac{1-x}{(1+x)^2} \ge \frac{-1}{8},$$

and $x \in [-5, 1/3] \setminus \{-1\}$ involves

$$\frac{1-x}{(1+x)^2} \ge \frac{3}{8}.$$

Indeed, we have

$$\frac{1-x}{(1+x)^2}+\frac{1}{8}=\frac{(x-3)^2}{8(1+x)^2} \ge 0$$

and

$$\frac{1-x}{(1+x)^2}-\frac{3}{8}=\frac{(5+x)(1-3x)}{8(1+x)^2} \ge 0.$$

Therefore, if $a \le 1/3$, then

$$\frac{1-a}{(1+a)^2}+\frac{1-b}{(1+b)^2}+\frac{1-c}{(1+c)^2}+\frac{1-d}{(1+d)^2} \ge \frac{3}{8}-\frac{1}{8}-\frac{1}{8}-\frac{1}{8}=0.$$

Assume now that $1/3 \le a \le b \le c \le d$. Since

$$1-a \ge 1-b \ge 1-c \ge 1-d$$

and

$$\frac{1}{(1+a)^2} \ge \frac{1}{(1+b)^2} \ge \frac{1}{(1+c)^2} \ge \frac{1}{(1+d)^2},$$

by Chebyshev's inequality, we have

$$\frac{1-a}{(1+a)^2}+\frac{1-b}{(1+b)^2}+\frac{1-c}{(1+c)^2}+\frac{1-d}{(1+d)^2} \ge$$

$$\ge \frac{1}{4}\left[\sum(1-a)\right]\left[\sum\frac{1}{(1+a)^2}\right]=0.$$

The equality holds for $a=b=c=d=1$, and also for $a=-5$ and $b=c=d=3$ (or any cyclic permutation).

□

P 1.198. *If a, b, c, d are nonnegative real numbers such that*

$$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}+\frac{1}{d+1}=3,$$

then

$$3(ab+ac+ad+bc+bd+cd)+\frac{4}{a+b+c+d}\le 5.$$

(Vasile Cîrtoaje and Leonard Giugiuc, 2022)

Solution. Let

$$S=a+b+c+d,$$

$$S_2=ab+ac+ad+bc+bd+cd=\sum ab.$$

By the AM-HM inequality

$$\left[\sum(a+1)\right]\left(\sum\frac{1}{a+1}\right)\ge 16,$$

we get

$$3(S+4)\ge 16,\qquad S\ge\frac{4}{3}.$$

Case 1: $\frac{4}{3}\le S\le 2$. Write the hypothesis as

$$\frac{a}{a+1}+\frac{b}{b+1}+\frac{c}{c+1}+\frac{d}{c+1}=1,$$

or

$$\sum\frac{a}{a+1}=1.$$

By the Cauchy-Schwarz inequality, we have

$$\left[\sum a(a+1)\right]\left(\sum\frac{a}{a+1}\right)\ge S^2,$$

hence

$$\sum a(a+1)\ge S^2,$$

$$\sum a^2+S\ge S^2,$$

$$S^2-2S_2+S\ge S^2,$$

$$S_2\le\frac{S}{2}.$$

Thus, it suffices to show that

$$\frac{3S}{2}+\frac{4}{S}\le 5,$$

which is equivalent to

$$(S-2)(3S-4) \le 0.$$

Clearly, the last inequality is true.

Case 2: $S \ge 2$. From

$$3 = \sum \frac{1}{a+1} = \frac{4+3S+2S_2+\sum abc}{1+S+S_2+\sum abc+abcd} \le \frac{4+3S+2S_2}{1+S+S_2},$$

we get $S_2 \le 1$. Thus it suffices to show that

$$3 + \frac{4}{S} \le 5,$$

which is equivalent to $S \ge 2$.

The proof is completed. The equality occurs when $a = b = c = d = \frac{1}{3}$, and also when $a = b = 1$ and $c = d = 0$ (or any permutation).

Remark. The following generalization holds:

- *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that*

$$\sum_{i=1}^{n} \frac{1}{a_i+1} = n-1,$$

then

$$(n-1)\sum_{1\le i<j\le n} a_i a_j + \frac{n}{a_1+a_2+\cdots+a_n} \le \frac{3n-2}{2},$$

with equality when $a_1 = a_2 = \cdots = a_n = \frac{1}{n-1}$, and also when $a_1 = a_2 = 1$ and $a_3 = \cdots = a_n = 0$ (or any permutation).

□

P 1.199. *If a, b, c, d are nonnegative real numbers such that*

$$ab+ac+ad+bc+bd+cd = 9,$$

then

$$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}+\frac{1}{d+1} \le 1 + \frac{a+b+c+d}{4}.$$

(Vasile Cîrtoaje, 2021)

Solution. Let

$$S = a + b + c + d,$$

$$S_2 = ab + ac + ad + bc + bd + cd = \sum ab.$$

Write the inequality as

$$\frac{S}{4} + \frac{a}{a+1} + \frac{a}{b+1} + \frac{a}{c+1} + \frac{a}{d+1} \ge 3.$$

By the Cauchy-Schwarz inequality, we have

$$\left[\sum a(a+1)\right]\left(\sum \frac{a}{a+1}\right) \ge S^2,$$

hence

$$\sum \frac{a}{a+1} \ge \frac{S^2}{a^2+b^2+c^2+d^2+S} = \frac{S^2}{S^2-2S_2+S} = \frac{S^2}{S^2+S-18}.$$

So, it suffices to show that

$$\frac{S}{4} + \frac{S^2}{S^2+S-18} \ge 3,$$

which is equivalent to

$$S^3 - 7S^2 - 30S + 216 \ge 0,$$

$$(6-S)(36+S-S^2) \ge 0.$$

Clearly, this inequality is true for $S \le 6$. Consider now that $S \ge 6$. Since

$$\sum \frac{a}{a+1} - \frac{3}{2} = \frac{S + 2S_2 + 3\sum abc + 4abcd}{1 + S + S_2 + \sum abc + abcd} - \frac{3}{2}$$

$$= \frac{-3 - S + S_2 + 3\sum abc + 5abcd}{1 + S + S_2 + \sum abc + abcd} = \frac{6 - S + 3\sum abc + 5abcd}{10 + S + \sum abc + abcd} \ge \frac{6-S}{10+S},$$

it suffices to show that

$$\frac{S}{4} + \frac{6-S}{10+S} + \frac{3}{2} \ge 3,$$

which is equivalent to the obvious inequality

$$(S-6)(S+6) \ge 0.$$

The equality occurs for $a = b = 3$ and $c = d = 0$ (or any permutation).

□

P 1.200. *If a, b, c, d are nonnegative real numbers such that*

$$ab + ac + ad + bc + bd + cd = 6,$$

then

$$\frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} + \frac{1}{d^2+1} \ge 2.$$

Solution. Assume that

$$a \ge b \ge c \ge d,$$

and denote

$$P = ab \ge 1, \qquad s = \frac{c+d}{2}.$$

Since

$$\frac{1}{a^2+1} + \frac{1}{b^2+1} - \frac{2}{P+1} = \frac{a(b-a)}{(a^2+1)(P+1)} + \frac{b(a-b)}{(b^2+1)(P+1)}$$

$$= \frac{(a-b)^2(ab-1)}{(a^2+1)(b^2+1)(P+1)} \ge 0,$$

it is sufficient to show that

$$\frac{2}{P+1} + \frac{1}{c^2+1} + \frac{1}{d^2+1} \ge 2.$$

From the hypothesis condition

$$P + 2(a+b)s + cd = 6,$$

we get

$$P + 4s\sqrt{P} + cd \le 6.$$

Since the left hand side of the required inequality decreases when each of P, c and d increases, it suffices to consider the hypothesis

$$P + 4s\sqrt{P} + cd = 6.$$

Write the required inequality as follows:

$$\frac{1}{c^2+1} + \frac{1}{d^2+1} \ge \frac{2P}{P+1},$$

$$\frac{2s^2+1-cd}{4s^2+(1-cd)^2} \ge \frac{P}{P+1}, \tag{*}$$

$$\frac{2s^2+4s\sqrt{P}+P-5}{4s^2+(4s\sqrt{P}+P-5)^2} \ge \frac{P}{P+1}.$$

For fixed P, we write this inequality in the form $F(s) \ge 0$, where

$$F(s) = As^2 + Bs + C,$$

with

$$A = 2(1-P) - 16P^2 < 0.$$

From

$$6 = P + 4s\sqrt{P} + cd \le P + 4s\sqrt{P} + s^2$$

and

$$6 = P + 4s\sqrt{P} + cd \geq P + 4s\sqrt{P},$$

we get $s \in [m, M]$, where

$$m = -2\sqrt{P} + \sqrt{3P+6}, \qquad M = \frac{6-P}{4\sqrt{P}}.$$

Since $F(s)$ is a concave function, it suffices to show that $F(s) \geq 0$ for $s = m$ and $s = M$, that is for $cd = s^2$ (when $c = d$) and $cd = 0$ (when d=0).

Case 1: $cd = s^2$. The required inequality (*) becomes

$$\frac{1}{s^2+1} \geq \frac{P}{P+1},$$

$$1 \geq Ps^2.$$

It follows by the AM-GM inequality as follows:

$$6 = P + 4s\sqrt{P} + s^2 \geq 6\sqrt[6]{P(s\sqrt{P})^4 s^2} \geq 6\sqrt[6]{P(s^4P^2)s^2} = 6\sqrt{Ps^2}.$$

Case 2: $cd = 0$. The required inequality (*) becomes

$$\frac{2s^2+1}{4s^2+1} \geq \frac{P}{P+1},$$

$$1 \geq 2(P-1)s^2.$$

From

$$6 = P + 4s\sqrt{P},$$

we get

$$s = \frac{6-P}{4\sqrt{P}},$$

hence

$$1 - 2(P-1)s^2 \geq 1 - \frac{(P-1)(6-P)^2}{8P} = \frac{(2-P)^2(9-P)}{8P} \geq 0.$$

The equality holds for $a = b = c = d = 1$, and also for $a = b = c = \sqrt{2}$ and $d = 0$ (or any cyclic permutation).

Remark 1. The following generalization is valid (*Vasile Cîrtoaje*, 2021):

- *If $a_1, a_2, \dots, a_n$ are nonnegative real numbers such that*

$$\sum_{1 \leq i < j \leq n} a_i a_j = \frac{n(n-1)}{2},$$

then

$$\frac{1}{a_1^2+1} + \frac{1}{a_2^2+1} + \cdots + \frac{1}{a_n^2+1} \geq \frac{n}{2},$$

with equality for $a_1 = a_2 = \cdots + a_n = 1$*, and also for* $a_1 = 0$ *and* $a_2 = a_3 = \cdots = a_n = \sqrt{\frac{n}{n-2}}$ *(or any cyclic permutation).*

We can prove this inequality by the induction method (*Vo Quoc Ba Can*). For $n = 2$, the inequality is an identity. Assume now that the statement holds for $n \geq 2$ nonnegative real numbers a_i and show that it also holds for $n+1$ nonnegative numbers a_i, that is, if

$$\sum_{1\leq i<j\leq n+1} a_i a_j = \frac{n(n+1)}{2},$$

then

$$\sum_{i=1}^{n+1} \frac{1}{a_i^2+1} \geq \frac{n+1}{2}.$$

Without loss of generality, assume that $a_{n+1} = \min\{a_1, a_2, \ldots, a_n\}$. We claim that this assumption implies

$$\sum_{1\leq i<j\leq n} a_i a_j \geq \frac{n(n-1)}{2},$$

hence

$$\sum_{1\leq i<j\leq n} a_i a_j = \frac{n(n-1)t^2}{2}, \qquad t \geq 1.$$

To prove this claim, we denote a_{n+1} by y, and write the desired inequality as follows:

$$(n+1)\sum_{1\leq i<j\leq n} a_i a_j \geq (n-1)\sum_{1\leq i<j\leq n+1} a_i a_j,$$

$$(n+1)\sum_{1\leq i<j\leq n} a_i a_j \geq (n-1)\left(\sum_{1\leq i<j\leq n} a_i a_j + y\sum_{i=1}^{n} a_i\right),$$

$$2\sum_{1\leq i<j\leq n} a_i a_j \geq (n-1)y\sum_{i=1}^{n} a_i.$$

Using the substitutions $a_i = x_i + y$ for $i = 1, 2, \ldots, n$, we have all $x_i \geq 0$ and

$$2\sum_{1\leq i<j\leq n} a_i a_j - (n-1)y\sum_{i=1}^{n} a_i = 2\sum_{1\leq i<j\leq n} (x_i+y)(x_j+y) - (n-1)y\sum_{i=1}^{n}(x_i+y)$$

$$= 2\sum_{1\leq i<j\leq n} x_i x_j + (n-1)y\sum_{i=1}^{n} x_i \geq 0.$$

Next, from the known inequality

$$a_1^2 + a_2^2 + \cdots + a_n^2 \geq \frac{1}{n}(a_1 + a_2 + \cdots + a_n)^2,$$

we get

$$(a_1 + a_2 + \cdots + a_n)^2 - n(n-1)t^2 \ge \frac{1}{n}(a_1 + a_2 + \cdots + a_n)^2,$$

therefore

$$a_1 + a_2 + \cdots + a_n \ge nt.$$

Since

$$\frac{n(n+1)}{2} = \sum_{1\le i<j\le n+1} a_i a_j = \sum_{1\le i<j\le n} a_i a_j + (a_1 + a_2 + \cdots + a_n)a_{n+1}$$
$$\ge \frac{n(n-1)t^2}{2} + nta_{n+1},$$

we obtain

$$a_{n+1} \le \frac{T}{2t}, \qquad T = n + 1 - (n-1)t^2.$$

Let us define the nonnegative real numbers $b_i = \frac{a_i}{t}$ for $i = 1, 2, \ldots, n$. Since

$$\sum_{1\le i<j\le n} b_i b_j = \frac{1}{t^2} \sum_{1\le i<j\le n} a_i a_j = \frac{n(n-1)}{2},$$

by the induction hypothesis we have

$$\sum_{i=1}^{n} \frac{1}{b_i^2 + 1} \ge \frac{n}{2},$$

hence

$$\sum_{i=1}^{n} \frac{1}{a_i^2 + t^2} \ge \frac{n}{2t^2}.$$

By the Cauchy-Schwarz inequality, we have

$$[(a_i^2 + 1) + (t^2 - 1)]\left[\frac{(t^2+1)^2}{a_i^2 + 1} + (t^2 - 1)\right] \ge [(t^2 + 1) + (t^2 - 1)]^2,$$

which is equivalent to

$$\frac{(t^2+1)^2}{a_i^2 + 1} + t^2 - 1 \ge \frac{4t^4}{a_i^2 + t^2}.$$

By summing these inequalities for all $i \le n$, we obtain

$$(t^2+1)^2 \sum_{i=1}^{n} \frac{1}{a_i^2 + 1} + n(t^2 - 1) \ge 4t^2 \sum_{i=1}^{n} \frac{1}{a_i^2 + t^2} \ge 2nt^2,$$

hence

$$\sum_{i=1}^{n} \frac{1}{a_i^2 + 1} \ge \frac{n}{t^2 + 1}.$$

Finally, we have

$$\sum_{i=1}^{n+1} \frac{1}{a_i^2+1} = \sum_{i=1}^{n} \frac{1}{a_i^2+1} + \frac{1}{a_{n+1}^2+1} \ge \frac{n}{t^2+1} + \frac{1}{a_{n+1}^2+1} \ge \frac{n}{t^2+1} + \frac{1}{T^2/(4t^2)+1}$$

$$= \frac{n+1}{2} - \frac{(n^2-1)[(n-1)t^6-(3n-1)t^4+(3n+1)t^2-n-1]}{2(t^2+1)(T^2+4t^2)}$$

$$= \frac{n+1}{2} - \frac{(n^2-1)(t^2-1)^2[(n-1)t^2-n-1]}{2(t^2+1)(T^2+4t^2)}$$

$$= \frac{n+1}{2} + \frac{(n^2-1)(t^2-1)^2 T}{2(t^2+1)(T^2+4t^2)} \ge \frac{n+1}{2}.$$

Remark 2. Another way to prove the inequality in Remark 1 is based on the following lemma (*Vasile Cîrtoaje*):

Lemma. *If $x \ge y \ge 0$ such that*

$$xy + a(x+y) = b,$$

where a and b are positive real constants, then the expression

$$E = \frac{1}{x^2+1} + \frac{1}{y^2+1}$$

has the minimum value for $y = 0$ or $x = y$.

□

P 1.201. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n = n$. Prove that*

$$\sum \frac{1}{(n+1)a_1^2 + a_2^2 + \cdots + a_n^2} \le \frac{1}{2}.$$

(Vasile Cîrtoaje, 2008)

First Solution. By the Cauchy-Schwarz inequality, we have

$$\sum \frac{n^2}{(n+1)a_1^2 + a_2^2 + \cdots + a_n^2} = \sum \frac{(a_1+a_2+\cdots+a_n)^2}{2a_1^2 + (a_1^2+a_2^2) + \cdots + (a_1^2+a_n^2)}$$

$$\le \sum \left(\frac{1}{2} + \frac{a_2^2}{a_1^2+a_2^2} + \cdots + \frac{a_n^2}{a_1^2+a_n^2} \right)$$

$$= \frac{n}{2} + \frac{n(n-1)}{2} = \frac{n^2}{2},$$

from which the conclusion follows. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Second Solution. Write the inequality as

$$\sum \frac{a_1^2+a_2^2+\cdots+a_n^2}{(n+1)a_1^2+a_2^2+\cdots+a_n^2} \le \frac{a_1^2+a_2^2+\cdots+a_n^2}{2}.$$

Since

$$\frac{a_1^2+a_2^2+\cdots+a_n^2}{(n+1)a_1^2+a_2^2+\cdots+a_n^2} = 1 - \frac{na_1^2}{(n+1)a_1^2+a_2^2+\cdots+a_n^2},$$

we need to prove that

$$\sum \frac{a_1^2}{(n+1)a_1^2+a_2^2+\cdots+a_n^2} + \frac{a_1^2+a_2^2+\cdots+a_n^2}{2n} \ge 1.$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\sum \frac{a_1^2}{(n+1)a_1^2+a_2^2+\cdots+a_n^2} &\ge \frac{(a_1+a_2+\cdots+a_n)^2}{\sum[(n+1)a_1^2+a_2^2+\cdots+a_n^2]}\\ &= \frac{n}{2(a_1^2+a_2^2+\cdots+a_n^2)}.\end{aligned}$$

Then, it suffices to prove that

$$\frac{n}{a_1^2+a_2^2+\cdots+a_n^2} + \frac{a_1^2+a_2^2+\cdots+a_n^2}{n} \ge 2,$$

which follows immediately from the AM-GM inequality.

□

P 1.202. *Let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1+a_2+\cdots+a_n = n$. Prove that*

$$\frac{a_1^2}{a_1^2-2a_1+n} + \frac{a_2^2}{a_2^2-2a_2+n} + \cdots + \frac{a_n^2}{a_n^2-2a_n+n} \ge \frac{n}{n-1}.$$

(Vasile Cîrtoaje, 2010)

Solution. Without loss of generality, assume that

$$(a_n-1)^2 = \max\{(a_1-1)^2, (a_2-1)^2, \cdots, (a_n-1)^2\}.$$

Since

$$\frac{n}{n-1} - \frac{a_n^2}{a_n^2-2a_n+n} = \frac{(n-a_n)^2}{(n-1)(a_n^2-2a_n+n)} = \frac{(a_1+a_2+\cdots+a_{n-1})^2}{(n-1)[(a_n-1)^2+n-1]},$$

we can write the inequality as

$$\sum_{i=1}^{n-1} \frac{a_i^2}{a_i^2-2a_i+n} \ge \frac{(a_1+a_2+\cdots+a_{n-1})^2}{(n-1)[(a_n-1)^2+n-1]}.$$

From the Cauchy-Schwarz inequality

$$\sum_{i=1}^{n-1} \frac{a_i^2}{a_i^2 - 2a_i + n} \ge \frac{(a_1 + a_2 + \cdots + a_{n-1})^2}{\sum_{i=1}^{n-1}(a_i^2 - 2a_i + n)} = \frac{(a_1 + a_2 + \cdots + a_{n-1})^2}{\sum_{i=1}^{n-1}(a_i - 1)^2 + (n-1)^2}$$

it suffices to show that

$$\frac{1}{\sum_{i=1}^{n-1}(a_i - 1)^2 + (n-1)^2} \ge \frac{1}{(n-1)[(a_n - 1)^2 + n - 1]},$$

which is equivalent to the obvious inequality

$$\sum_{i=1}^{n-1}[(a_n - 1)^2 - (a_i - 1)^2] \ge 0.$$

The proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = a_2 = \cdots = a_{n-1} = 0$ and $a_n = n$ (or any cyclic permutation).

□

P 1.203. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 a_2 \cdots a_n = 1$. Prove that*

(*a*)
$$\frac{1}{1 + (n-1)a_1} + \frac{1}{1 + (n-1)a_2} + \cdots + \frac{1}{1 + (n-1)a_n} \ge 1;$$

(*b*)
$$\frac{1}{a_1 + n - 1} + \frac{1}{a_2 + n - 1} + \cdots + \frac{1}{a_n + n - 1} \le 1.$$

(Vasile Cîrtoaje, 1991)

Solution. (a) ***First Solution.*** Let $k = (n-1)/n$. We can get the required inequality by summing the inequalities

$$\frac{1}{1 + (n-1)a_i} \ge \frac{a_i^{-k}}{a_1^{-k} + a_2^{-k} + \cdots + a_n^{-k}}$$

for $i = 1, 2, \cdots, n$. The inequality is equivalent to

$$a_1^{-k} + \cdots + a_{i-1}^{-k} + a_{i+1}^{-k} + \cdots + a_n^{-k} \ge (n-1)a_i^{1-k},$$

which follows from the AM-GM inequality. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Second Solution. Replacing all a_i by $1/a_i$, the inequality becomes

$$\frac{a_1}{a_1 + n - 1} + \frac{a_2}{a_2 + n - 1} + \cdots + \frac{a_n}{a_n + n - 1} \ge 1.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a_i}{a_i+n-1} \ge \frac{\left(\sum \sqrt{a_1}\right)^2}{\sum(a_1+n-1)}.$$

Thus, we still have to prove that

$$\left(\sum \sqrt{a_1}\right)^2 \ge \sum a_1 + n(n-1),$$

which is equivalent to

$$\sum_{1\le i<j\le n} 2\sqrt{a_i a_j} \ge n(n-1).$$

Since $a_1 a_2 \cdots a_n = 1$, this inequality follows from the AM-GM inequality.

Third Solution. Use the contradiction method. Assume that

$$\frac{1}{1+(n-1)a_1} + \frac{1}{1+(n-1)a_2} + \cdots + \frac{1}{1+(n-1)a_n} < 1$$

and show that $a_1 a_2 \cdots a_n > 1$ (which contradicts the hypothesis $a_1 a_2 \cdots a_n = 1$). Let

$$x_i = \frac{1}{1+(n-1)a_i}, \qquad 0 < x_i < 1, \qquad i = 1, 2, \cdots, n.$$

Since

$$a_i = \frac{1-x_i}{(n-1)x_i}, \qquad i = 1, 2, \cdots, n,$$

we need to show that

$$x_1 + x_2 + \cdots + x_n < 1$$

implies

$$(1-x_1)(1-x_2)\cdots(1-x_n) > (n-1)^n x_1 x_2 \cdots x_n.$$

Using the AM-GM inequality, we have

$$1 - x_i > \sum_{k\ne i} x_k \ge (n-1)\left(\prod_{k\ne i} x_k\right)^{1/(n-1)}.$$

Multiplying the inequalities

$$1 - x_i > (n-1)\left(\prod_{k\ne i} x_k\right)^{1/(n-1)}, \qquad i = 1, 2, \cdots, n,$$

the conclusion follows.

(b) This inequality follows from the inequality in (a) by replacing all a_i with $1/a_i$. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Remark. The inequalities in P 1.203 are particular cases of the following more general results (*Vasile Cîrtoaje*, 2005):

- *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1a_2\cdots a_n = 1$. If*

$$0 < k \le n-1, \qquad p \ge n^{1/k} - 1,$$

then

$$\frac{1}{(1+pa_1)^k} + \frac{1}{(1+pa_2)^k} + \cdots + \frac{1}{(1+pa_n)^k} \ge \frac{n}{(1+p)^k}.$$

- *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1a_2\cdots a_n = 1$. If*

$$k \ge \frac{1}{n-1}, \qquad 0 < p \le \left(\frac{n}{n-1}\right)^{1/k} - 1,$$

then

$$\frac{1}{(1+pa_1)^k} + \frac{1}{(1+pa_2)^k} + \cdots + \frac{1}{(1+pa_n)^k} \le \frac{n}{(1+p)^k}.$$

□

P 1.204. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1a_2\cdots a_n = 1$. Prove that*

$$\frac{1}{1-a_1+na_1^2} + \frac{1}{1-a_2+na_2^2} + \cdots + \frac{1}{1-a_n+na_n^2} \ge 1.$$

(Vasile Cîrtoaje, 2009)

Solution. First, we show that

$$\frac{1}{1-x+nx^2} \ge \frac{1}{1+(n-1)x^k},$$

where $x > 0$ and $k = 2 + \dfrac{1}{n-1}$. Write the inequality as

$$(n-1)x^k + x \ge nx^2.$$

We can get this inequality using the AM-GM inequality as follows:

$$(n-1)x^k + x \ge n\sqrt[n]{x^{(n-1)k}x} = nx^2.$$

Thus, it suffices to show that

$$\frac{1}{1+(n-1)a_1^k} + \frac{1}{1+(n-1)a_2^k} + \cdots + \frac{1}{1+(n-1)a_n^k} \ge 1,$$

which follows immediately from the inequality (a) in P 1.203. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Remark 1. Similarly, we can prove the following more general statement.

• *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1a_2\cdots a_n = 1$. If p and q are real numbers such that $p+q = n-1$ and $n-1 \le q \le (\sqrt{n}+1)^2$, then*

$$\frac{1}{1+pa_1+qa_1^2} + \frac{1}{1+pa_2+qa_2^2} + \cdots + \frac{1}{1+pa_n+qa_n^2} \ge 1.$$

Remark 2. We can extend the inequality in Remark 1 as follows (*Vasile Cîrtoaje*, 2009).

• *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1a_2\cdots a_n = 1$. If p and q are real numbers such that $p+q = n-1$ and $0 \le q \le (\sqrt{n}+1)^2$, then*

$$\frac{1}{1+pa_1+qa_1^2} + \frac{1}{1+pa_2+qa_2^2} + \cdots + \frac{1}{1+pa_n+qa_n^2} \ge 1.$$

□

P 1.205. *Let $n \ge 3$ and $a_1, a_2, \ldots, a_n \ge \dfrac{2(n-3)}{2n-3}$ such that $a_1a_2\cdots a_n = 1$. Prove that*

$$\frac{1}{a_1+2} + \frac{1}{a_2+2} + \cdots + \frac{1}{a_n+2} \le \frac{n}{3}.$$

(Vasile Cîrtoaje, 2005)

Solution. Let

$$E_n(a_1, a_2, \ldots, a_n) = \frac{1}{a_1+2} + \frac{1}{a_2+2} + \cdots + \frac{1}{a_n+2} - \frac{n}{3}.$$

To show that $E_n(a_1, a_2, \ldots, a_n) \ge 0$, we use the induction method. For $n = 3$, the inequality follows from P 1.203, (b). For $n \ge 4$, assume that the inequality is true for $n-1$ numbers and prove that $E_n(a_1, a_2, \ldots, a_n) \ge 0$ for $a_1a_2\cdots a_n = 1$ and $a_1, a_2, \ldots, a_n \ge p_n$, where

$$p_n = \frac{2(n-3)}{2n-3}.$$

Due to symmetry, we may assume that $a_1 \ge 1$ and $a_2 \le 1$. There are two cases to consider.

Case 1: $a_1a_2 \le 4$. From $a_1a_2 \ge a_2$, $p_{n-1} < p_n$ and $a_1, a_2, \ldots, a_n \ge p_n$, it follows that

$$a_1a_2,\ a_3, \ldots, a_n > p_{n-1}.$$

Then, by the induction hypothesis, we have $E_{n-1}(a_1a_2, a_3, \ldots, a_n) \le 0$; thus, it suffices to show that

$$E_n(a_1, a_2, \ldots, a_n) \le E_{n-1}(a_1a_2, a_3, \ldots, a_n).$$

This is equivalent to
$$\frac{1}{a_1+2}+\frac{1}{a_2+2}-\frac{1}{a_1a_2+2}-\frac{1}{3}\le 0,$$
which reduces to the obvious inequality
$$(a_1-1)(1-a_2)(a_1a_2-4)\le 0.$$

Case 2: $a_1a_2\ge 4$. Since
$$\frac{1}{a_1+2}+\frac{1}{a_2+2}=\frac{a_1+a_2+4}{a_1a_2+2(a_1+a_2)+4}\le\frac{a_1+a_2+4}{4+2(a_1+a_2)+4}=\frac{1}{2}$$
and
$$\frac{1}{a_3+2}+\cdots+\frac{1}{a_n+2}\le\frac{n-2}{p_n+2}=\frac{2n-3}{6},$$
we have
$$E_n(a_1,a_2,\ldots,a_n)\le\frac{1}{2}+\frac{2n-3}{6}-\frac{n}{3}=0.$$
Thus, the proof is completed. The equality holds for $a_1=a_2=\cdots=a_n=1$.

□

P 1.206. *If* $a_1,a_2,\ldots,a_n\ge 0$, *then*
$$\frac{1}{1+na_1}+\frac{1}{1+na_2}+\cdots+\frac{1}{1+na_n}\ge\frac{n}{n+a_1a_2\cdots a_n}.$$

(Vasile Cîrtoaje, 2013)

Solution. If one of $a_1,a_2,\ldots,a_n$ is zero, the inequality is obvious. Consider further $a_1,a_2,\ldots,a_n>0$, and let
$$r=\sqrt[n]{a_1a_2\cdots a_n}.$$
By the Cauchy-Schwarz inequality, we have
$$\sum\frac{1}{1+na_1}\ge\frac{\left(\sum\sqrt{a_2a_3\cdots a_n}\right)^2}{\sum(1+na_1)a_2a_3\cdots a_n}=\frac{\left(\sum\sqrt{a_2a_3\cdots a_n}\right)^2}{\sum a_2a_3\cdots a_n+n^2r^n}.$$
Therefore, it suffices to show that
$$(n+r^n)\left(\sum\sqrt{a_2a_3\cdots a_n}\right)^2\ge n\sum a_2a_3\cdots a_n+n^3r^n.$$
By the AM-GM inequality, we have
$$\left(\sum\sqrt{a_2a_3\cdots a_n}\right)^2\ge\sum a_2a_3\cdots a_n+n(n-1)r^{n-1}.$$

Thus, it is enough to prove that

$$(n+r^n)\left[\sum a_2a_3\cdots a_n + n(n-1)r^{n-1}\right] \ge n\sum a_2a_3\cdots a_n + n^3r^n,$$

which is equivalent to

$$r^n\sum a_2a_3\cdots a_n + n(n-1)r^{2n-1} + n^2(n-1)r^{n-1} \ge n^3r^n.$$

Also, by the AM-GM inequality,

$$\sum a_2a_3\cdots a_n \ge nr^{n-1},$$

and it suffices to show the inequality

$$nr^{2n-1} + n(n-1)r^{2n-1} + n^2(n-1)r^{n-1} \ge n^3r^n,$$

which can be rewritten as

$$n^2r^{n-1}(r^n - nr + n - 1) \ge 0.$$

Indeed, by the AM-GM inequality, we get

$$r^n + n - 1 = r^n + 1 + \cdots + 1 \ge n\sqrt[n]{r^n\cdot 1\cdots 1} = nr.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 1.207. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that*

$$a_1 + a_2 + \cdots + a_n = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n},$$

then

$$\frac{1}{(n-1)a_1+1} + \frac{1}{(n-1)a_2+1} + \cdots + \frac{1}{(n-1)a_n+1} \ge 1.$$

(Vasile Cîrtoaje, AMM, 5, 1996)

Solution. By the contradiction method, it is enough to show that

$$\frac{1}{(n-1)a_1+1} + \frac{1}{(n-1)a_2+1} + \cdots + \frac{1}{(n-1)a_n+1} < 1$$

implies

$$a_1 + a_2 + \cdots + a_n > \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}.$$

This is true if

$$\frac{1}{(n-1)a_1+1} + \frac{1}{(n-1)a_2+1} + \cdots + \frac{1}{(n-1)a_n+1} = 1$$

implies

$$a_1 + a_2 + \cdots + a_n \ge \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}.$$

Use the substitution

$$\frac{1}{(n-1)a_1+1} = \frac{x_i}{S}, \qquad S = x_1 + x_2 + \cdots + x_n, \qquad i = 1, 2, \dots, n,$$

which leads to

$$a_i = \frac{S - x_i}{(n-1)x_i}, \qquad i = 1, 2, \dots, n.$$

So, we need to show that

$$\frac{S-x_1}{x_1} + \frac{S-x_2}{x_2} + \cdots + \frac{S-x_n}{x_n} \ge (n-1)^2 \left(\frac{x_1}{S-x_1} + \frac{x_2}{S-x_2} + \cdots + \frac{x_n}{S-x_n} \right),$$

which is equivalent to

$$\frac{x_2 + x_3 + \cdots + x_n}{x_1} + \frac{x_3 + \cdots + x_n + x_1}{x_2} + \cdots + \frac{x_1 + x_2 + \cdots + x_{n-1}}{x_n} \ge$$

$$\ge (n-1)^2 \left(\frac{x_1}{x_2 + x_3 \cdots + x_n} + \frac{x_2}{x_3 + \cdots + x_n + x_1} + \cdots + \frac{x_n}{x_1 + x_2 + \cdots + x_{n-1}} \right),$$

or

$$x_1 A_1 + x_2 A_2 + \cdots x_n A_n \ge 0,$$

where

$$A_1 = \frac{1}{x_2} + \cdots + \frac{1}{x_n} - \frac{(n-1)^2}{x_2 + \cdots + x_n}\ , \ \dots \ , \ A_n = \frac{1}{x_1} + \cdots + \frac{1}{x_{n-1}} - \frac{(n-1)^2}{x_1 + \cdots + x_{n-1}}.$$

The last inequality is true because $A_i \ge 0$ for $i = 1, 2, \dots, n$ (by the AM-HM inequality). The equality occurs for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 1.208. *If $a_1, a_2, \dots, a_n$ are nonnegative real numbers such that*

$$\sum_{1 \le i < j \le n} a_i a_j = \frac{n(n-1)}{2}\ ,$$

then

$$\frac{1}{a_1 + n - 1} + \frac{1}{a_2 + n - 1} + \cdots + \frac{1}{a_n + n - 1} \ge 1.$$

(Vasile Cîrtoaje and Vo Quoc Ba Can, Crux Mathematicorum, 3, 2025)

Solution. By the contradiction method, it is enough to show that

$$\frac{1}{a_1+n-1}+\frac{1}{a_2+n-1}+\cdots+\frac{1}{a_n+n-1}<1$$

implies

$$\sum_{1\le i<j\le n} a_i a_j > \frac{n(n-1)}{2}.$$

Write the hypothesis as

$$\frac{a_1}{a_1+n-1}+\frac{a_2}{a_2+n-1}+\cdots+\frac{a_n}{a_n+n-1}>1,$$

and use the substitution

$$b_i=\frac{a_i}{a_i+n-1}\in[0,1), \qquad i=1,2,\ldots,n.$$

Since $a_i=\dfrac{(n-1)b_i}{1-b_i}$, we need to show that $\sum_{i=1}^n b_i>1$ implies

$$\sum_{1\le i<j\le n}\frac{b_ib_j}{(1-b_i)(1-b_j)}>\frac{n}{2(n-1)}.$$

By the Cauchy-Schwarz inequality, we have

$$\left[\sum_{1\le i<j\le n} b_ib_j(1-b_i)(1-b_j)\right]\left[\sum_{1\le i<j\le n}\frac{b_ib_j}{(1-b_i)(1-b_j)}\right]\ge\left(\sum_{1\le i<j\le n} b_ib_j\right)^2.$$

Thus, it suffices to show that

$$\left(\sum_{1\le i<j\le n} b_ib_j\right)^2>\frac{n}{2(n-1)}\sum_{1\le i<j\le n} b_ib_j(1-b_i)(1-b_j).$$

Indeed, using the substitution $x_i=b_i(1-b_i)$ for $i=1,2,\ldots,n$, as well as the well-known inequality

$$n\sum_{i=1}^n x_i^2\ge\left(\sum_{i=1}^n x_i\right)^2,$$

we have

$$2\sum_{1\le i<j\le n} b_ib_j(1-b_i)(1-b_j)=2\sum_{1\le i<j\le n} x_ix_j=\left(\sum_{i=1}^n x_i\right)^2-\sum_{i=1}^n x_i^2$$

$$\le\left(\sum_{i=1}^n x_i\right)^2-\frac{1}{n}\left(\sum_{i=1}^n x_i\right)^2=\frac{n-1}{n}\left(\sum_{i=1}^n x_i\right)^2=\frac{n-1}{n}\left(\sum_{i=1}^n b_i-\sum_{i=1}^n b_i^2\right)^2$$

$$< \frac{n-1}{n}\left[\left(\sum_{i=1}^n b_i\right)^2 - \sum_{i=1}^n b_i^2\right]^2 = \frac{4(n-1)}{n}\left(\sum_{1\le i<j\le n} b_ib_j\right)^2.$$

The proof is completed. For $n \ge 3$, the equality occurs when $a_1 = a_2 = \cdots = a_n = 1$.

Remark. Actually, $n-1$ is the largest positive value of the constant k such that the inequality

$$\frac{1}{a_1+k} + \frac{1}{a_2+k} + \cdots + \frac{1}{a_n+k} \ge \frac{n}{1+k}.$$

holds for any nonnegative real numbers $a_1, a_2, \ldots, a_n$ with $\sum_{1\le i<j\le n} a_ia_j = \dfrac{n(n-1)}{2}$.

For $a_2 = \cdots = a_n := x \le 1$, from the constraint we get

$$a_1 = \frac{n-(n-2)x^2}{2x}, \qquad x \in (0,1],$$

while the inequality becomes $F(x) \ge 0$, where

$$F(x) = \frac{2x}{n+2kx-(n-2)x^2} + \frac{n-1}{x+k} - \frac{n}{1+k}.$$

From the necessary condition $\lim_{x\to 0} F(x) \ge 0$, we obtain $k \le n-1$.

□

P 1.209. *Let a, b, c be nonnegative real numbers such that*

$$a \ge b \ge 1 \ge c, \qquad a+b+c = 3.$$

Prove that

$$\frac{1}{a^2+3} + \frac{1}{b^2+3} + \frac{1}{c^2+3} \le \frac{3}{4}.$$

(Vasile Cîrtoaje, 2005)

First Solution. Let

$$r = abc, \qquad q = ab+bc+ca.$$

From

$$(a-1)(b-1)(c-1) \le 0,$$

we get

$$r \le q-2.$$

The desired inequality is equivalent to

$$3a^2b^2c^2 + 5(a^2b^2+b^2c^2+c^2a^2) + 3(a^2+b^2+c^2) - 27 \ge 0,$$

$$3r^2 - 30r + 5q^2 - 6q \ge 0,$$

$$3(5-r)^2+5q^2-6q-75 \geq 0.$$

Since

$$3q \leq (a+b+c)^2 = 9$$

and

$$5-r \geq 5-(q-2) = 7-q > 0,$$

it suffices to show that

$$3(7-q)^2+5q^2-6q-75 \geq 0.$$

This is equivalent to the obvious inequality

$$(q-3)^2 \geq 0.$$

The proof is completed. The equality holds for $a=b=c=1$.

Second Solution (by *Nguyen Van Quy*). Write the inequality as follows:

$$\left(\frac{1}{a^2+3}-\frac{3-a}{8}\right)+\left(\frac{1}{b^2+3}-\frac{3-b}{8}\right)+\left(\frac{1}{c^2+3}-\frac{3-c}{8}\right) \leq 0,$$

$$\frac{(a-1)^3}{a^2+3}+\frac{(b-1)^3}{b^2+3} \leq \frac{(1-c)^3}{c^2+3}.$$

Indeed, we have

$$\frac{(1-c)^3}{c^2+3} = \frac{(a-1+b-1)^3}{c^2+3} \geq \frac{(a-1)^3+(b-1)^3}{c^2+3} \geq \frac{(a-1)^3}{a^2+3}+\frac{(b-1)^3}{b^2+3}.$$

Third Solution. Denoting

$$d = 2-c,$$

we have

$$a+b = 1+d, \quad d \geq a \geq b \geq 1.$$

We claim that

$$\frac{1}{c^2+3}+\frac{1}{d^2+3} \leq \frac{1}{2}.$$

Indeed,

$$\frac{1}{2}-\frac{1}{c^2+3}-\frac{1}{d^2+3} = \frac{(cd-1)^2}{2(c^2+3)(d^2+3)} \geq 0.$$

Thus, it suffices to show that

$$\frac{1}{a^2+3}+\frac{1}{b^2+3} \leq \frac{1}{d^2+3}+\frac{1}{4}.$$

Since

$$\frac{1}{a^2+3}-\frac{1}{d^2+3} = \frac{(d-a)(d+a)}{(a^2+3)(d^2+3)} = \frac{(b-1)(d+a)}{(a^2+3)(d^2+3)},$$

$$\frac{1}{4}-\frac{1}{b^2+3}=\frac{(b-1)(b+1)}{4(b^2+3)},$$

we need to prove that

$$\frac{d+a}{(a^2+3)(d^2+3)}\le\frac{b+1}{4(b^2+3)}.$$

We can get this inequality by multiplying the inequalities

$$\frac{d+a}{d^2+3}\le\frac{a+1}{4},$$

$$\frac{a+1}{a^2+3}\le\frac{b+1}{b^2+3}.$$

We have

$$\frac{a+1}{4}-\frac{d+a}{d^2+3}=\frac{(d-1)(ad+a+d-3)}{4(d^2+3)}\ge 0,$$

$$\frac{b+1}{b^2+3}-\frac{a+1}{a^2+3}=\frac{(a-b)(ab+a+b-3)}{(a^2+3)(b^2+3)}\ge 0.$$

□

P 1.210. *Let a,b,c be nonnegative real numbers such that*

$$a\ge 1\ge b\ge c,\qquad a+b+c=3.$$

Prove that

$$\frac{1}{a^2+2}+\frac{1}{b^2+2}+\frac{1}{c^2+2}\ge 1.$$

(Vasile Cîrtoaje, 2005)

First Solution. Let

$$r=abc,\qquad q=ab+bc+ca.$$

From

$$(a-1)(b-1)(c-1)\ge 0,$$

we get

$$r\ge q-2.$$

Also, we have

$$r\le\frac{(a+b+c)^3}{27}=1.$$

$$q\le\frac{1}{3}(a+b+c)^3=3.$$

The desired inequality is equivalent to

$$3\ge a^2b^2c^2+a^2b^2+b^2c^2+c^2a^2,$$

$$4 \ge r^2 - 6r + q^2,$$
$$(3-r)^2 + q^2 \le 13.$$

Consider further two cases: $q \le 2$ and $2 \le q \le 3$.

Case 1: $q \le 2$. We have
$$(3-r)^2 + q^2 \le 3^2 + 2^2 = 13.$$

Case 2: $2 \le q \le 3$. From $r \le q - 2$, we get
$$(3-r)^2 + q^2 \le (5-q)^2 + q^2 = 2(q-3)(q-2) \le 0.$$

The proof is completed. The equality holds for $a = b = c = 1$, as well as for $a = 2$, $b = 1$ and $c = 0$.

Second Solution. First, we can check that the desired inequality becomes an equality for $a = b = c = 1$, and also for $a = 2$, $b = 1$, $c = 0$. Consider then the inequality $f(x) \ge 0$, where
$$f(x) = \frac{1}{x^2+2} - A - Bx.$$
We have the derivative
$$f'(x) = \frac{-2x}{(x^2+2)^2} - B.$$
From the conditions $f(1) = 0$ and $f'(1) = 0$, we get $A = 5/9$ and $B = -2/9$. Also, from the conditions $f(2) = 0$ and $f'(2) = 0$, we get $A = 7/18$ and $B = -1/9$. Using these values of A and B, we obtain the relations
$$\frac{1}{x^2+2} - \frac{5-2x}{9} = \frac{(x-1)^2(2x-1)}{9(x^2+2)},$$
$$\frac{1}{x^2+2} - \frac{7-2x}{18} = \frac{(x-2)^2(2x+1)}{18(x^2+2)},$$
which involve
$$\frac{1}{x^2+2} \ge \frac{5-2x}{9}, \qquad x \ge \frac{1}{2},$$
$$\frac{1}{x^2+2} \ge \frac{7-2x}{18}, \qquad x \ge 0.$$

Consider further two cases: $c \ge 1/2$ and $c \le 1/2$.

Case 1: $c \ge \frac{1}{2}$. By summing the inequalities
$$\frac{1}{a^2+2} \ge \frac{5-2a}{9}, \quad \frac{1}{b^2+2} \ge \frac{5-2b}{9}, \quad \frac{1}{c^2+2} \ge \frac{5-2c}{9},$$
we get
$$\frac{1}{a^2+2} + \frac{1}{b^2+2} + \frac{1}{c^2+2} \ge \frac{15 - 2(a+b+c)}{9} = 1.$$

Case 2: $c \le \frac{1}{2}$. We have

$$\frac{1}{a^2+2} \ge \frac{7-2a}{18}.$$

Consider now the similar inequalities

$$\frac{1}{b^2+2} \ge \frac{B-2b}{18},$$

$$\frac{1}{c^2+2} \ge \frac{C-2c}{18},$$

which are satisfied as equalities for $b=1$ and $c=0$ if $B=8$ and $C=9$:

$$\frac{1}{b^2+2} \ge \frac{8-2b}{18},$$

$$\frac{1}{c^2+2} \ge \frac{9-2c}{18}$$

Since

$$\frac{1}{b^2+2} - \frac{8-2b}{18} = \frac{(1-b)(1+3b-b^2)}{9(b^2+2)}$$

and

$$\frac{1}{c^2+2} - \frac{9-2c}{18} = \frac{c(1-2c)(4-c)}{18(c^2+2)},$$

these inequalities holds for $0 \le b \le 1$ and $0 \le c \le 1/2$. Therefore, we have

$$\frac{1}{a^2+2} + \frac{1}{b^2+2} + \frac{1}{c^2+2} \ge \frac{7-2a}{18} + \frac{8-2b}{18} + \frac{9-2c}{18} = 1.$$

□

P 1.211. *If $a \ge 1 \ge b \ge c > -3$ such that $ab+bc+ca=3$, then*

$$\frac{1}{a^2+ab+b^2} + \frac{1}{b^2+bc+c^2} + \frac{1}{c^2+ca+a^2} \ge 1.$$

(Vasile Cîrtoaje, 2015)

Solution. We will show first that $p > 0$, where $p = a+b+c$. We have

$$p \ge 1+c+c = 1+2c,$$

hence

$$p - c \ge c+1.$$

On the other hand, from

$$(a-1)(b-1) \le 0,$$

we find

$$ab - (a + b) + 1 \le 0,$$
$$3 - c(a + b) - (a + b) + 1 \le 0,$$
$$4 \le (c + 1)(a + b),$$
$$4 \le (c + 1)(p - c),$$

hence

$$p(c + 1) \ge c^2 + c + 4 > 0.$$

From $p(c + 1) > 0$, it follows that $c > -1$ involves $p > 0$. To show that $c > -1$, we use the contradiction method. The case $c = -1$ contradicts the inequality $(c + 1)(p - c) \ge 4$, and the case $c < -1$ leads to

$$p - c \le \frac{4}{c + 1},$$
$$c + 1 \le \frac{4}{c + 1},$$
$$(c + 1)^2 \ge 4,$$

hence $c \le -3$, which is false. Therefore, we have $p > 0$. According Lemma below, we can write the inequality as

$$p^3abc - 27 + (p^2 - 9)^2 \ge 0.$$

From $(a - 1)(b - 1)(c - 1) \ge 0$, we get

$$abc \ge 4 - p.$$

Thus,

$$p^3abc - 27 + (p^2 - 9)^2 \ge p^3(4 - p) - 27 + (p^2 - 9)^2 = 2(2p + 3)(p - 3)^2 \ge 0.$$

The equality holds for $a = b = c = 1$.

Lemma. *Let a, b, c be real numbers, $p = a + b + c$ and $q = ab + bc + ca$. If $q > 0$, then the inequality*

$$\frac{1}{a^2 + ab + b^2} + \frac{1}{b^2 + bc + c^2} + \frac{1}{c^2 + ca + a^2} \ge \frac{3}{ab + bc + ca}$$

is equivalent to

$$3(p^3abc - q^3) + q(p^2 - 3q)^2 \ge 0.$$

Proof. Write the inequality as

$$q\sum(x + ab - c^2)(x + ac - b^2) \ge 3\prod(x + bc - a^2),$$

where

$$x = a^2 + b^2 + c^2 = p^2 - 2q.$$

From

$$\sum(ab-c^2)(ac-b^2) = q^2 - xq,$$

$$\sum(x+ab-c^2)(x+ac-b^2) = x^2 + xq + q^2$$

and

$$\prod(bc-a^2) = q^3 - p^3abc,$$

$$\prod(x+bc-a^2) = xq^2 + q^3 - p^3abc,$$

the conclusion follows.

□

P 1.212. *If $a \ge b \ge 1 \ge c \ge 0$ such that $a+b+c=3$, then*

$$\frac{1}{a^2+ab+b^2} + \frac{1}{b^2+bc+c^2} + \frac{1}{c^2+ca+a^2} \le \frac{3}{ab+bc+ca}.$$

(Vasile Cîrtoaje, 2015)

Solution. By Lemma from the previous P 1.211, we need to show that

$$3(p^3abc - q^3) + q(p^2-3q)^2 \le 0,$$

where $p=3$ and $q=ab+bc+ca$; that is

$$27abc - q^3 + 3q(3-q)^2 \le 0.$$

From $p^2 \ge 3q$, we get $q \le 3$, and from $(a-1)(b-1)(c-1) \le 0$, we get

$$abc \le q-2, \quad q \ge 2.$$

Thus,

$$27abc - q^3 + 3q(3-q)^2 \le 27(q-2) - q^3 + 3q(3-q)^2 = 2(q-3)^3 \le 0.$$

Thus, the proof is completed. The equality holds for $a=b=c=1$.

Remark. Actually, the inequality holds for

$$a \ge b \ge 1 \ge c \ge 1-\sqrt{3}.$$

To prove this, it suffices to show that $ab+bc+ca \ge 0$. Indeed, we have

$$\begin{aligned} ab+bc+ca &= (a-1)(b-1) - 1 + a + b + c(a+b) \ge -1 + (1+c)(a+b) \\ &= -1 + (1+c)(3-c) \ge 0. \end{aligned}$$

□

P 1.213. *If a, b, c are positive real numbers such that*

$$a \geq 1 \geq b \geq c, \qquad abc = 1,$$

then

$$\frac{1-a}{3+a^2} + \frac{1-b}{3+b^2} + \frac{1-c}{3+c^2} \geq 0.$$

(Vasile Cîrtoaje, 2009)

First Solution. Denote the left side of the inequality by $E(a, b, c)$. We will show that

$$E(a, b, c) \geq E(ab, 1, c) \geq 0.$$

Let

$$a + b = s, \quad ab = p.$$

We have

$$p \geq abc = 1, \quad s \geq 2\sqrt{p} \geq 2.$$

Therefore,

$$\begin{aligned} E(a, b, c) - E(ab, 1, c) &= \frac{1-a}{3+a^2} + \frac{1-b}{3+b^2} + \frac{ab-1}{3+a^2b^2} \\ &= \frac{s^2 - (3+p)s + 2(3-p)}{3s^2 + (p-3)^2} + \frac{p-1}{3+p^2} \\ &= \frac{(3+p)(s-p-1)(ps+p-3)}{(3+p^2)[3s^2 + (p-3)^2]}. \end{aligned}$$

Since

$$s - p - 1 = (a-1)(1-b) \geq 0, \quad ps + p - 3 \geq 2p + p - 3 \geq 0,$$

it follows that

$$E(a, b, c) - E(ab, 1, c) \geq 0.$$

Also, we have

$$E(ab, 1, c) = E(1/c, 1, c) = \frac{(1-c)^4}{(3c^2+1)(3+c^2)} \geq 0.$$

The equality holds for $a = b = c = 1$.

Second Solution. Let $p = a + b + c$ and $q = ab + bc + ca$. From

$$(a-1)(b-1)(c-1) \geq 0,$$

we get

$$p \geq q.$$

The desired inequality is true because it is equivalent to

$$\sum (1-a)(9 + 3b^2 + 3c^2 + b^2c^2) \geq 0,$$

$$27+6\sum a^2+\sum b^2c^2-9p-3pq+9-q\ge 0,$$
$$27+6(p^2-2q)+(q^2-2p)-9p-3pq+9-q\ge 0,$$
$$6p^2+q^2-3pq-11p-13q+36\ge 0,$$
$$(p+q-6)^2+5p^2-5pq+p-q\ge 0,$$
$$(p+q-6)^2+(5p+1)(p-q)\ge 0.$$

□

P 1.214. *If a,b,c are positive real numbers such that*

$$a\ge 1\ge b\ge c,\qquad abc=1,$$

then

$$\frac{1}{a^2+4ab+b^2}+\frac{1}{b^2+4bc+c^2}+\frac{1}{c^2+4ca+a^2}\ge\frac{1}{2}.$$

(Vasile Cîrtoaje, 2015)

Solution. Write the inequality as

$$2E\ge F,$$

where

$$E=\sum(a^2+4ab+b^2)(a^2+4ac+c^2),\qquad F=\prod(b^2+4bc+c^2).$$

Using Lemma below for $k=4$ and $r=1$, we get

$$E=18pr+p^4-3q^2=18p+p^4-3q^2,$$

$$F=27r^2+2p^3r+p^2q^2+2q^3=27+2p^3+p^2q^2+2q^3,$$

hence

$$2E-F=2p^4-2p^3+36p-27-(p^2+6)q^2-2q^3.$$

From $(a-1)(b-1)(c-1)\ge 0$, we get

$$p\ge q.$$

Thus,

$$\begin{aligned}2E-F&\ge 2p^4-2p^3+36p-27-(p^2+6)p^2-2p^3\\&=p^4-4p^3-6p^2+36p-27=(p-1)(p-3)^2(p+3)\ge 0.\end{aligned}$$

Thus, the proof is completed. The equality holds for $a=b=c=1$.

Lemma. *If a,b,c are real numbers,*

$$p=a+b+c,\qquad q=ab+bc+ca,\qquad r=abc$$

and

$$E = \sum (a^2 + kab + b^2)(a^2 + kac + c^2), \qquad F = \prod (b^2 + kbc + c^2),$$

then

$$E = (k-1)(k+2)pr + p^4 + (k-4)p^2q + (5-2k)q^2,$$

$$F = (k-1)^3r^2 + [(k-2)p^2 + (k-1)(k-4)q]pr + p^2q^2 + (k-2)q^3.$$

Proof. Let

$$x = a^2 + b^2 + c^2 = p^2 - 2q.$$

Since

$$E = \sum (x + kab - c^2)(x + kac - b^2)$$
$$= x^2 + kxq + (k-1)(k+2)pr + q^2$$

and

$$F = \prod (x + kbc - a^2)$$
$$= x[(k-1)(k+2)pr + q^2] + (k-1)^3r^2 - k[kp^2 - 3(k-1)q]pr + kq^3,$$

the conclusion follows.

□

P 1.215. *If $a_1, a_2, \ldots, a_n$ are real number such that at most one of them is less than 1 and $a_1 + a_2 + \cdots + a_n = n$, then*

(a)
$$\frac{a_1+1}{a_1^2+1} + \frac{a_2+1}{a_2^2+1} + \cdots + \frac{a_n+1}{a_n^2+1} \le n;$$

(b)
$$\frac{1}{a_1^2+3} + \frac{1}{a_2^2+3} + \cdots + \frac{1}{a_1^2+3} \le \frac{n}{4}.$$

(Vasile Cîrtoaje, 2009)

Solution. Assume that $a_1 \le 1 \le a_2 \le \cdots \le a_n$.

(a) Write the inequality as

$$\left(1 - \frac{a_1+1}{a_1^2+1}\right) + \left(1 - \frac{a_2+1}{a_2^2+1}\right) + \cdots + \left(1 - \frac{a_n+1}{a_n^2+1}\right) \ge 0,$$

$$\frac{a_1(a_1-1)}{a_1^2+1} + \frac{a_2(a_2-1)}{a_2^2+1} + \cdots + \frac{a_n(a_n-1)}{a_n^2+1} \ge 0,$$

$$\frac{a_2(a_2-1)}{a_2^2+1} + \cdots + \frac{a_n(a_n-1)}{a_n^2+1} \ge \frac{a_1(1-a_1)}{a_1^2+1},$$

$$\frac{a_2(a_2-1)}{a_2^2+1}+\cdots+\frac{a_n(a_n-1)}{a_n^2+1}\ge\frac{a_1[(a_2-1)+\cdots+(a_n-1)]}{a_1^2+1},$$

$$(a_2-1)\left(\frac{a_2}{a_2^2+1}-\frac{a_1}{a_1^2+1}\right)+\cdots+(a_n-1)\left(\frac{a_n}{a_n^2+1}-\frac{a_1}{a_1^2+1}\right)\ge 0,$$

$$\frac{(a_2-1)(a_2-a_1)(1-a_1a_2)}{a_2^2+1}+\cdots+\frac{(a_n-1)(a_n-a_1)(1-a_1a_n)}{a_n^2+1}\ge 0.$$

For $a_1\ge 0$, it suffices to show that $1-a_1a_n\ge 0$. Indeed,

$$2\sqrt{a_1a_n}\le a_1+a_n=2+(1-a_2)+\cdots+(1-a_{n-1})\le 2.$$

For $a_1\le 0$, the inequality is also true because

$$1-a_1a_2>0,\quad \cdots,\quad 1-a_1a_n>0.$$

The equality holds for $a_1=a_2=\cdots=a_n=1$.

(b) As in the case (a), we write the inequality as

$$\frac{(a_2-1)(a_2-a_1)(3-a_1a_2-a_1-a_2)}{a_2^2+3}+\cdots+\frac{(a_n-1)(a_n-a_1)(3-a_1a_n-a_1-a_n)}{a_2^2+3}\ge 0.$$

For $a_1\ge 0$, it suffices to show that $3-a_1a_n-a_1-a_n\ge 0$. From $(1-a_1)(a_n-1)\ge 0$, we get $3-a_1a_n\ge 4-a_1-a_n$, hence

$$\frac{1}{2}(3-a_1a_n-a_1-a_n)\ge 2-a_1-a_n=(a_2-1)+\cdots+(a_{n-1}-1)\ge 0.$$

For $a_1\le 0$, the inequality is also true because

$$3-a_1a_2-a_1-a_2>2-a_1-a_2=(a_3-1)+\cdots+(a_n-1)\ge 0,$$

$$\cdots$$

$$3-a_1a_n-a_1-a_n>2-a_1-a_n=(a_2-1)+\cdots+(a_{n-1}-1)\ge 0.$$

The equality holds for $a_1=a_2=\cdots=a_n=1$.

□

P 1.216. *If $a_1,a_2,\ldots,a_n$ are nonnegative real numbers such that at most one of them is less than 1 and $a_1+a_2+\cdots+a_n=n$, then*

$$\frac{a_1^2-1}{(a_1+3)^2}+\frac{a_2^2-1}{(a_2+3)^2}+\cdots+\frac{a_n^2-1}{(a_n+3)^2}\ge 0.$$

(Vasile Cîrtoaje, 2009)

Solution. Assume that $a_1 \le 1 \le a_2 \le \cdots \le a_n$, and write the inequality as follows:

$$\frac{a_2^2-1}{(a_2+3)^2}+\cdots+\frac{a_n^2-1}{(a_n+3)^2} \ge \frac{1-a_1^2}{(a_1+3)^2},$$

$$\frac{a_2^2-1}{(a_2+3)^2}+\cdots+\frac{a_n^2-1}{(a_n+3)^2} \ge \frac{[(a_2-1)+\cdots+(a_n-1)](1+a_1)}{(a_1+3)^2},$$

$$(a_2-1)\left[\frac{a_2+1}{(a_2+3)^2}-\frac{a_1+1}{(a_1+3)^2}\right]+\cdots+(a_n-1)\left[\frac{a_n+1}{(a_n+3)^2}-\frac{a_1+1}{(a_1+3)^2}\right] \ge 0,$$

$$\frac{(a_2-1)(a_2-a_1)(3-a_1-a_2-a_1a_2)}{(a_1+3)^2(a_2+3)^2}+\cdots+\frac{(a_n-1)(a_n-a_1)(3-a_1-a_n-a_1a_n)}{(a_1+3)^2(a_n+3)^2} \ge 0.$$

It suffices to show that $3-a_1-a_n-a_1a_n \ge 0$. Since

$$3-a_1-a_n-a_1a_n \ge 3-a_1-a_n-\frac{1}{4}(a_1+a_n)^2=\frac{1}{4}(2-a_1-a_n)(6+a_1+a_n) \ge 0,$$

we only need to show that $2-a_1-a_n \ge 0$. Indeed, we have

$$2-a_1-a_n=(a_2-1)+\cdots+(a_{n-1}-1) \ge 0.$$

The equality holds for $a_1=a_2=\cdots=a_n=1$.

□

P 1.217. *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that at most one of them is larger than 1 and $a_1+a_2+\cdots+a_n=n$, then*

$$\frac{1}{3a_1^3+4}+\frac{1}{3a_2^3+4}+\cdots+\frac{1}{3a_n^3+4} \ge \frac{n}{7}.$$

(Vasile Cîrtoaje, 2009)

Solution. Assume that $a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n$ and write the inequality as follows:

$$\left(\frac{1}{3a_2^3+4}-\frac{1}{7}\right)+\cdots+\left(\frac{1}{3a_n^3+4}-\frac{1}{7}\right) \ge \frac{1}{7}-\frac{1}{3a_1^3+4},$$

$$\frac{1-a_2^3}{3a_2^3+4}+\cdots+\frac{1-a_n^3}{3a_n^3+4} \ge \frac{a_1^3-1}{3a_1^3+4},$$

$$\frac{1-a_2^3}{3a_2^3+4}+\cdots+\frac{1-a_n^3}{3a_n^3+4} \ge \frac{[(1-a_2)+\cdots+(1-a_n)](1+a_1+a_1^2)}{3a_1^3+4},$$

$$(1-a_2)\left(\frac{1+a_2+a_2^2}{3a_2^3+4}-\frac{1+a_1+a_1^2}{3a_1^3+4}\right)+\cdots+(1-a_n)\left(\frac{1+a_n+a_n^2}{3a_n^3+4}-\frac{1+a_1+a_1^2}{3a_1^3+4}\right) \ge 0.$$

It suffices to show that

$$\frac{1+a_i+a_i^2}{3a_i^3+4}-\frac{1+a_1+a_1^2}{3a_1^3+4} \ge 0$$

for $i = 2, \ldots, n$. Write these inequalities as

$$(a_1 - a_i)E_i \ge 0,$$

where

$$\begin{aligned} E_i &= 3a_1^2a_i^2 + 3a_1a_i(a_1 + a_i) + 3(a_1^2 + a_1a_i + a_i^2) - 4(a_1 + a_i) - 4 \\ &= (a_1 + a_i)(3a_1 + 3a_i - 4 + 3a_1a_i) + 3a_1^2a_i^2 - 3a_1a_i - 4. \end{aligned}$$

Since

$$a_1 + a_i \ge a_1 + a_n = 2 + (1 - a_2) + \cdots + (1 - a_{n-1}) \ge 2,$$

we have

$$E_i \ge 2(6 - 4 + 3a_1a_i) + 3a_1^2a_i^2 - 3a_1a_i - 4 = 3a_1a_i + 3a_1^2a_i^2 \ge 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = 2$, $a_2 = \cdots = a_{n-1} = 1$, $a_n = 0$.

□

P 1.218. *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that at most one of them is less than 1 and $a_1^2 + a_2^2 + \cdots + a_n^2 = n$, then*

$$\frac{1}{3 - a_1} + \frac{1}{3 - a_2} + \cdots + \frac{1}{3 - a_n} \le \frac{n}{2}.$$

(Vasile Cîrtoaje, 2009)

Solution. Assume that $a_1 \le 1 \le a_2 \le \cdots \le a_n$ and write the inequality as follows:

$$\left(\frac{2}{3 - a_1} - 1\right) + \left(\frac{2}{3 - a_1} - 1\right) + \cdots + \left(\frac{2}{3 - a_1} - 1\right) \le 0.$$

$$\frac{a_1 - 1}{3 - a_1} + \frac{a_2 - 1}{3 - a_2} + \cdots + \frac{a_n - 1}{3 - a_n} \le 0,$$

$$\frac{a_2 - 1}{3 - a_2} + \cdots + \frac{a_n - 1}{3 - a_n} \le \frac{1 - a_1}{3 - a_1},$$

$$\frac{a_2^2 - 1}{(1 + a_2)(3 - a_2)} + \cdots + \frac{a_n^2 - 1}{(1 + a_n)(3 - a_n)} \le \frac{1 - a_1^2}{(1 + a_1)(3 - a_1)},$$

$$\frac{a_2^2 - 1}{(1 + a_2)(3 - a_2)} + \cdots + \frac{a_n^2 - 1}{(1 + a_n)(3 - a_n)} \le \frac{(a_2^2 - 1) + \cdots + (a_n^2 - 1)}{(1 + a_1)(3 - a_1)},$$

$$(a_2^2 - 1)E_2 + \cdots + (a_n^2 - 1)E_n \le 0,$$

where

$$E_j = \frac{1}{(1 + a_j)(3 - a_j)} - \frac{1}{(1 + a_1)(3 - a_1)}, \qquad j = 2, \ldots, n.$$

It suffices to show that $E_j \le 0$, which is equivalent to

$$(a_j - a_1)(a_1 + a_j - 2) \le 0.$$

This is true because

$$a_1 + a_i - 2 \le a_1 + a_n - 2 = (1 - a_2) + \cdots + (1 - a_{n-1}) \le 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 1.219. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that at most one of them is larger than 1 and $a_1a_2\cdots a_n = 1$, then*

$$\frac{1}{1+a_1} + \frac{1}{1+a_2} + \cdots + \frac{1}{1+a_n} \ge \frac{n}{2}.$$

(Vasile Cîrtoaje, 2009)

Solution. For $n = 2$, the inequality is an identity. Assume now $n \ge 3$.

First Solution. Assume $a_n \ge 1$ and $a_1, \ldots, a_{n-1} \le 1$, and write the inequality as

$$E(a_1, a_2, \ldots, a_{n-1}) \ge 0,$$

where

$$E(a_1, a_2, \ldots, a_{n-1}) = \frac{1}{1+a_1} + \frac{1}{1+a_2} + \cdots + \frac{1}{1+a_{n-1}} - \frac{1}{1+a_1a_2\cdots a_{n-1}} - \frac{n-2}{2}.$$

For fixed $a_2, \ldots, a_{n-1}$, we will show that $E(a_1, a_2, \ldots, a_{n-1})$ has the minimum value for $a_1 = 1$. Indeed, denoting $x = a_2 \cdots a_{n-1}$, we have $x \le 1$ and

$$E(a_1, a_2, \ldots, a_{n-1}) - E(1, a_2, \ldots, a_{n-1}) = \frac{1}{1+a_1} - \frac{1}{2} - \left(\frac{1}{1+a_1x} - \frac{1}{1+x}\right)$$

$$= \frac{1-a_1}{2(1+a_1)} - \frac{x(1-a_1)}{(1+x)(1+a_1x)} = \frac{(1-a_1)(1-x)(1-a_1x)}{2(1+a_1)(1+x)(1+a_1x)} \ge 0.$$

By extending this result, $E(a_1, a_2, \ldots, a_{n-1})$ has the minimum value when $a_1 = a_2 = \cdots = a_{n-1} = 1$. Therefore,

$$E(a_1, a_2, \ldots, a_{n-1} \ge E(1, 1, \ldots, 1) = 0.$$

The proof is finished. The equality occurs when $n - 2$ numbers a_i are equal to 1 and the product of the other two is 1.

Second Solution. We use the induction method. Assume $a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n > 0$ and denote

$$E_n(a_1, a_2, \ldots, a_n) = \frac{1}{a_1+1} + \frac{1}{a_2+1} + \cdots + \frac{1}{a_n+1} - \frac{n}{2}.$$

We will show that

$$E_n(a_1, a_2, a_3, \ldots, a_n) \ge E_n(a_1a_2, 1, a_3, \ldots, a_{n-1}, a_n) \ge 0.$$

The right inequality can be written as

$$E_{n-1}(a_1a_2, a_3, \ldots, a_{n-1}, a_n) \ge 0.$$

Since

$$a_1a_2 = \frac{1}{a_3 \cdots a_{n-1}a_n} \ge 1$$

and

$$(a_1a_2)a_3 \cdots a_{n-1}a_n = 1,$$

the inequality follows by the induction hypothesis.

The left inequality is equivalent to

$$\frac{1}{a_1+1} + \frac{1}{a_2+1} \ge \frac{1}{a_1a_2+1} + \frac{1}{2},$$

$$\frac{1-a_2}{2(a_2+1)} \ge \frac{a_1(1-a_2)}{(a_1+1)(a_1a_2+1)},$$

which is true if

$$(a_1+1)(a_1a_2+1) \ge 2a_1(a_2+1).$$

This inequality can be written in the obvious form

$$(a_1-1)(a_1a_2-1) \ge 0.$$

Remark. Replacing $a_1, a_2, \ldots, a_n$ with $1/a_1, 1/a_2, \ldots, 1/a_n$, we get the following statement:

- *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that at most one of them is less than 1 and $a_1a_2 \cdots a_n = 1$, then*

$$\frac{1}{1+a_1} + \frac{1}{1+a_2} + \cdots + \frac{1}{1+a_n} \le \frac{n}{2}.$$

□

P 1.220. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that at most one of them is larger than 1 and $a_1a_2 \cdots a_n = 1$, then*

$$\frac{1}{(a_1+2)^2} + \frac{1}{(a_2+2)^2} + \cdots + \frac{1}{(a_n+2)^2} \ge \frac{n}{9}.$$

(Vasile Cîrtoaje, 2009)

Solution. Assume that $a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n$ and use the induction method. For $n = 2$, the desired inequality is equivalent to

$$(a_1 - 1)^4 \ge 0.$$

Let us denote

$$E_n(a_1, a_2, \ldots, a_n) = \frac{1}{(a_1+2)^2} + \frac{1}{(a_2+2)^2} + \cdots + \frac{1}{(a_n+2)^2} - \frac{n}{9}.$$

To end the proof, it suffices to show that

$$E_n(a_1, a_2, a_3, \ldots, a_{n-1}, a_n) \ge E_n(a_1, 1, a_3, \ldots, a_{n-1}, a_2 a_n) \ge 0$$

for $n \ge 3$.

The right inequality can be written as

$$E_{n-1}(a_1, a_3, \ldots, a_{n-1}, a_2 a_n) \ge 0.$$

Since

$$a_2 a_n \le a_n \le a_{n-1}$$

and

$$a_1 a_3 \ldots a_{n-1}(a_2 a_n) = 1,$$

the inequality follows by the induction hypothesis.

The left inequality is equivalent to

$$\frac{1}{(a_2+2)^2} + \frac{1}{(a_n+2)^2} \ge \frac{1}{9} + \frac{1}{(a_2 a_n + 2)^2}.$$

Denoting

$$s = a_2 + a_n, \quad p = a_2 a_n, \qquad s \le 2, \quad p \le 1,$$

the inequality becomes

$$\frac{s^2 + 4s + 8 - 2p}{(2s+4+p)^2} \ge \frac{p^2 + 4p + 13}{9(p+2)^2},$$

$$(1 + p - s)(As + B) \ge 0,$$

where

$$A = 16 - 20p - 5p^2, \qquad B = 80 - 32p - 29p^2 - p^3 > 0.$$

Since

$$1 + p - s = (1 - a_2)(1 - a_n) \ge 0,$$

we only need to show that $As + B \ge 0$. For the nontrivial case $A < 0$, we get

$$As + B \ge 2A + B = 112 - 72p - 39p^2 - p^3 = (1-p)(112 + 40p + p^2) \ge 0.$$

This completes the proof. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Remark. Similarly, we can prove the following generalization:

- *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that*

$$a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n, \qquad a_1 a_2 \cdots a_n = 1.$$

If $k \ge 1$, then

$$\frac{1}{(a_1+k)^k} + \frac{1}{(a_2+k)^k} + \cdots + \frac{1}{(a_n+k)^k} \ge \frac{n}{(1+k)^k}.$$

For $n = 2$, the desired inequality is true if $g(x) \ge 0$ for $x \ge 1$, where

$$g(x) = \frac{1}{(x+k)^k} + \frac{x^k}{(kx+1)^k} - \frac{2}{(1+k)^k},$$

$$\frac{g'(x)}{k} = \frac{x^{k-1}(x+k)^{k+1} - (kx+1)^{k+1}}{(x+k)^{k+1}(kx+1)^{k+1}}.$$

It suffices to show that $h(x) \ge 0$ for $x \ge 1$, where

$$h(x) = (k-1)\ln x + (k+1)\ln(x+k) - (k+1)\ln(kx+1),$$

$$h'(x) = \frac{k-1}{x} + \frac{k+1}{x+k} - \frac{k(k+1)}{kx+1} = \frac{k(k-1)(x-1)^2}{(x+k)(kx+1)}.$$

Since $h'(x) \ge 0$, $h(x)$ is increasing for $x \ge 1$, hence

$$h(x) \ge h(1) = 0.$$

Let

$$E_n(a_1, a_2, \ldots, a_n) = \frac{1}{(a_1+k)^k} + \frac{1}{(a_2+k)^k} + \cdots + \frac{1}{(a_n+k)^k} - \frac{n}{(1+k)^k}.$$

It suffices to show that

$$E_n(a_1, a_2, a_3, \ldots, a_{n-1}, a_n) \ge E_n(a_1, 1, a_3, \ldots, a_{n-1}, a_2 a_n) \ge 0.$$

The right inequality follows by the induction hypothesis, while the left inequality is equivalent to

$$f_1(a_2) + f_1(a_n) \ge f_1(1) + f_2(a_2 a_n),$$

where

$$f_1(x) = \frac{1}{(x+k)^k}.$$

Using the substitution

$$a_2 = e^a, \quad a_n = e^b,$$

the inequality becomes

$$f(a) + f(b) \ge f(0) + f(a+b),$$

where

$$f(x) = \frac{1}{(e^x + k)^k}.$$

From

$$f''(x) = \frac{k^2 e^x (e^x - 1)}{(e^x + k)^{k+2}},$$

it follows that f is concave on $(-\infty, 0]$. Since

$$0 \ge a \ge b \ge a + b,$$

the inequality $f(a) + f(b) \ge f(0) + f(a + b)$ follows from Karamata's inequality.

□

P 1.221. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that at most one of them is larger than 1 and $a_1 a_2 \cdots a_n = 1$, then*

$$\frac{1 - a_1}{3 + a_1^2} + \frac{1 - a_2}{3 + a_2^2} + \cdots + \frac{1 - a_n}{3 + a_n^2} \ge 0.$$

(Vasile Cîrtoaje, 2013)

Solution. Assume that $a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n$ and use the induction method. For $n = 2$, the desired inequality is equivalent to

$$(a_1 - 1)^4 \ge 0.$$

Let us denote

$$E_n(a_1, a_2, \ldots, a_n) = \frac{1 - a_1}{3 + a_1^2} + \frac{1 - a_2}{3 + a_2^2} + \cdots + \frac{1 - a_n}{3 + a_n^2}.$$

We will show that

$$E_n(a_1, \ldots, a_{n-2}, a_{n-1}, a_n) \ge E_n(a_1, \ldots, a_{n-2}, 1, a_{n-1}a_n) \ge 0.$$

The right inequality can be written as

$$E_{n-1}(a_1, a_2, \ldots, a_{n-2}, a_{n-1}a_n) \ge 0.$$

Since

$$a_1 \ge 1 \ge a_2 \ge \cdots \ge a_{n-2} \ge a_{n-1}a_n,$$

and

$$a_1 a_2 \cdots a_{n-2}(a_{n-1}a_n) = 1,$$

the inequality follows by the induction hypothesis.

The left inequality reduces to

$$\frac{1 - a_{n-1}}{3 + a_{n-1}^2} + \frac{1 - a_n}{3 + a_n^2} \ge \frac{1 - a_{n-1}a_n}{3 + a_{n-1}^2 a_n^2},$$

which is equivalent to the obvious inequality

$$(1-a_{n-1})(1-a_n)(3+a_{n-1}a_n)(3-a_{n-1}a_n-a_{n-1}^2a_n-a_{n-1}a_n^2)\geq 0.$$

Thus, the proof is completed. The equality holds for $a_1=a_2=\cdots=a_n=1$.

□

P 1.222. *Let a,b,c be nonnegative real numbers (no two of which are zero) such that $a+b+c=3$.*

(a) If $k\geq 3$, then

$$\frac{(k-2)a^2+3a}{bc+ka}+\frac{(k-2)b^2+3b}{ca+kb}+\frac{(k-2)c^2+3c}{ab+kc}\geq 3;$$

(b) If $0<k\leq 3$, then

$$\frac{(k-2)a^2+3a}{bc+ka}+\frac{(k-2)b^2+3b}{ca+kb}+\frac{(k-2)c^2+3c}{ab+kc}\leq 3.$$

(Vasile Cîrtoaje, 2021)

Solution. Denote

$$q=ab+bc+ca,\qquad r=abc.$$

We have

$$a^3+b^3+c^3=3abc+(a+b+c)^3-2(a+b+c)(ab+bc+ca)=3(9+r-3q)$$

and

$$3q\leq(a+b+c)^2=9,\qquad q\leq 3.$$

In addition, by Schur's inequality

$$(a+b+c)^3+9abc\geq 4(a+b+c)(ab+bc+ca),$$

we have

$$9+3r-4q\geq 0.$$

(a) Write the inequality as

$$(k-2)\sum a^2(ca+kb)(ab+kc)+3\sum a(ca+kb)(ab+kc)$$

$$-3(bc+ka)(ca+kb)(ab+kc)\geq 0,$$

$$(k-2)\left[abc\sum a^3+k\sum b^2c^2(b+c)+k^2abc\sum a\right]+$$

$$+3\left(abc\sum a^2 + 2k\sum b^2c^2 + 3k^2abc\right]$$
$$-3\left(a^2b^2c^2 + kabc\sum a^2 + k^2\sum b^2c^2 + k^3abc\right) \geq 0,$$

$$Ak^2 + Bk + C \geq 0,$$

where

$$\begin{aligned} A &= \sum b^2c^2(b+c) - 2abc\sum a - 3\sum b^2c^2 + 9abc \\ &= \sum b^2c^2(3-a) - 2abc\sum a - 3\sum b^2c^2 + 9abc \\ &= -abc\sum bc - 2abc\sum a + 9abc = r(3-q), \end{aligned}$$

$$\begin{aligned} B &= abc\sum a^3 - 2\sum b^2c^2(b+c) - 3abc\sum a^2 + 6\sum b^2c^2 \\ &= abc\sum a^3 - 2\sum b^2c^2(3-a) - 3abc\sum a^2 + 6\sum b^2c^2 \\ &= abc\sum a^3 + 2abc\sum bc - 3abc\sum a^2 \\ &= 3r(9+r-3q) + 2rq - 3r(9-2q) = r(3r-q), \end{aligned}$$

$$\begin{aligned} C &= -2abc\sum a^3 - 3a^2b^2c^2 + 3abc\sum a^2 \\ &= -6r(9+r-3q) - 3r^2 + 3r(9-2q) = -3r(9+3r-4q). \end{aligned}$$

Therefore,

$$\begin{aligned} Ak^2 + Bk + C &= r[(3-q)k^2 + (3r-q)k - 3(9+3r-4q)] \\ &= (k-3)r[(3-q)k + (9+3r-4q)] \geq 0. \end{aligned}$$

(b) From the proof of (a), it follows that the inequality is equivalent to

$$Ak^2 + Bk + C \leq 0,$$

where

$$Ak^2 + Bk + C = (k-3)r[(3-q)k + (9+3r-4q)] \leq 0.$$

For both inequality, the equality holds when $a = b = c = 1$, and also when one of a, b, c is zero. Moreover, for $k = 3$, both inequalities are identities.

□

P 1.223. *Let a, b, c be nonnegative real numbers (no two of which are zero) such that $a+b+c=3$.*

(a) If $0<k\le 3$, then

$$\frac{(k-1)a^2+(k+3)a}{bc+ka}+\frac{(k-1)b^2+(k+3)b}{ca+kb}+\frac{(k-1)c^2+(k+3)c}{ab+kc}\ge 6;$$

(b) If $k\ge 3$, then

$$\frac{(k-1)a^2+(k+3)a}{bc+ka}+\frac{(k-1)b^2+(k+3)b}{ca+kb}+\frac{(k-1)c^2+(k+3)c}{ab+kc}\le 6.$$

(Vasile Cîrtoaje, 2021)

Solution. Denote

$$q=ab+bc+ca, \qquad r=abc.$$

We have

$$a^3+b^3+c^3=3abc+(a+b+c)^3-2(a+b+c)(ab+bc+ca)=3(9+r-3q)$$

and

$$3q\le (a+b+c)^2=9, \qquad q\le 3,$$
$$(a+b+c)(ab+bc+ca)\ge 9abc, \qquad q\le 3r.$$

(a) Write the inequality as

$$(k-1)\sum a^2(ca+kb)(ab+kc)+(k+3)\sum a(ca+kb)(ab+kc)$$
$$-6(bc+a)(ca+b)(ab+c)\ge 0,$$

$$(k-1)\left[abc\sum a^3+k\sum b^2c^2(b+c)+k^2abc\sum a\right]+$$
$$+(k+3)\left(abc\sum a^2+2k\sum b^2c^2+3k^2abc\right)$$
$$-6\left(a^2b^2c^2+kabc\sum a^2+\sum k^2b^2c^2+k^3abc\right)\ge 0,$$

$$Ak^2+Bk+C\ge 0,$$

where

$$\begin{aligned}A&=\sum b^2c^2(b+c)-abc\sum a-4\sum b^2c^2+9abc\\&=\sum b^2c^2(3-a)-4\sum b^2c^2+6abc\\&=-abc\sum bc-\sum b^2c^2+6abc=-qr-(q^2-6r)+6r=12r-qr-q^2,\end{aligned}$$

$$\begin{aligned} B &= abc\sum a^3 - \sum b^2c^2(b+c) - 5abc\sum a^2 + 6\sum b^2c^2 \\ &= abc\sum a^3 - \sum b^2c^2(3-a) - 5abc\sum a^2 + 6\sum b^2c^2 \\ &= abc\sum a^3 + abc\sum bc - 5abc\sum a^2 + 3\sum b^2c^2 \\ &= 3r(9+r-3q) + qr - 5r(9-2q) + 3(q^2-6r) = 3r^2 + 2qr - 36r + 3q^2, \end{aligned}$$

$$\begin{aligned} C &= -abc\sum a^3 + 3abc\sum a^2 - 6a^2b^2c^2 \\ &= -3r(9+r-3q) + 3r(9-2q) - 6r^2 = 3r(q-3r). \end{aligned}$$

Therefore,

$$\begin{aligned} Ak^2 + Bk + C &= (12r - qr - q^2)k^2 + (3r^2 + 2qr - 36r + 3q^2)k + 3r(q-3r) \\ &= (3-k)[(q^2 + qr - 12r)k + r(q-3r)]. \end{aligned}$$

Since $q - 3r \ge 0$, it suffices to show that $q^2 + qr - 12r \ge 0$, that is

$$q^2 \ge (12-q)r.$$

According to P 3.57 in Volume 1, for fixed q, the product r is maximum when two of a, b, c are equal. Thus, we only need to consider $b = c$, when

$$a = 3 - 2b, \qquad q = 2ab + b^2 = 3b(2-b), \qquad r = b^2(3-2b),$$

and

$$\begin{aligned} q^2 - (12-q)r &= 9b^2(2-b)^2 - 3b^2(4 - 2b + b^2)(3-2b) \\ &= 6b^3(b-1)^2 \ge 0. \end{aligned}$$

(b) From the proof of (a), it follows that the inequality is equivalent to

$$Ak^2 + Bk + C \le 0,$$

where

$$Ak^2 + Bk + C = (3-k)[(q^2 + qr - 12r)k + r(q-3r)] \le 0.$$

For both inequality, the equality holds when $a = b = c = 1$. Moreover, for $k = 3$, both inequalities are identities.

□

P 1.224. *If a, b, c are positive real numbers such that $abc = 1$, then*

$$\frac{1}{(4a+b+c)^2} + \frac{1}{(4b+c+a)^2} + \frac{1}{(4c+a+b)^2} \le \frac{1}{12}.$$

(Choy Fai Lam and Vasile Cîrtoaje, Recreatii Matematice, no. 2, 2022)

Solution. Assume that $a \ge b \ge c$. Since

$$(4a+b+c)^2 = (3a+a+b+c)^2 \ge 12a(a+b+c), \qquad (4b+c+a)^2 \ge 12b(a+b+c),$$

it is enough to show that

$$\frac{1}{a(a+b+c)} + \frac{1}{b(a+b+c)} + \frac{12}{(4c+a+b)^2} \le 1,$$

or

$$\frac{c(a+b)}{a+b+c} + \frac{12}{(a+b+4c)^2} \le 1.$$

Denote

$$x = \frac{a+b}{2}.$$

From

$$4 = 4abc \le (a+b)^2c = 4x^2c,$$

we get

$$x \ge \frac{1}{\sqrt{c}} \ge 1.$$

Write the required inequality as follows:

$$\frac{2cx}{2x+c} + \frac{3}{(x+2c)^2} \le 1,$$

$$2(1-c)x^3 + (9c-8c^2)x^2 - 2(4c^3-6c^2+3)x + c(4c^2-3) \ge 0.$$

Since

$$2(1-c)x^3 \ge 2(1-c)x^2,$$

it suffices to show that $f(x) \ge 0$, where

$$f(x) = (2+7c-8c^2)x^2 - 2(4c^3-6c^2+3)x + c(4c^2-3).$$

Since

$$\frac{1}{2}f'(x) = (2+7c-8c^2)x - (4c^3-6c^2+3) \ge \frac{2+7c-8c^2}{\sqrt{c}} - (4c^3-6c^2+3)$$

$$= 2\left(\frac{1}{\sqrt{c}} + \sqrt{c}\right) + 5\sqrt{c} - 8c\sqrt{c} - (4c^3-6c^2+3) \ge 4 + 5c - 8c - (4c^3-6c^2+3)$$

$$= 1 - 3c + 6c^2 - 4c^3 = (1-c)(1-2c+4c^2) \ge 0,$$

f is an increasing function, hence

$$f(x) \ge f\left(\frac{1}{\sqrt{c}}\right).$$

So, we need to show that $f\left(\frac{1}{\sqrt{c}}\right) \geq 0$, that is

$$4c^4 - 11c^2 + 7c + 2 - 2\sqrt{c}(4c^3 - 6c^2 + 3) \geq 0.$$

Setting

$$t = \sqrt{c}, \qquad 0 < t \leq 1,$$

the inequality becomes

$$4t^8 - 8t^7 + 12t^5 - 11t^4 + 7t^2 - 6t + 2 \geq 0,$$

$$(1-t)^2[4t^6 + t^2 + 2(1-t)(1+2t^3)] \geq 0.$$

The equality holds when $a = b = c = 1$.

□

P 1.225. *Let a, b, c be nonnegative real numbers satisfying*

$$a^2 + b^2 + c^2 = 3.$$

If $\frac{1}{3} \leq k \leq 11$, *then*

$$\frac{15 - 7a^2}{1 + kbc} + \frac{15 - 7b^2}{1 + kca} + \frac{15 - 7c^2}{1 + kab} \geq \frac{24}{1 + k}.$$

(Vasile Cîrtoaje, 2021)

Solution. We use the *highest coefficient method*. Write the inequality in the form $f_6(a, b, c) \geq 0$, where

$$f_6(a, b, c) = (1 + k)\sum(15 - 7a^2)(1 + kab)(1 + kac) - 24(1 + kbc)(1 + kca)(1 + kab).$$

Clearly, $f_6(a, b, c)$ has the same highest coefficient A as

$$-7(1 + k)k^2abc\sum a^3 - 24k^3a^2b^2c^2,$$

i.e.

$$A = -21(1 + k)k^2 - 24k^3.$$

Since $A < 0$, according to P 3.76-(a) in Volume 1, it suffices to prove the homogeneous inequality

$$\frac{5(b^2 + c^2) - 2a^2}{a^2 + b^2 + c^2 + 3kbc} + \frac{5(c^2 + a^2) - 2b^2}{a^2 + b^2 + c^2 + 3kca} + \frac{5(a^2 + b^2) - 2c^2}{a^2 + b^2 + c^2 + 3kab} \geq \frac{8}{1 + k}$$

for $b = c = 1$ and for $a = 0$.

In the first case ($b=c=1$), the inequality is equivalent to

$$\frac{5-a^2}{a^2+3k+2}+\frac{5a^2+3}{a^2+3ka+2}\ge \frac{4}{1+k},$$

$$k(a-1)^2[4a^2-(3k+7)a+9k+1]\ge 0.$$

It is true since

$$4a^2-(3k+7)a+9k+1\ge 4\sqrt{9k+1}a-(3k+7)a=\frac{3(3k-1)(11-k)a}{4\sqrt{9k+1}+(3k+7)}\ge 0.$$

In the second case ($a=0$), the inequality is equivalent to

$$\frac{5(b^2+c^2)}{b^2+c^2+3kbc}+\frac{5c^2-2b^2}{b^2+c^2}+\frac{5b^2-2c^2}{b^2+c^2}\ge \frac{8}{1+k},$$

$$\frac{5(b^2+c^2)}{b^2+c^2+3kbc}+3\ge \frac{8}{1+k},$$

$$8(b^2+c^2)+3(3k-5)bc\ge 0.$$

Indeed, we have

$$8(b^2+c^2)+3(3k-5)bc\ge 16bc+3(3k-5)bc=(9k+1)bc\ge 0.$$

The equality occurs for $a=b=c=1$. If $k=11$, then the equality occurs also for $a=\frac{5}{3}$ and $b=c=\frac{1}{3}$ (or any cyclic permutation).

□

P 1.226. *If a,b,c are nonnegative real numbers such that $ab+bc+ca=3$, then*

$$\frac{1}{(ka+b+c)^2}+\frac{1}{(a+kb+c)^2}+\frac{1}{(a+b+kc)^2}\le \frac{3}{(k+2)^2}$$

for $1\le k\le k_0$, where $k_0\approx 2.82374$ is the positive root of the equation

$$k^3+6k^2-15k-28=0.$$

(Vasile Cîrtoaje and Leonard Giugiuc, MATINF, 7, 2021)

Solution. Write the inequality in the homogeneous form

$$\frac{1}{(ka+b+c)^2}+\frac{1}{(a+kb+c)^2}+\frac{1}{(a+b+kc)^2}\le \frac{9}{(k+2)^2(ab+bc+ca)}.$$

Since this inequality is well known for $k = 1$, consider next $k \in (1, k_0]$. Without loss of generality, assume that $a+b+c = 1$ and $a \le b \le c$. Denoting $m = k-1 > 0$, $q = ab+bc+ca$ and $r = abc$, we may write the inequality as follows:

$$\frac{1}{(ma+1)^2} + \frac{1}{(mb+1)^2} + \frac{1}{(mc+1)^2} \le \frac{9}{(m+3)^2 q},$$

$$9(ma+1)^2(mb+1)^2(mc+1)^2 \ge (m+3)^2 q \sum (mb+1)^2(mc+1)^2,$$

$$9(m^3 r + m^2 q + m + 1)^2 \ge (m+3)^2 q \sum (m^2 bc - ma + m + 1)^2,$$

$$f(q,r) + g(q) \ge 0,$$

where

$$f(q,r) = 9m^6 r^2 + 18m^3(m^2 q + m + 1)r + 2m^3(m+3)^2 q(m+3)r.$$

It is known that for $a+b+c = 1$ and fixed q, the product r has the minimum value when $a = 0$ or $b = c$. Since $f(q,r)$ is increasing with respect to r, it suffices to consider these cases.

Case 1: $a = 0$. The homogeneous inequality becomes

$$\frac{1}{(b+c)^2} + \frac{1}{(kb+c)^2} + \frac{1}{(b+kc)^2} \le \frac{9}{(k+2)^2 bc}.$$

Using the inequality

$$\frac{1}{(b+c)^2} \le \frac{1}{4bc}$$

and Lemma below, it suffices to show that

$$\frac{1}{4bc} + \frac{2}{(k+1)^2 bc} \le \frac{9}{(k+2)^2 bc},$$

which is equivalent to $k(k^3 + 6k^2 - 15k - 28) \le 0$.

Case 2: $b = c$. For $b = c = 1$, the homogeneous inequality becomes

$$\frac{9}{(k+2)^2(2a+1)} - \frac{1}{(ka+2)^2} - \frac{2}{(a+k+1)^2} \ge 0,$$

which is equivalent to

$$(a-1)^2(9k^2a^2 + 2\gamma a + k\beta) \ge 0,$$

where

$$\beta = -k^3 - 6k^2 + 15k + 28,$$

$$\gamma = -2k^4 + k^3 + 9k^2 + 14k - 4.$$

Since $\beta \ge 0$, it suffices to show that $\gamma \ge 0$. Since

$$2\gamma = \beta + (k-1)\alpha, \qquad \alpha = -4k^3 - k^2 + 23k + 36,$$

it suffices to show that $\alpha \ge 0$. We have

$$2\alpha = 3\beta - 5k^3 + 16k^2 + k - 12 = 3\beta + (k-1)(3-k)(5k+4) > 0.$$

For $k \in [1, k_0]$, the equality occurs when $a = b = c = 1$. For $k = k_0$, the equality occurs also when $a = 0$ and $b = c = \sqrt{3}$ (or any cyclic permutation).

Lemma. If b, c are positive real numbers and $1 \le k \le 3$, then

$$\frac{1}{(kb+c)^2} + \frac{1}{(b+kc)^2} \le \frac{2}{(k+1)^2bc}.$$

Proof. The inequality is equivalent to

$$(b-c)^2[2k^2(b^2+c^2) + \beta bc] \ge 0,$$

where

$$\beta = -k^4 + 2k^3 + 2k^2 + 2k - 1 > k^2(-k^2 + 2k - 1).$$

It is true because

$$2k^2(b^2+c^2) + \beta bc \ge 4k^2bc + \beta bc > k^2(-k^2+3k+3)bc = k^2(3-k)(1+k) \ge 0.$$

Remark. We have

$$k_0 = 6cos\left(\frac{\pi - arccos(1/3)}{3}\right) - 2.$$

□

P 1.227. *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = 1$. Prove that*

$$\frac{a_1^2}{(1-a_1)^2} + \frac{a_2^2}{(1-a_2)^2} + \cdots + \frac{a_n^2}{(1-a_n)^2} \ge \left(\frac{a_1}{1-a_1} + \frac{a_2}{1-a_2} + \cdots + \frac{a_n}{1-a_n} - \frac{\sqrt{n}}{\sqrt{n}+1}\right)^2.$$

(Vasile Cîrtoaje and Leonard Giugiuc, Crux Mathematicorum, 12, 2022)

Solution. Using the substitution

$$x_i = \frac{a_i}{1-a_i}, \qquad i = 1, 2, \ldots, n\ ,$$

we need to prove that

$$\frac{x_1}{x_1+1} + \frac{x_2}{x_2+1} + \cdots + \frac{x_n}{x_n+1} = 1$$

involves

$$x_1^2 + x_2^2 + \cdots + x_n^2 \ge \left(x_1 + x_2 + \cdots + x_n - \frac{\sqrt{n}}{\sqrt{n}+1}\right)^2.$$

Let

$$S = x_1 + x_2 + \cdots + x_n.$$

By the Cauchy-Schwarz inequality, we have

$$\left[\sum_{i=1}^n x_i(x_i+1)\right]\left(\sum_{i=1}^n \frac{x_i}{x_i+1}\right) \ge \left(\sum_{i=1}^n x_i\right)^2,$$

therefore

$$\sum_{i=1}^n x_i^2 + S \ge S^2.$$

Thus, it is sufficient to show that

$$S^2 - S \ge \left(S - \frac{\sqrt{n}}{\sqrt{n}+1}\right)^2,$$

which is equivalent to $S \ge \dfrac{n}{n-1}$. To prove this, we rewrite the hypothesis in the form

$$\frac{1}{x_1+1} + \frac{1}{x_2+1} + \cdots + \frac{1}{x_n+1} = n-1.$$

By the AM-HM inequality, we have

$$\left[\sum_{i=1}^n (x_i+1)\right]\left(\sum_{i=1}^n \frac{1}{x_i+1}\right) \ge n^2,$$

$$(S+n)(n-1) \ge n^2,$$

hence

$$S \ge \frac{n}{n-1}.$$

The proof is completed. The equality occurs for $a_1 = a_2 = \cdots = a_n = \dfrac{1}{n}$.

□

P 1.228. *If a, b, c, d are nonnegative real numbers such that at most one of them is less than 1 and $ab + ac + ad + bc + bd + cd = 6$, then*

$$\frac{1}{a+5} + \frac{1}{b+5} + \frac{1}{c+5} + \frac{1}{d+5} \ge \frac{2}{3}.$$

(Vasile Cîrtoaje, AMM, 2, 2025)

Solution. Without loss of generality, assume that

$$a \ge b \ge c \ge 1 \ge d.$$

For fixed a and b, we may consider that d is a function of c. By differentiating the equality constraint, we get $d'(a+b+c)+a+b+d=0$. Denoting by $F(c)$ the left side of the inequality, we get

$$F'(c) = \frac{-d'}{(d+5)^2} - \frac{1}{(c+5)^2} = \frac{a+b+d}{(a+b+c)(d+5)^2} - \frac{1}{(c+5)^2}.$$

We will show that $F'(c) \ge 0$, that is

$$\frac{(c+5)^2}{(d+5)^2} \ge \frac{a+b+c}{a+b+d}, \qquad \frac{(c+5)^2}{(d+5)^2} - 1 \ge \frac{a+b+c}{a+b+d} - 1, \qquad \frac{(c-d)(c+d+10)}{(d+5)^2} \ge \frac{c-d}{a+b+d}.$$

We need to show that $(a+b+d)(c+d+10) \ge (d+5)^2$, i.e.

$$(a+b)(c+d+10)+cd \ge 25, \qquad 10(a+b) \ge ab+19.$$

Indeed,

$$10(a+b)-ab-19 \ge 20\sqrt{ab}-ab-19 = (\sqrt{ab}-1)(19-\sqrt{ab}) \ge 0.$$

Since $F'(c) \ge 0$, $F(c)$ is increasing and is minimum when c is minimum, hence when $c=1$. So, it suffices to consider the case $c=1$, when we need to show that

$$\frac{1}{a+5} + \frac{1}{b+5} + \frac{1}{d+5} \ge \frac{1}{2}$$

for

$$ab+bd+da+a+b+d=6, \qquad a \ge b \ge 1 \ge d \ge 0.$$

For fixed d, we consider that a is a function of b. By differentiating the equality constraint, we get $a'(b+d+1)+a+d+1=0$. Denoting by $E(b)$ the left side of the inequality, we get

$$E'(b) = \frac{-a'}{(a+5)^2} - \frac{1}{(b+5)^2} = \frac{a+d+1}{(b+d+1)(a+5)^2} - \frac{1}{(b+5)^2}.$$

We will show that $E'(b) \ge 0$, that is

$$\frac{a+d+1}{b+d+1} \ge \frac{(a+5)^2}{(b+5)^2}, \qquad \frac{a+d+1}{b+d+1} - 1 \ge \frac{(a+5)^2}{(b+5)^2} - 1, \qquad \frac{a-b}{b+d+1} \ge \frac{(a-b)(a+b+10)}{(b+5)^2}.$$

We need to show that $(b+5)^2 \ge (b+d+1)(a+b+10)$, i.e.

$$15 \ge ab+bd+da+a+b+10d, \qquad 15 \ge 6+9d, \qquad 9(1-d) \ge 0.$$

Since $E'(b) \geq 0$, $E(b)$ is increasing and is minimum when b is minimum, hence when $b = 1$. Therefore, it suffices to consider $b = 1$, when we need to show that

$$\frac{1}{a+5} + \frac{1}{d+5} \geq \frac{1}{3}$$

for

$$ad + 2(a+d) = 5, \qquad a \geq 1 \geq d \geq 0.$$

It is easy to show that the desired inequality is an identity.

For $a \geq b \geq c \geq 1 \geq d$, the equality occurs when $b = c = 1$ and $ad + 2(a+d) = 5$ with $a \geq 1 \geq d$.

□

P 1.229. *For $n \geq 3$, let $a_1, a_2, \ldots, a_n$ be nonnegative real number such that at most one of them is less than 1 and*

$$\sum_{1 \leq i < j \leq n} a_i a_j = \frac{n(n-1)}{2}.$$

Then,

$$\frac{1}{a_1 + 2n - 3} + \frac{1}{a_2 + 2n - 3} + \cdots + \frac{1}{a_n + 2n - 3} \geq \frac{n}{2(n-1)}.$$

(Vasile Cîrtoaje, AMM, 2, 2025)

Solution. We need to show that $E \geq \dfrac{n}{1+k}$ for $k = 2n - 3$, where

$$E = \frac{1}{a_1 + k} + \frac{1}{a_2 + k} + \cdots + \frac{1}{a_n + k}.$$

Assume that $a_1 = \min\{a_1, a_2, \ldots, a_n\}$ and $a_n = \max\{a_1, a_2, \ldots, a_n\}$. If $a_1 = 1$ or $a_n = 1$, then $a_1 = a_2 = \cdots = a_n = 1$, and the inequality becomes an equality. Consider now $a_1 < 1$, $a_n > 1$ and $a_2, \ldots, a_{n-1} \in [1, a_n]$. For fixed $a_3, \ldots, a_n$, we may suppose that a_1 and E are functions of a_2. We claim that $E'(a_2) \geq 0$. If this claim is true, then $E(a_2)$ is increasing and it is minimum when a_2 is minimum, hence when $a_2 = 1$. Differentiating the equality constraint yields

$$(S - a_1)a_1' + S - a_2 = 0,$$

where $S = a_1 + a_2 + \cdots + a_n$. Therefore,

$$E'(a_2) = \frac{-a_1'}{(a_1+k)^2} - \frac{1}{(a_2+k)^2} = \frac{S - a_2}{(S-a_1)(a_1+k)^2} - \frac{1}{(a_2+k)^2}.$$

To prove that $E'(a_2) \geq 0$, we write it as follows:

$$\left(\frac{a_2+k}{a_1+k}\right)^2 \geq \frac{S-a_1}{S-a_2}, \qquad \left(\frac{a_2+k}{a_1+k}\right)^2 - 1 \geq \frac{a_2 - a_1}{S - a_2}.$$

Since

$$\left(\frac{a_2+k}{a_1+k}\right)^2 - 1 \ge \frac{2(a_2+k)}{a_1+k} - 2 = \frac{2(a_2-a_1)}{a_1+k},$$

it suffices to show that

$$\frac{2}{a_1+k} \ge \frac{1}{S-a_2},$$

i.e.

$$a_1 + 2(a_3 + \cdots + a_n) - k \ge 0.$$

From

$$(a_1+a_2+\cdots+a_n)^2 = a_1^2+a_2^2+\cdots+a_n^2+n^2-n > (a_1+a_2+\cdots+a_n)^2/n+n^2-n,$$

we obtain $a_1+a_2+\cdots+a_n > n$. Therefore,

$$a_1+2(a_3+\cdots+a_n)-k > (n-a_2-\cdots-a_n)+2(a_3+\cdots+a_n)-k = n-a_2+(a_3+\cdots+a_{n-1})+a_n-k$$

$$\ge n-a_2+(n-3)+a_n-k \ge 2n-3-k = 0.$$

Similarly, we can prove that for fixed $a_2,\ldots,a_{i-1},a_{i+1},\ldots,a_n$, where $i \in \{3,\ldots,n-1\}$, the expression E is minimum when $a_i = 1$. So, we only need to prove the original inequality for $a_2 = a_3 = \cdots = a_{n-1} = 1$, i.e. to show that

$$\frac{1}{a_1+2n-3} + \frac{1}{a_n+2n-3} \ge \frac{1}{n-1}$$

for $a_1a_n + (n-2)(a_1+a_n) = 2n-3$. It is easy to verify that this inequality is an identity. For $a_1 = \min\{a_1,a_2,\ldots,a_n\}$ and $a_n = \max\{a_1,a_2,\ldots,a_n\}$, the equality occurs when $a_2 = \cdots = a_{n-1} = 1$ and $a_1a_n + (n-2)(a_1+a_n) = 2n-3$ with $0 \le a_1 \le 1 \le a_n$.

Remark. For given $n \ge 3$, $2n-3$ is the largest positive value of k_n so that

$$\frac{1}{a_1+k_n} + \frac{1}{a_2+k_n} + \cdots + \frac{1}{a_n+k_n} \ge \frac{n}{1+k_n}$$

for any nonnegative real numbers $a_1,a_2,\ldots,a_n$ such that at most one of them is less than 1 and $\sum\limits_{1\le i<j\le n} a_ia_j = \frac{n(n-1)}{2}$. Indeed, assuming $a_1 = \frac{2n-3}{n-2}$, $a_2 = \cdots = a_{n-1} = 1$ and $a_n = 0$, the constraints are satisfied and the inequality becomes

$$\frac{n-2}{2n-3+(n-2)k_n} + \frac{1}{k_n} \ge \frac{2}{1+k_n},$$

which is equivalent to $k_n \le 2n-3$.

□

P 1.230. *If a, b, c, d are nonnegative real numbers such that at most one of them is larger than 1 and $ab + ac + ad + bc + bd + cd = 6$, then*

$$\frac{1}{a+5} + \frac{1}{b+5} + \frac{1}{c+5} + \frac{1}{d+5} \le \frac{2}{3}.$$

(Vasile Cîrtoaje, GMA, no. 1-2, 2024)

Solution. Without loss of generality, assume that

$$a \ge 1 \ge b \ge c \ge d.$$

For fixed c and d, we may consider that a is a decreasing function of b. By differentiating the equality constraint, we get $a'(b+c+d) + a + c + d = 0$. Denoting by $F(b)$ the left side of the inequality, we get

$$F'(b) = \frac{-a'}{(a+5)^2} - \frac{1}{(b+5)^2} = \frac{a+c+d}{(b+c+d)(a+5)^2} - \frac{1}{(b+5)^2}.$$

We will show that $F'(b) \ge 0$, that is

$$\frac{a+c+d}{b+c+d} \ge \frac{(a+5)^2}{(b+5)^2}, \qquad \frac{a+c+d}{b+c+d} - 1 \ge \frac{(a+5)^2}{(b+5)^2} - 1, \qquad \frac{a-b}{b+c+d} \ge \frac{(a-b)(a+b+10)}{(b+5)^2}.$$

We need to show that $(b+5)^2 \ge (b+c+d)(a+b+10)$, i.e.

$$25 \ge ab + (c+d)(a+b+10), \qquad cd + 19 \ge 10(c+d), \qquad (c-1)(d-1) + 9(2-c-d) \ge 0.$$

The last inequality is clearly true. Since $F'(b) \ge 0$, $F(b)$ is increasing and is maximum when b is maximum (a is minimum), i.e. when $b = 1$ (because $a = 1$ involves $b = c = d = 1$). Therefore, it suffices to consider $b = 1$, when we need to show that

$$\frac{1}{a+5} + \frac{1}{c+5} + \frac{1}{d+5} \le \frac{1}{2}$$

for

$$ac + cd + da + a + c + d = 6, \qquad a \ge 1 \ge c \ge d \ge 0.$$

For fixed d, we consider that a is a decreasing function of c. Differentiating the equality constraint yields $a'(c+d+1) + a + d + 1 = 0$. Denoting by $E(c)$ the left side of the inequality, we get

$$E'(c) = \frac{-a'}{(a+5)^2} - \frac{1}{(c+5)^2} = \frac{a+d+1}{(c+d+1)(a+5)^2} - \frac{1}{(c+5)^2}.$$

We will show that $E'(c) \ge 0$, that is

$$\frac{a+d+1}{c+d+1} \ge \frac{(a+5)^2}{(c+5)^2}, \qquad \frac{a+d+1}{c+d+1} - 1 \ge \frac{(a+5)^2}{(c+5)^2} - 1, \qquad \frac{a-c}{c+d+1} \ge \frac{(a-c)(a+c+10)}{(c+5)^2}.$$

We need to show that $(c+5)^2 \ge (c+d+1)(a+c+10)$, i.e.

$$15 \ge ac+cd+da+a+c+10d, \qquad 15 \ge 6+9d, \qquad 9(1-d) \ge 0.$$

Since $E'(c) \ge 0$, $E(c)$ is increasing and is maximum when c is maximum (a is minimum), i.e. when $c=1$ (because $a=1$ involves $b=c=d=1$). Therefore, it suffices to consider $c=1$, when we need to show that

$$\frac{1}{a+5}+\frac{1}{d+5} \le \frac{1}{3}$$

for

$$ad+2(a+d)=5, \qquad a \ge 1 \ge d \ge 0.$$

Actually, the desired inequality is an identity.

The equality occurs for $b=c=1$ and $ad+2(a+d)=5$ with $a \ge 1 \ge d$.

□

P 1.231. *If $a_1, a_2, \dots, a_n$ $(n \ge 3)$ are nonnegative real number such that at most one of them is larger than 1 and*

$$\sum_{1\le i<j\le n} a_i a_j = \frac{n(n-1)}{2},$$

then

$$\frac{1}{a_1+2n-3}+\frac{1}{a_2+2n-3}+\cdots+\frac{1}{a_n+2n-3} \le \frac{n}{2(n-1)}.$$

(Vasile Cîrtoaje, GMA, no. 1-2, 2024)

***Solution*.** We need to show that $E \le \dfrac{n}{1+k}$ for $k=2n-3$, where

$$E=\frac{1}{a_1+k}+\frac{1}{a_2+k}+\cdots+\frac{1}{a_n+k}.$$

Assume that $a_1=\max\{a_1,a_2,\dots,a_n\}$ and $a_n=\min\{a_1,a_2,\dots,a_n\}$. If $a_1=1$ or $a_n=1$, then $a_1=a_2=\cdots=a_n=1$, and the inequality becomes an equality. Consider now $a_1>1$, $a_n<1$ and $a_2,\dots,a_{n-1} \in [a_n,1]$. For fixed $a_3,\dots,a_n$, we may suppose that a_1 and E are functions of a_2. We claim that $E'(a_2) \ge 0$. If this claim is true, then $E(a_2)$ is increasing and it is maximum when a_2 is maximum, hence when $a_2=1$. Differentiating the equality constraint yields

$$(S+a_2)a_1'+S+a_1=0,$$

where $S=a_3+\cdots+a_n$. Therefore,

$$E'(a_2)=\frac{-a_1'}{(a_1+k)^2}-\frac{1}{(a_2+k)^2}=\frac{S+a_1}{(S+a_2)(a_1+k)^2}-\frac{1}{(a_2+k)^2}.$$

To prove that $E'(a_2) \geq 0$, we write it as follows:

$$\left(\frac{a_2+k}{a_1+k}\right)^2 \geq \frac{S+a_2}{S+a_1}, \qquad \left(\frac{a_2+k}{a_1+k}\right)^2 - 1 \geq \frac{S+a_2}{S+a_1} - 1, \qquad \frac{(a_2-a_1)(a_1+a_2+2k)}{(a_1+k)^2} \geq \frac{a_2-a_1}{S+a_1}.$$

It is true if

$$\frac{a_1+a_2+2k}{(a_1+k)^2} \leq \frac{1}{S+a_1},$$

i.e.

$$k^2 \geq a_1a_2 + (a_1+a_2)S + 2kS.$$

Write now the equality constraint as

$$a_1a_2 + (a_1+a_2)S + Q = \frac{n(n-1)}{2},$$

where $Q = \displaystyle\sum_{3\leq i<j\leq n} a_ia_j$. So, we need to show that

$$k^2 \geq \frac{n(n-1)}{2} - Q + 2kS.$$

Denoting

$$x_i = 1 - a_i \geq 0, \qquad i = 3, \dots, n,$$

we have

$$S = n-2-\sum_{i=3}^n x_i, \quad Q = \sum_{3\leq i<j\leq n} (1-x_i)(1-x_j) = \sum_{3\leq i<j\leq n} x_ix_j - (n-3)\sum_{i=3}^n x_i + \frac{(n-2)(n-3)}{2},$$

and the inequality becomes

$$\sum_{3\leq i<j\leq n} x_ix_j + 3(n-1)\sum_{i=3}^n x_i \geq 0.$$

Similarly, we can prove that for fixed $a_2, \dots, a_{i-1}, a_{i+1}, \dots, a_n$, where $i \in \{3, \dots, n-1\}$, the expression E is maximum when $a_i = 1$. So, we only need to prove the original inequality for $a_2 = a_3 = \cdots = a_{n-1} = 1$, i.e. to show that

$$\frac{1}{a_1+2n-3} + \frac{1}{a_n+2n-3} \leq \frac{1}{n-1}$$

for $a_1a_n + (n-2)(a_1+a_n) = 2n-3$. It is easy to verify that this inequality is an identity. For $k_n = 2n-3$, $a_1 = \max\{a_1, a_2, \dots, a_n\}$ and $a_n = \min\{a_1, a_2, \dots, a_n\}$, the equality occurs when $a_2 = \cdots = a_{n-1} = 1$ and $a_1a_n + (n-2)(a_1+a_n) = 2n-3$ with $a_1 \geq 1 \geq a_n$.

Remark. For given $n \geq 3$, $2n-3$ is the least positive value of k_n so that

$$\frac{1}{a_1+k_n} + \frac{1}{a_2+k_n} + \cdots + \frac{1}{a_n+k_n} \leq \frac{n}{1+k_n}$$

for any nonnegative real numbers $a_1, a_2, \ldots, a_n$ such that at most one of them is larger than 1 and $\displaystyle\sum_{1\le i<j\le n} a_ia_j = \frac{n(n-1)}{2}$. Indeed, assuming $a_1 = \dfrac{2n-3}{n-2}$, $a_2 = \cdots = a_{n-1} = 1$ and $a_n = 0$, the constraints are satisfied and the inequality becomes

$$\frac{n-2}{2n-3+(n-2)k_n} + \frac{1}{k_n} \le \frac{2}{1+k_n},$$

which is equivalent to $k_n \ge 2n-3$.

□

P 1.232. *Let a,b,c be the lengths of the sides of a triangle such that $a+b+c=2$. Prove that*

$$\frac{a}{a^2+bc} + \frac{b}{b^2+ca} + \frac{c}{c^2+ab} \ge 2.$$

Solution. By the Cauchy-Schwartz inequality, we have

$$\sum \frac{a}{a^2+bc} \ge \frac{(a+b+c)^2}{\sum a(a^2+bc} = \frac{4}{a^3+b^3+c^3+3abc}.$$

So, it suffices to show that

$$2 \ge a^3+b^3+c^3+3abc,$$

which is equivalent to the homogeneous inequality

$$(a+b+c)^3 \ge 4(a^3+b^3+c^3)+12abc.$$

Substituting $a=y+z$, $b=z+x$ and $c=x+y$, where $x,y,z \ge 0$, the inequality becomes

$$2(x+y+z)^2 \ge (y+z)^2+(z+x)^2+(x+y)^2+3(y+z)(z+x)(x+y),$$

$$xy(x+y)+yz(y+z)+zx(z+x)+xyz \ge 0.$$

For $a \ge b \ge c$, the equality occurs for a degenerate triangle with $a=b=1$ and $c=0$.

□

P 1.233. *Let a,b,c be the lengths of the sides of a triangle such that $a+b+c=\sqrt{2}$. Prove that*

$$\frac{a}{a+bc} + \frac{b}{b+ca} + \frac{c}{c+ab} \ge 2.$$

Solution. Let

$$p = a + b + c, \qquad q = ab + bc + ca.$$

Write the inequality in the homogeneous form

$$\frac{a}{ap + bc\sqrt{2}} + \frac{b}{bp + ca\sqrt{2}} + \frac{c}{cp + ab\sqrt{2}} \ge \frac{2}{p}.$$

To prove it, use the *highest coefficient method*. We need to prove that $f_6(a, b, c) \ge 0$, where

$$f_6(a, b, c) = p\sum a(bp + ca\sqrt{2})(cp + ab\sqrt{2}) - (ap + bc\sqrt{2})(bp + ca\sqrt{2})(cp + ab\sqrt{2}).$$

Clearly, $f_6(a, b, c)$ has the highest coefficient $A = -2\sqrt{2}$. Since $A < 0$, by P 3.76-(b) in Volume 1, it suffices to prove the homogeneous inequality for $b = c = 1$ and $0 \le a \le 2$, and for $a = b + c$.

Case 1: $b = c = 1$ and $0 \le a \le 2$. The homogeneous inequality reduces to

$$\frac{a}{a(a+2) + \sqrt{2}} + \frac{2}{(1 + \sqrt{2})a + 2} \ge \frac{2}{a + 2},$$

$$\frac{a}{a^2 + 2a + \sqrt{2}} \ge \frac{2\sqrt{2}a}{(a + 2)[(1 + \sqrt{2})a + 2]},$$

$$a[(1 - \sqrt{2})a + 4 - 2\sqrt{2}] \ge 0.$$

It is true because

$$(1 - \sqrt{2})a + 4 - 2\sqrt{2} \ge 2(1 - \sqrt{2}) + 4 - 2\sqrt{2} = 2(3 - 2\sqrt{2}) > 0.$$

Case 2: $a = b + c$. The homogeneous inequality reduces to

$$\frac{b + c}{2(b + c)^2 + bc\sqrt{2}} + \frac{b}{2b(b + c) + c(b + c)\sqrt{2}} + \frac{c}{2c(b + c) + b(b + c)\sqrt{2}} \ge \frac{1}{b + c},$$

$$\frac{(b + c)^2}{2(b + c)^2 + bc\sqrt{2}} + \frac{b}{2b + c\sqrt{2}} + \frac{c}{2c + b\sqrt{2}} \ge 1,$$

$$\frac{2(b + c)^2}{2(b + c)^2 + bc\sqrt{2}} + \frac{4bc + \sqrt{2}(b^2 + c^2)}{3bc + \sqrt{2}(b^2 + c^2)} \ge 2,$$

$$\frac{2(b + c)^2}{2(b + c)^2 + bc\sqrt{2}} \ge \frac{2bc + \sqrt{2}(b^2 + c^2)}{3bc + \sqrt{2}(b^2 + c^2)}.$$

Due to homogeneity, we may consider $b + c = 1$. Denoting $x = bc$, the inequality becomes

$$\frac{2}{2 + x\sqrt{2}} \ge \frac{\sqrt{2} - 2(\sqrt{2} - 1)x}{\sqrt{2} + (3 - 2\sqrt{2})x},$$

$$(4-2\sqrt{2})x^2 \ge 0.$$

The proof is completed. For $a \ge b \ge c$, the equality occurs for a degenerate triangle with $a = b = \dfrac{1}{\sqrt{2}}$ and $c = 0$.

□

P 1.234. *Let*

$$x = \frac{2a}{b+c}, \qquad y = \frac{2b}{c+a}, \qquad z = \frac{2c}{a+b},$$

where a, b, c are positive real numbers such that at most one of them is less than 1 and $abc = 1$. Prove that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + 3 \ge 2(x+y+z).$$

(Vasile Cîrtoaje, Math. Reflections, 1, 2025)

Solution. Assume that $a \ge b \ge 1 \ge c$ and denote $p = a+b+c$ and $q = ab+bc+ca$. By the AM-GM inequality, we have

$$p \ge 3(abc)^{1/3} = 3.$$

In addition, from $(a-1)(b-1)(c-1) \le 0$, we get

$$q \ge p.$$

Write the inequality as follows:

$$\sum_{cyc} \frac{b+c}{a} + 6 \ge \sum_{cyc} \frac{8a}{b+c},$$

$$\sum_{cyc} bc(b+c) + 6 \ge \frac{8}{(b+c)(c+a)(a+b)} \sum_{cyc} a(a+b)(a+c),$$

$$pq + 3 \ge \frac{8(p^3 - 2pq + 3)}{pq - 1}.$$

So, for fixed p, we need to show that $f(q) \ge 0$, where

$$f(q) = p^2q^2 + 18pq - 8p^3 - 27.$$

Since $f(q)$ is increasing, we have

$$f(q) \ge f(p) = p^4 - 8p^3 + 18p^2 - 27 = (p+1)(p^3 - 9p^2 + 27p - 27) = (p+1)(p-3)^3 \ge 0.$$

The equality occurs for $a = b = c = 1$.

□

P 1.235. *If* $a_1 \ge a_2 \ge \cdots \ge a_n > 0$, *then*

$$\frac{1}{n}\left(\sqrt{a_1}-\sqrt{a_n}\right)^2 \le \frac{a_1+a_2+\cdots+a_n}{n} - \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}} \le \left(\sqrt{a_1}-\sqrt{a_n}\right)^2.$$

(Leonard Giugiuc and Vasile Cîrtoaje, GMA, no. 3-4, 2022)

Solution. (a) Since

$$\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}} \le \sqrt[n]{a_1a_2\cdots a_n},$$

the left inequality is true if

$$\left(\sqrt{a_1}-\sqrt{a_n}\right)^2 \le a_1+a_2+\cdots+a_n - n\sqrt[n]{a_1a_2\cdots a_n},$$

which is just the inequality in P 3.217 in Volume 1 for $k=1$ and $j=n$. The equality holds for $a_1=a_2=\cdots=a_n$.

(b) For fixed $a_n=1$ and $a_1=\alpha>1$ (if $a_1=1$, then the right inequality is an equality), we need to show that $f(a_2,\ldots,a_{n-1}) \ge 0$ for $\alpha \ge a_2 \ge \cdots \ge a_{n-1} \ge 1$, where

$$f(a_2,\ldots,a_{n-1}) = \frac{n^2}{\frac{1}{\alpha}+\frac{1}{a_2}+\cdots+\frac{1}{a_{n-1}}+1} + n\left(\sqrt{\alpha}-1\right)^2 - \alpha - a_2 - \cdots - a_{n-1} - 1.$$

Since f is convex with respect to each variable, it has the maximum value when $a_2,\ldots,a_{n-1} \in \{1,\alpha\}$, that is

$$\alpha = a_2 = \ldots = a_j, \qquad a_{j+1} = \ldots = a_{n-1} = 1,$$

where $j \in \{1,2,\ldots,n-1\}$ (see also Remark from P 2.104 in Volume 1). So, it suffices to show that

$$\frac{n^2}{\frac{j}{\alpha}+n-j} + n\left(\sqrt{\alpha}-1\right)^2 - j\alpha - n + j \ge 0,$$

which is equivalent to

$$n\left(\sqrt{\alpha}-1\right)^2 \ge \frac{j(n-j)(\alpha-1)^2}{(n-j)\alpha+j},$$

$$\left(\sqrt{\alpha}-1\right)^2\left[(n-j)\sqrt{\alpha}-j\right]^2 \ge 0.$$

For $a_n=1$, the equality occurs when $a_1=a_2=\cdots=a_n=1$, and also when $a_1=\cdots=a_j=\left(\frac{j}{n-j}\right)^2$ and $a_{j+1}=\cdots=a_n=1$, with $\frac{n}{2} \le j \le n-1$.

Remark. Similarly, we can prove the following generalization:

- *If* $a_1 \ge a_2 \ge \cdots \ge a_n > 0$, *then*

$$\frac{a_1+a_2+\cdots+a_n}{n} - \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}} \le \frac{(n-1)(a_1-a_n)^2}{n(a_1+a_n)},$$

with equality for $a_1=a_2=\cdots=a_n$.

□

P 1.236. *Prove that* $25/17$ *is the largest positive value of* k *such that*

$$\frac{1}{a^2+k}+\frac{1}{b^2+k}+\frac{1}{c^2+k}+\frac{1}{d^2+k}\geq\frac{4}{1+k}$$

for any nonnegative real numbers a,b,c,d *with* $ab+ac+ad+bc+bd+cd=6$ *and at most one of them larger than* 1.

(Vasile Cîrtoaje, RMM, 39, 2025)

Solution. Without loss of generality, assume that $a,b,c\in[0,1]$ and $d\geq 1$. For $b=c=1$, the inequality becomes

$$\frac{1}{a^2+k}+\frac{1}{d^2+k}\geq\frac{2}{1+k},$$

under the constraint $ad+4S=5$, where $S=(a+d)/2$. From $5=ad+4S\leq S^2+4S$ and $5=ad+4S\geq 4S$, we get $1\leq S\leq 5/4$. Write the inequality as follows:

$$\frac{4S^2-2ad+2k}{(ad-k)^2+4kS^2}\geq\frac{2}{1+k},$$

$$\frac{4S^2-2(5-4S)+2k}{(5-k-4S)^2+4kS^2}\geq\frac{2}{1+k},$$

$$(S-1)[15-3k-(k+7)S]\geq 0.$$

Choosing $S=5/4$, we get the necessary condition $k\leq 25/17$. So, we only need to prove the original inequality for $k=25/17$.

If c,d are fixed, then the expression

$$F=\frac{1}{a^2+k}+\frac{1}{b^2+k}+\frac{1}{c^2+k}+\frac{1}{d^2+k}$$

has the minimum value when

$$E(a,b)=\frac{1}{a^2+k}+\frac{1}{b^2+k}$$

has the minimum value. Denoting

$$x=a,\qquad y=b,\qquad A=c+d,\qquad B=6-cd,$$

we have $A>0$, $B\geq 0$, $x,y\in[0,1]$ and $xy+A(x+y)=B$. By Lemma below, it follows that $E(a,b)$ has the minimum value for $\min\{a,b\}=0$ or $a=b$. By extending this result to any two of a,b,c, it suffices to prove the original inequality for $a=b=c$, for $a=0$ and $b=c$, and for $a=b=0$.

Case 1: $a=b=c$. We need to show that

$$\frac{3}{c^2+k}+\frac{1}{d^2+k}\geq\frac{4}{1+k}$$

for $0 \le c \le 1 \le d$ such that $c^2 + cd = 2$. The inequality is equivalent to

$$(1-c^2)^2(3-k-c^2) \ge 0.$$

It is true because $3 - k - c^2 \ge 2 - k > 0$.

Case 2: $a = 0$ and $b = c$. We need to show that

$$\frac{1}{k} + \frac{2}{c^2+k} + \frac{1}{d^2+k} \ge \frac{4}{1+k}$$

for $0 \le c \le 1 \le d$ such that $c^2 + 2cd = 6$. Write the inequality as follows:

$$\frac{2}{c^2+k} - \frac{2}{1+k} \ge \frac{2}{1+k} - \frac{1}{k} - \frac{1}{d^2+k},$$

$$\frac{2(1-c^2)}{(1+k)(c^2+k)} \ge \frac{(k-1)(1-c^2)(36-c^2)}{k(1+k)[c^4-4(3-k)c^2+36]}.$$

It is true if

$$\frac{2}{c^2+k} \ge \frac{(k-1)(36-c^2)}{k[c^4-4(3-k)c^2+36]},$$

i.e.

$$\frac{25}{c^2+k} \ge \frac{4(36-c^2)}{c^4-4(3-k)c^2+36}.$$

It is true if

$$\frac{24}{1+k} \ge \frac{4\cdot 36}{0-4(3-k)+36},$$

i.e.

$$\frac{2}{1+k} \ge \frac{3}{6+k}, \qquad 9 \ge k.$$

Case 3: $a = b = 0$. We need to show that

$$\frac{2}{k} + \frac{1}{c^2+k} + \frac{1}{d^2+k} \ge \frac{4}{1+k}$$

for $0 < c \le 1 \le d$ such that $cd = 6$. It is true if

$$\frac{2}{k} + \frac{1}{1+k} + 0 \ge \frac{4}{1+k},$$

i.e. $k \le 2$.

The proof is completed. For $k = 25/17$, the equality occurs when $a = b = c = d = 1$, and also for $a = 0$, $b = c = 1$ and $d = 5/2$ (or any cyclic permutation).

Lemma. *Let $A > 0$, $B \ge 0$ and $k > 1$ be real constants, and let $x, y \in [0,1]$ such that $xy + A(x+y) = B$. Then, the expression*

$$E = \frac{1}{x^2+k} + \frac{1}{y^2+k}$$

has the minimum value for $\min\{x,y\}=0$ *or* $x=y$.

Proof. For $B=0$, we have $x=y=0$. Assume further $B>0$, and denote $s=x+y$ and $p=xy$. We need to show that if $0\le 4p\le s^2$ and $p+As=B$, then the expression

$$E=\frac{s^2-2p+2k}{ks^2+(p-k)^2}$$

has the minimum value for $p=0$ (when $\min\{x,y\}=0$) or $4p=s^2$ (when $x=y$). From $B=p+As\ge p+2A\sqrt{p}$, we get

$$p\le p_1=(\sqrt{A^2+B}-A)^2,$$

with equality for $4p=s^2$. Since

$$kE=1+\frac{k^2-p^2}{ks^2+(p-k)^2}=1+F(p),\qquad F(p)=\frac{A^2(k^2-p^2)}{(A^2+k)p^2-2k(A^2+B)p+k^2A^2+kB^2},$$

we need to show that for $p\in[0,p_1]$, $F(p)$ has the minimum value for $p=0$ or $p=p_1$. Let m be this minimum value. Since $k>1\ge p$, hence $k^2-p^2>0$, we have $m>0$. The inequality $F(p)-m\ge 0$ is equivalent to $F_1(p)\ge 0$, where

$$F_1(p)=-[(m+1)A^2+km]p^2+2km(A^2+B)p-(m-1)k^2A^2-kmB^2.$$

Since $F_1(p)$ is concave, the inequality $F_1(p)\ge 0$ is an equality for $p=0$ or $p=p_1$. Therefore, $F(p)$ has the minimum value for $p=0$ or $p=p_1$.

□

P 1.237. *Let a,b,c be nonnegative real numbers such that $a+b+c=3$. Prove that*

$$\frac{1}{32a^2-48a+25}+\frac{1}{32b^2-48b+25}+\frac{1}{32c^2-48c+25}\le\frac{1}{3}.$$

(Vasile Cîrtoaje, Math. Reflections, 5, 2024)

Solution. We see that $32x^2-48x+25=2(4x-3)^2+7>0$ for any real x. Assume that $a\ge b\ge c$ and write the inequality as $F\le\dfrac{1}{3}$.

Case 1: $c>\dfrac{3}{4}$. For fixed a, assume that b is a function of c, i.e. b=3-a-c. We have

$$\frac{F'(c)}{16}=\frac{-(4b-3)b'}{(32b^2-48b+25)^2}-\frac{4c-3}{(32c^2-48c+25)^2}=\frac{4b-3}{(32b^2-48b+25)^2}-\frac{4c-3}{(32c^2-48c+25)^2}.$$

We will show that $F'(c)\ge 0$, i.e.

$$\left(\frac{32c^2-48c+25}{32b^2-48b+25}\right)^2\ge\frac{4c-3}{4b-3}.$$

Since

$$\left(\frac{32c^2-48c+25}{32b^2-48b+25}\right)^2 \ge \frac{2(32c^2-48c+25)}{32b^2-48b+25}-1,$$

it suffices to show that

$$\frac{2(32c^2-48c+25)}{32b^2-48b+25}-1 \ge \frac{4c-3}{4b-3},$$

i.e.

$$1-\frac{4c-3}{4b-3} \ge 2\left(1-\frac{32c^2-48c+25}{32b^2-48b+25}\right), \qquad \frac{4(b-c)}{4b-3} \ge \frac{32(b-c)(2b+2c-3)}{32b^2-48b+25}.$$

This is true if

$$\frac{1}{4b-3} \ge \frac{8(2b+2c-3)}{32b^2-48b+25}.$$

Since $2b+2c-3=3-2a \le 3-2b$, it suffices to prove that

$$\frac{1}{4b-3} \ge \frac{8(3-2b)}{32b^2-48b+25}.$$

Indeed,

$$32b^2-48b+25-8(3-2b)(4b-3)=96b^2-192b+97>96b^2-192b+96=96(b-1)^2 \ge 0.$$

Since $F'(c) \ge 0$, $F(c)$ is increasing and has the maximum value when c is maximum, hence when $c=b$. So, we only need to prove the original inequality for $c=b$, i.e. to show that $a+2b=3$ involves

$$\frac{1}{32a^2-48a+25}+\frac{2}{32b^2-48b+25} \le \frac{1}{3}.$$

Write the inequality as follows

$$\frac{1}{128b^2-288b+119} \le \frac{32b^2-48b+19}{3(32b^2-48b+25)},$$

$$64b^4-240b^3+337b^2-210b+49 \ge 0, \qquad (b-1)^2(8b-7)^2 \ge 0.$$

Case 2: $0 \le c \le \frac{3}{4}$. For fixed b, assume that a is a decreasing function of c, i.e. a=3-b-c. We have

$$\frac{F'(c)}{16}=\frac{-(4a-3)a'}{(32a^2-48a+25)^2}-\frac{4c-3}{(32c^2-48c+25)^2}=\frac{4a-3}{(32a^2-48a+25)^2}-\frac{4c-3}{(32c^2-48c+25)^2}$$

$$\ge \frac{4a-3}{(32a^2-48a+25)^2}>0.$$

Since $F'(c)>0$, $F(c)$ is increasing and has the maximum value when c is maximum (a is minimum), hence when $c=b$ or $a=b$. Therefore, we only need to prove the original inequality for these cases. In both cases, the original inequality becomes

$$(b-1)^2(8b-7)^2 \ge 0.$$

So, the proof is completed. The equality occurs when $a=b=c=1$, and also for $a=\frac{5}{4}$ and $b=c=\frac{7}{8}$ (or any cyclic permutation).

□

P 1.238. *If a,b,c are nonnegative real numbers such that at most one of them is less than 1 and $ab+bc+ca=3$, then*

$$\frac{1}{3a^2+5}+\frac{1}{3b^2+5}+\frac{1}{3c^2+5}\le\frac{3}{8}.$$

(Vasile Cîrtoaje, Math. Reflections, 6, 2024)

Solution. Assume that $a\ge b\ge 1\ge d$, and denote $x=\frac{(a+b)^2}{4}$ and $p=ab$. From $(a-1)(b-1)\ge 0$, we obtain $a+b\le ab+1$, hence $(a+b)^2\le(ab+1)^2$. Therefore,

$$1\le p\le x\le\frac{(p+1)^2}{4}.$$

Since

$$\frac{1}{3a^2+5}+\frac{1}{3b^2+5}=\frac{3(a^2+b^2)+10}{9a^2b^2+15(a^2+b^2)+25}=\frac{2(6x-3p+5)}{60x+9p^2-30p+25}$$

and

$$c^2=\frac{(3-ab)^2}{(a+b)^2}=\frac{(3-p)^2}{4x},$$

the desired inequality can be written as $F\le 0$, where

$$F=\frac{6x-3p+5}{60x+9p^2-30p+25}+\frac{2x}{20x+3p^2-18p+27}-\frac{3}{16}.$$

Moreover, for fixed p, the inequality $F\le 0$ is equivalent to $f(x)\ge 0$, where

$$f(x)=-240x^2+B(p)x+C(p).$$

Since $f(x)$ is concave and $x\in[p,(p+1)^2/4]$, it suffices to prove that $F\ge 0$ for $x=p$ and $x=(p+1)^2/4$. For $x=p$, we have

$$F=\frac{1}{3p+5}+\frac{2p}{3p^2+2p+27}-\frac{3}{16}=\frac{27(1-p)^3}{16(3p+5)(3p^2+2p+27)}\le 0,$$

and for $x=(p+1)^2/4$, we have

$$16F=\frac{3p^2+13}{3p^2+5}+\frac{p^2+2p+1}{p^2-p+4}-3=\frac{-3(p-1)^4}{(3p^2+5)(p^2-p+4)}\le 0.$$

The proof is completed. The equality occurs for $a=b=c=1$.

□

P 1.239. *Let a, b, c, d be nonnegative real numbers such that $ab+ac+ad+bc+bd+cd = 1$. Prove that*

$$\frac{1}{a+b+c}+\frac{1}{b+c+d}+\frac{1}{c+d+a}+\frac{1}{d+a+b}\ge 3.$$

(Vasile Cîrtoaje, Math. Reflections, 3, 2025)

Solution. Assume that $a = \max\{a, b, c, d\}$, and write the inequality as $E \ge 3$. For fixed a and d, we may assume that $b \ge c$, and b and E are functions of c. By differentiating the equality constraint, we get

$$(p-b)b' + p - c = 0, \qquad -b' = \frac{p-c}{p-b}.$$

Let

$$A = \frac{1}{(p-a)^2}+\frac{1}{(p-d)^2} > \frac{1}{(p-a)^2} \ge \frac{1}{(p-b)(p-c)}.$$

We have

$$E'(c) = \left(A+\frac{1}{(p-c)^2}\right)(-b') - A - \frac{1}{(p-b)^2} = A(-b'-1) - \frac{b'}{(p-c)^2} - \frac{1}{(p-b)^2}$$

$$= \frac{(b-c)A}{p-b} - \frac{b-c}{(p-b)^2(p-c)} \ge \frac{b-c}{(p-b)^2(p-c)} - \frac{b-c}{(p-b)^2(p-c)} = 0.$$

Therefore, $E(c)$ is increasing and has the minimum value when c is minimum possible (to respect the constraint $a \ge b \ge c \ge 0$), hence when $c = 0$ or $b = a$ (because $b(c)$ is a decreasing function). So, abandoning the condition $b \ge c$, it suffices to prove the required inequality for the case when one of b and c is equal to 0 or a. By extending this result to any two numbers from b, c, d, it suffices to consider the case when at most one of b, c, d is different from 0 or a. So, we may assume the cases $(b, c, d) = (a, x, 0, 0)$, $(b, c, d) = (a, a, x, 0)$ and $(b, c, d) = (a, a, a, x)$, where $x \le a$.

Case 1: $(b, c, d) = (a, x, 0, 0)$. We need to show that $ax = 1$ implies

$$\frac{2}{a+x}+\frac{1}{a}+\frac{1}{x}\ge 3.$$

The inequality is equivalent to

$$\frac{2a}{a^2+1}+\frac{1}{a}+a\ge 3,$$

$$(a-1)^2(a^2-a+1)\ge 0.$$

Case 2: $(b, c, d) = (a, a, x, 0)$. We need to show that $a^2 + 2ax = 1$ implies

$$\frac{1}{2a+x}+\frac{2}{a+x}+\frac{1}{2a}\ge 3,$$

which is equivalent to

$$\frac{2a}{3a^2+1}+\frac{4a}{a^2+1}+\frac{1}{2a}\ge 3,$$

$$1+2a+8a^2+7a^4-18a^5 \ge 0,$$

$$(1-a^5)+2a(1-a^4)+8a^2(1-a^3)+7a^4(1-a) \ge 0.$$

Since $1=a^2+2ax \ge a^2$, hence $a \le 1$, the last inequality is true.

Case 3: $(b,c,d)=(a,a,a,x)$. We need to show that $3a^2+3ax=1$ implies

$$\frac{1}{3a}+\frac{3}{2a+x} \ge 3.$$

From $1=3a^2+3ax \le 3a^2$ and $1=3a^2+3ax \ge 3a^2+3a^2$, we get $\dfrac{1}{\sqrt{6}} \le a \le \dfrac{1}{\sqrt{3}}$. The inequality is equivalent to

$$\frac{1}{3a}+\frac{9a}{3a^2+1} \ge 3,$$

$$1-9a+30a^2-27a^3 \ge 0.$$

We have

$$1-9a+30a^2-27a^3 > 1-9a+30a^2-28a^3 = (2a-1)^2(5-7a)+2(2-3a)(3a-1) > 0,$$

since $5-7a>0$, $2-3a>0$ and $3a-1>0$.

The proof is completed. The equality occurs when two of a,b,c,d are equal to 1 and all the others to 0.

□

P 1.240. *Let a,b,c be nonnegative real numbers such that $a^2+b^2+c^2+ab+bc+ca=6$. Prove that*

$$\frac{1}{a^2+5}+\frac{1}{b^2+5}+\frac{1}{c^2+5} \le \frac{1}{2}.$$

(Vasile Cîrtoaje, Math. Reflections, 2, 2025)

Solution. Assume that $a \ge b \ge c$ and write the inequality as $E \le 0$. For fixed a, we may assume that b and E are functions of c. By differentiating the constraint, we have

$$(2b+a+c)b'+2c+a+b=0,$$

hence

$$E'(c)=\frac{-2bb'}{(b^2+5)^2}-\frac{2c}{(c^2+5)^2}=\frac{2b(a+b+2c)}{(a+2b+c)(b^2+5)}-\frac{2c}{(c^2+5)^2}.$$

We will show that $E'(c) \ge 0$. For the non-trivial case $c>0$, it is equivalent to

$$\frac{b(c^2+5)^2}{c(b^2+5)^2} \ge \frac{a+2b+c}{a+b+2c}, \qquad \frac{b(c^2+5)^2}{c(b^2+5)^2}-1 \ge \frac{a+2b+c}{a+b+2c}-1,$$

$$\frac{(b-c)[25-10bc-bc(b^2+bc+c^2)]}{c(b^2+5)^2} \ge \frac{b-c}{a+b+2c}.$$

It is true if
$$\frac{25-10bc-bc(b^2+bc+c^2)}{c(b^2+5)^2} \ge \frac{1}{a+b+2c}.$$
Since $bc \le 1$ and $a+b+2c \ge 4c$, it suffices to show that
$$\frac{25-10bc-(b^2+bc+c^2)}{(b^2+5)^2} \ge \frac{1}{4},$$
that is
$$75 \ge 14b^2+44bc+4c^2+b^4.$$
From
$$6 = a^2+b^2+c^2+ab+bc+ca \ge b^2+b^2+c^2+b^2+bc+bc = 3b^2+2bc+c^2 > 3b^2,$$
we get $3b^2+2bc+c^2 \le 6$ and $b^2 < 2$. So, we have
$$75-(14b^2+44bc+4c^2+b^4) > 72-(14b^2+44bc+4c^2+2b^2) = 4[18-(4b^2+11bc+c^2)]$$
$$\ge 4[3(3b^2+2bc+c^2)-(4b^2+11bc+c^2)] = 4(5b^2-5bc+2c^2) > 0.$$
Since $E'(c) \ge 0$, $E(c)$ is increasing and has the maximum value when c is maximum, hence when $c=b$. So, we only need to prove the original inequality for $c=b$, that is to show that $a^2+3b^2+2ab=6$ implies
$$\frac{1}{a^2+5}+\frac{2}{b^2+5} \le \frac{1}{2}.$$
The inequality can be written as
$$a^2b^2+a^2+3b^2 \ge 5.$$
Indeed, we have
$$a^2b^2+(a^2+3b^2)-5 = a^2b^2+(6-2ab)-5 = (ab-1)^2 \ge 0.$$
The equality occurs when $a=b=c=1$, and also when $a=\sqrt{3}$ and $b=c=\dfrac{1}{\sqrt{3}}$ (or any cyclic permutation).

□

P 1.241. *If* $a_1, a_2, \ldots, a_{13}$ *are real numbers such that*
$$a_1+a_2+\cdots+a_{13}=13,$$
then
$$\frac{4a_1+7}{a_1^2-2a_1+4}+\frac{4a_2+7}{a_2^2-2a_2+4}+\cdots+\frac{4a_{13}+7}{a_{13}^2-2a_{13}+4} \le \frac{143}{3}.$$
(Vasile Cîrtoaje, 2018)

Solution. Substituting

$$a_i = x_i + 1, \qquad i = 1, 2, \ldots, 13,$$

we need to show that

$$\frac{4x_1 + 11}{x_1^2 + 3} + \frac{4x_2 + 11}{x_2^2 + 3} + \cdots + \frac{4x_{13} + 11}{x_{13}^2 + 3} \le \frac{143}{3}$$

for

$$x_1 + x_2 + \cdots + x_{13} = 0.$$

The inequality is equivalent to

$$\frac{(2x_1 - 1)^2}{x_1^2 + 3} + \frac{(2x_2 - 1)^2}{x_2^2 + 3} + \cdots + \frac{(2x_{13} - 1)^2}{x_{13}^2 + 3} \ge \frac{13}{3}.$$

Let

$$S = x_1^2 + x_2^2 + \cdots + x_{13}^2.$$

Since

$$13(x_1^2+3) = 12x_1^2+x_1^2+39 = 12x_1^2+(x_2+\cdots+x_{13})^2+39 \le 12x_1^2+12(x_2^2+\cdots+x_{13}^2)+39 = 12S+39,$$

it is enough to show that

$$\frac{(2x_1 - 1)^2}{12S + 39} + \frac{(2x_2 - 1)^2}{12S + 39} + \cdots + \frac{(2x_2 - 1)^2}{12S + 39} \ge \frac{1}{3},$$

which is an identity.

The equality occurs for $a_1 = a_2 = \cdots = a_{13} = 1$, and for $a_1 = -5$, $a_2 = \cdots = a_{13} = \dfrac{3}{2}$ (or any cyclic permutation).

□

Chapter 2

Symmetric Nonrational Inequalities

2.1 Applications

2.1. If a, b are nonnegative real numbers such that $a^2 + b^2 \le 1 + \frac{2}{\sqrt{3}}$, then

$$\frac{a}{2a^2+1} + \frac{b}{2b^2+1} \le \frac{\sqrt{2(a^2+b^2)}}{a^2+b^2+1}.$$

2.2. If a, b, c are real numbers, then

$$\sum \sqrt{a^2 - ab + b^2} \le \sqrt{6(a^2+b^2+c^2) - 3(ab+bc+ca)}.$$

2.3. If a, b, c are positive real numbers, then

$$a\sqrt{b+c} + b\sqrt{c+a} + c\sqrt{a+b} \ge \frac{2bc}{\sqrt{b+c}} + \frac{2ca}{\sqrt{c+a}} + \frac{2ab}{\sqrt{a+b}}.$$

2.4. If a, b, c are nonnegative real numbers, then

$$\sqrt{a^2-ab+b^2} + \sqrt{b^2-bc+c^2} + \sqrt{c^2-ca+a^2} \le 3\sqrt{\frac{a^2+b^2+c^2}{2}}.$$

2.5. If a, b, c are nonnegative real numbers, then

$$\sqrt{a^2+b^2-\frac{2}{3}ab} + \sqrt{b^2+c^2-\frac{2}{3}bc} + \sqrt{c^2+a^2-\frac{2}{3}ca} \ge 2\sqrt{a^2+b^2+c^2}.$$

2.6. If a,b,c are nonnegative real numbers, then

$$\sum \sqrt{a^2+ab+b^2} \ge \sqrt{4(a^2+b^2+c^2)+5(ab+bc+ca)}.$$

2.7. If a,b,c are positive real numbers, then

$$\sum \sqrt{a^2+ab+b^2} \le \sqrt{5(a^2+b^2+c^2)+4(ab+bc+ca)}.$$

2.8. If a,b,c are nonnegative real numbers, then

$$\sum \sqrt{a^2+ab+b^2} \le 2\sqrt{a^2+b^2+c^2}+\sqrt{ab+bc+ca}.$$

2.9. If a,b,c are nonnegative real numbers, then

$$\sqrt{a^2+2bc}+\sqrt{b^2+2ca}+\sqrt{c^2+2ab} \le \sqrt{a^2+b^2+c^2}+2\sqrt{ab+bc+ca}.$$

2.10. If a,b,c are nonnegative real numbers, then

$$\frac{1}{\sqrt{a^2+2bc}}+\frac{1}{\sqrt{b^2+2ca}}+\frac{1}{\sqrt{c^2+2ab}} \ge \frac{1}{\sqrt{a^2+b^2+c^2}}+\frac{2}{\sqrt{ab+bc+ca}}.$$

2.11. If a,b,c are positive real numbers, then

$$\sqrt{2a^2+bc}+\sqrt{2b^2+ca}+\sqrt{2c^2+ab} \le 2\sqrt{a^2+b^2+c^2}+\sqrt{ab+bc+ca}.$$

2.12. Let a,b,c be nonnegative real numbers such that $a+b+c=3$. If $k=\sqrt{3}-1$, then

$$\sum \sqrt{a(a+kb)(a+kc)} \le 3\sqrt{3}.$$

2.13. If a,b,c are nonnegative real numbers such that $a+b+c=3$, then

$$\sum \sqrt{a(2a+b)(2a+c)} \ge 9.$$

2.14. Let a,b,c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$\sqrt{b^2+c^2+a(b+c)}+\sqrt{c^2+a^2+b(c+a)}+\sqrt{a^2+b^2+c(a+b)} \ge 6.$$

2.15. Let a, b, c be nonnegative real numbers such that $a+b+c=3$. Prove that

(a) $$\sqrt{a(3a^2+abc)}+\sqrt{b(3b^2+abc)}+\sqrt{c(3c^2+abc)}\ge 6;$$

(b) $$\sqrt{3a^2+abc}+\sqrt{3b^2+abc}+\sqrt{3c^2+abc}\ge 3\sqrt{3+abc}.$$

2.16. Let a, b, c be positive real numbers such that $ab+bc+ca=3$. Prove that

$$a\sqrt{(a+2b)(a+2c)}+b\sqrt{(b+2c)(b+2a)}+c\sqrt{(c+2a)(c+2b)}\ge 9.$$

2.17. Let a, b, c be nonnegative real numbers such that $a+b+c=1$. Prove that

$$\sqrt{a+(b-c)^2}+\sqrt{b+(c-a)^2}+\sqrt{c+(a-b)^2}\ge\sqrt{3}.$$

2.18. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\sqrt{\frac{a(b+c)}{a^2+bc}}+\sqrt{\frac{b(c+a)}{b^2+ca}}+\sqrt{\frac{c(a+b)}{c^2+ab}}\ge 2.$$

2.19. Let a, b, c be positive real numbers such that $abc=1$. Prove that

$$\frac{1}{\sqrt[3]{a^2+25a+1}}+\frac{1}{\sqrt[3]{b^2+25b+1}}+\frac{1}{\sqrt[3]{c^2+25c+1}}\ge 1.$$

2.20. If a, b, c are nonnegative real numbers, then

$$\sqrt{a^2+bc}+\sqrt{b^2+ca}+\sqrt{c^2+ab}\le\frac{3}{2}(a+b+c).$$

2.21. If a, b, c are nonnegative real numbers, then

$$\sqrt{a^2+9bc}+\sqrt{b^2+9ca}+\sqrt{c^2+9ab}\ge 5\sqrt{ab+bc+ca}.$$

2.22. If a, b, c are nonnegative real numbers, then

$$\sum\sqrt{(a^2+4bc)(b^2+4ca)}\ge 5(ab+ac+bc).$$

2.23. If a, b, c are nonnegative real numbers, then

$$\sum \sqrt{(a^2+9bc)(b^2+9ca)} \ge 7(ab+ac+bc).$$

2.24. If a, b, c are nonnegative real numbers, then

$$\sum \sqrt{(a^2+b^2)(b^2+c^2)} \le (a+b+c)^2.$$

2.25. If a, b, c are nonnegative real numbers, then

$$\sum \sqrt{(a^2+ab+b^2)(b^2+bc+c^2)} \ge (a+b+c)^2.$$

2.26. If a, b, c are nonnegative real numbers, then

$$\sum \sqrt{(a^2+7ab+b^2)(b^2+7bc+c^2)} \ge 7(ab+ac+bc).$$

2.27. If a, b, c are nonnegative real numbers, then

$$\sum \sqrt{\left(a^2+\frac{7}{9}ab+b^2\right)\left(b^2+\frac{7}{9}bc+c^2\right)} \le \frac{13}{12}(a+b+c)^2.$$

2.28. If a, b, c are nonnegative real numbers, then

$$\sum \sqrt{\left(a^2+\frac{1}{3}ab+b^2\right)\left(b^2+\frac{1}{3}bc+c^2\right)} \le \frac{61}{60}(a+b+c)^2.$$

2.29. If a, b, c are nonnegative real numbers, then

$$\frac{a}{\sqrt{4b^2+bc+4c^2}} + \frac{b}{\sqrt{4c^2+ca+4a^2}} + \frac{c}{\sqrt{4a^2+ab+4b^2}} \ge 1.$$

2.30. If a, b, c are nonnegative real numbers, then

$$\frac{a}{\sqrt{b^2+bc+c^2}} + \frac{b}{\sqrt{c^2+ca+a^2}} + \frac{c}{\sqrt{a^2+ab+b^2}} \ge \frac{a+b+c}{\sqrt{ab+bc+ca}}.$$

2.31. If a, b, c are nonnegative real numbers, then

$$\frac{a}{\sqrt{a^2+2bc}}+\frac{b}{\sqrt{b^2+2ca}}+\frac{c}{\sqrt{c^2+2ab}} \le \frac{a+b+c}{\sqrt{ab+bc+ca}}.$$

2.32. If a, b, c are nonnegative real numbers, then

$$a^3+b^3+c^3+3abc \ge a^2\sqrt{a^2+3bc}+b^2\sqrt{b^2+3ca}+c^2\sqrt{c^2+3ab}.$$

2.33. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a}{\sqrt{4a^2+5bc}}+\frac{b}{\sqrt{4b^2+5ca}}+\frac{c}{\sqrt{4c^2+5ab}} \le 1.$$

2.34. Let a, b, c be nonnegative real numbers. Prove that

$$a\sqrt{4a^2+5bc}+b\sqrt{4b^2+5ca}+c\sqrt{4c^2+5ab} \ge (a+b+c)^2.$$

2.35. Let a, b, c be nonnegative real numbers. Prove that

$$a\sqrt{a^2+3bc}+b\sqrt{b^2+3ca}+c\sqrt{c^2+3ab} \ge 2(ab+bc+ca).$$

2.36. Let a, b, c be nonnegative real numbers. Prove that

$$a\sqrt{a^2+8bc}+b\sqrt{b^2+8ca}+c\sqrt{c^2+8ab} \le (a+b+c)^2.$$

2.37. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a^2+2bc}{\sqrt{b^2+bc+c^2}}+\frac{b^2+2ca}{\sqrt{c^2+ca+a^2}}+\frac{c^2+2ab}{\sqrt{a^2+ab+b^2}} \ge 3\sqrt{ab+bc+ca}.$$

2.38. Let a, b, c be nonnegative real numbers, no two of which are zero. If $k \ge 1$, then

$$\frac{a^{k+1}}{2a^2+bc}+\frac{b^{k+1}}{2b^2+ca}+\frac{c^{k+1}}{2c^2+ab} \le \frac{a^k+b^k+c^k}{a+b+c}.$$

2.39. If a, b, c are positive real numbers, then

(a) $$\frac{a^2-bc}{\sqrt{3a^2+2bc}}+\frac{b^2-ca}{\sqrt{3b^2+2ca}}+\frac{c^2-ab}{\sqrt{3c^2+2ab}} \ge 0;$$

(b) $$\frac{a^2-bc}{\sqrt{8a^2+(b+c)^2}}+\frac{b^2-ca}{\sqrt{8b^2+(c+a)^2}}+\frac{c^2-ab}{\sqrt{8c^2+(a+b)^2}} \ge 0.$$

2.40. Let a, b, c be positive real numbers. If $0 \le k \le 1+2\sqrt{2}$, then

$$\frac{a^2-bc}{\sqrt{ka^2+b^2+c^2}}+\frac{b^2-ca}{\sqrt{kb^2+c^2+a^2}}+\frac{c^2-ab}{\sqrt{kc^2+a^2+b^2}} \ge 0.$$

2.41. If a, b, c are nonnegative real numbers, then

$$(a^2-bc)\sqrt{b+c}+(b^2-ca)\sqrt{c+a}+(c^2-ab)\sqrt{a+b} \ge 0.$$

2.42. If a, b, c are nonnegative real numbers, then

$$(a^2-bc)\sqrt{a^2+4bc}+(b^2-ca)\sqrt{b^2+4ca}+(c^2-ab)\sqrt{c^2+4ab} \ge 0.$$

2.43. If a, b, c are nonnegative real numbers, then

$$\sqrt{\frac{a^3}{a^3+(b+c)^3}}+\sqrt{\frac{b^3}{b^3+(c+a)^3}}+\sqrt{\frac{c^3}{c^3+(a+b)^3}} \ge 1.$$

2.44. If a, b, c are positive real numbers, then

$$\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} \ge 1+\sqrt{1+\sqrt{(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}}.$$

2.45. If a, b, c are positive real numbers, then

$$5+\sqrt{2(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)-2} \ge (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

2.46. If a, b, c are real numbers, then

$$2(1+abc)+\sqrt{2(1+a^2)(1+b^2)(1+c^2)} \ge (1+a)(1+b)(1+c).$$

2.47. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\sqrt{\frac{a^2+bc}{b^2+c^2}}+\sqrt{\frac{b^2+ca}{c^2+a^2}}+\sqrt{\frac{c^2+ab}{a^2+b^2}} \ge 2+\frac{1}{\sqrt{2}}.$$

2.48. If a, b, c are nonnegative real numbers, then

$$\sqrt{a(2a+b+c)}+\sqrt{b(2b+c+a)}+\sqrt{c(2c+a+b)} \ge \sqrt{12(ab+bc+ca)}.$$

2.49. Let a, b, c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$a\sqrt{(4a+5b)(4a+5c)}+b\sqrt{(4b+5c)(4b+5a)}+c\sqrt{(4c+5a)(4c+5b)} \ge 27.$$

2.50. Let a, b, c be nonnegative real numbers such that $ab+bc+ca=3$. Prove that

$$a\sqrt{(a+3b)(a+3c)}+b\sqrt{(b+3c)(b+3a)}+c\sqrt{(c+3a)(c+3b)} \ge 12.$$

2.51. Let a, b, c be nonnegative real numbers such that $a^2+b^2+c^2=3$. Prove that

$$\sqrt{2+7ab}+\sqrt{2+7bc}+\sqrt{2+7ca} \ge 3\sqrt{3(ab+bc+ca)}.$$

2.52. Let a, b, c be nonnegative real numbers such that $a^2+b^2+c^2=3$. Prove that

$$\frac{a}{2a^2+1}+\frac{b}{2b^2+1}+\frac{c}{2c^2+1} \le 1.$$

2.53. Let a, b, c be nonnegative real numbers such that $ab+bc+ca=3$. Prove that

(a) $$\sum \sqrt{a(b+c)(a^2+bc)} \ge 6;$$

(b) $$\sum a(b+c)\sqrt{a^2+2bc} \ge 6\sqrt{3};$$

(c) $$\sum a(b+c)\sqrt{(a+2b)(a+2c)} \ge 18.$$

2.54. Let a, b, c be nonnegative real numbers such that $ab+bc+ca=3$. Prove that

$$a\sqrt{bc+3}+b\sqrt{ca+3}+c\sqrt{ab+3}\ge 6.$$

2.55. Let a, b, c be nonnegative real numbers such that $a+b+c=3$. Prove that

(a)
$$\sum(b+c)\sqrt{b^2+c^2+7bc}\ge 18;$$

(b)
$$\sum(b+c)\sqrt{b^2+c^2+10bc}\le 12\sqrt{3}.$$

2.56. Let a, b, c be nonnegative real numbers such then $a+b+c=2$. Prove that

$$\sqrt{a+4bc}+\sqrt{b+4ca}+\sqrt{c+4ab}\ge 4\sqrt{ab+bc+ca}.$$

2.57. If a, b, c are nonnegative real numbers, then

$$\sqrt{a^2+b^2+7ab}+\sqrt{b^2+c^2+7bc}+\sqrt{c^2+a^2+7ca}\ge 5\sqrt{ab+bc+ca}.$$

2.58. If a, b, c are nonnegative real numbers, then

$$\sqrt{a^2+b^2+5ab}+\sqrt{b^2+c^2+5bc}+\sqrt{c^2+a^2+5ca}\ge \sqrt{21(ab+bc+ca)}.$$

2.59. Let a, b, c be nonnegative real numbers such that $ab+bc+ca=3$. Prove that

$$a\sqrt{a^2+5}+b\sqrt{b^2+5}+c\sqrt{c^2+5}\ge \sqrt{\frac{2}{3}}(a+b+c)^2.$$

2.60. Let a, b, c be nonnegative real numbers such that $a^2+b^2+c^2=1$. Prove that

$$a\sqrt{2+3bc}+b\sqrt{2+3ca}+c\sqrt{2+3ab}\ge (a+b+c)^2.$$

2.61. Let a, b, c be nonnegative real numbers such that $a+b+c=3$. Prove that

(a)
$$a\sqrt{\frac{2a+bc}{3}}+b\sqrt{\frac{2b+ca}{3}}+c\sqrt{\frac{2c+ab}{3}}\ge 3;$$

(b)
$$a\sqrt{\frac{a(1+b+c)}{3}}+b\sqrt{\frac{b(1+c+a)}{3}}+c\sqrt{\frac{c(1+a+b)}{3}}\ge 3.$$

2.62. If a, b, c are nonnegative real numbers such that $a+b+c=3$, then

$$\sqrt{8(a^2+bc)+9}+\sqrt{8(b^2+ca)+9}+\sqrt{8(c^2+ab)+9} \geq 15.$$

2.63. Let a, b, c be nonnegative real numbers such that $a+b+c=3$. If $k \geq \frac{9}{8}$, then

$$\sqrt{a^2+bc+k}+\sqrt{b^2+ca+k}+\sqrt{c^2+ab+k} \geq 3\sqrt{2+k}.$$

2.64. If a, b, c are nonnegative real numbers such that $a+b+c=3$, then

$$\sqrt{a^3+2bc}+\sqrt{b^3+2ca}+\sqrt{c^3+2ab} \geq 3\sqrt{3}.$$

2.65. If a, b, c are positive real numbers, then

$$\frac{\sqrt{a^2+bc}}{b+c}+\frac{\sqrt{b^2+ca}}{c+a}+\frac{\sqrt{c^2+ab}}{a+b} \geq \frac{3\sqrt{2}}{2}.$$

2.66. If a, b, c are nonnegative real numbers, no two of which are zero,then

$$\frac{\sqrt{bc+4a(b+c)}}{b+c}+\frac{\sqrt{ca+4b(c+a)}}{c+a}+\frac{\sqrt{ab+4c(a+b)}}{a+b} \geq \frac{9}{2}.$$

2.67. If a, b, c are nonnegative real numbers, no two of which are zero,then

$$\frac{a\sqrt{a^2+3bc}}{b+c}+\frac{b\sqrt{b^2+3ca}}{c+a}+\frac{c\sqrt{c^2+3ab}}{a+b} \geq a+b+c.$$

2.68. If a, b, c are nonnegative real numbers, no two of which are zero,then

$$\sqrt{\frac{2a(b+c)}{(2b+c)(b+2c)}}+\sqrt{\frac{2b(c+a)}{(2c+a)(c+2a)}}+\sqrt{\frac{2c(a+b)}{(2a+b)(a+2b)}} \geq 2.$$

2.69. If a, b, c are nonnegative real numbers such that $ab+bc+ca=3$, then

$$\sqrt{\frac{bc}{3a^2+6}}+\sqrt{\frac{ca}{3b^2+6}}+\sqrt{\frac{ab}{3c^2+6}} \leq 1 \leq \sqrt{\frac{bc}{6a^2+3}}+\sqrt{\frac{ca}{6b^2+3}}+\sqrt{\frac{ab}{6c^2+3}}.$$

2.70. Let a, b, c be nonnegative real numbers such that $ab + bc + ca = 3$. If $k > 1$, than

$$a^k(b + c) + b^k(c + a) + c^k(a + b) \geq 6.$$

2.71. Let a, b, c be nonnegative real numbers such that $a + b + c = 2$. If

$$2 \leq k \leq 3,$$

than

$$a^k(b + c) + b^k(c + a) + c^k(a + b) \leq 2.$$

2.72. Let a, b, c be nonnegative real numbers, no two of which are zero. If $m > n \geq 0$, than

$$\frac{b^m + c^m}{b^n + c^n}(b + c - 2a) + \frac{c^m + a^m}{c^n + a^n}(c + a - 2b) + \frac{a^m + b^m}{a^n + b^n}(a + b - 2c) \geq 0.$$

2.73. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\sqrt{a^2 - a + 1} + \sqrt{a^2 - a + 1} + \sqrt{a^2 - a + 1} \geq a + b + c.$$

2.74. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\sqrt{16a^2 + 9} + \sqrt{16b^2 + 9} + \sqrt{16b^2 + 9} \geq 4(a + b + c) + 3.$$

2.75. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\sqrt{25a^2 + 144} + \sqrt{25b^2 + 144} + \sqrt{25c^2 + 144} \leq 5(a + b + c) + 24.$$

2.76. If a, b are positive real numbers such that $ab + bc + ca = 3$, then

(a) $$\sqrt{a^2 + 3} + \sqrt{b^2 + 3} + \sqrt{b^2 + 3} \geq a + b + c + 3;$$

(b) $$\sqrt{a + b} + \sqrt{b + c} + \sqrt{c + a} \geq \sqrt{4(a + b + c) + 6}.$$

2.77. If a, b, c are nonnegative real numbers such that $a + b + c = 3$, then

$$\sqrt{(5a^2 + 3)(5b^2 + 3)} + \sqrt{(5b^2 + 3)(5c^2 + 3)} + \sqrt{(5c^2 + 3)(5a^2 + 3)} \geq 24.$$

2.78. If a, b, c are nonnegative real numbers such that $a+b+c=3$, then

$$\sqrt{a^2+1}+\sqrt{b^2+1}+\sqrt{c^2+1} \ge \sqrt{\frac{4(a^2+b^2+c^2)+42}{3}}.$$

2.79. If a, b, c are nonnegative real numbers such that $a+b+c=3$, then

(a) $$\sqrt{a^2+3}+\sqrt{b^2+3}+\sqrt{c^2+3} \ge \sqrt{2(a^2+b^2+c^2)+30};$$

(b) $$\sqrt{3a^2+1}+\sqrt{3b^2+1}+\sqrt{3c^2+1} \ge \sqrt{2(a^2+b^2+c^2)+30}.$$

2.80. If a, b, c are nonnegative real numbers such that $a+b+c=3$, then

$$\sqrt{(32a^2+3)(32b^2+3)}+\sqrt{(32b^2+3)(32c^2+3)}+\sqrt{(32c^2+3)(32a^2+3)} \le 105.$$

2.81. If a, b, c are positive real numbers, then

$$\left|\frac{b+c}{a}-3\right|+\left|\frac{c+a}{b}-3\right|+\left|\frac{a+b}{c}-3\right| \ge 2.$$

2.82. If a, b, c are real numbers such that $abc \ne 0$, then

$$\left|\frac{b+c}{a}\right|+\left|\frac{c+a}{b}\right|+\left|\frac{a+b}{c}\right| \ge 2.$$

2.83. Let a, b, c be nonnegative real numbers, no two of which are zero, and let

$$x=\frac{2a}{b+c}, \quad y=\frac{2b}{c+a}, \quad z=\frac{2c}{a+b}.$$

Prove that

(a) $$\sqrt{xy}+\sqrt{yz}+\sqrt{zx} \ge xyz+2;$$

(b) $$x+y+z+\sqrt{xy}+\sqrt{yz}+\sqrt{zx} \ge 6;$$

(c) $$\sqrt{x}+\sqrt{y}+\sqrt{z} \ge \sqrt{8+xyz};$$

(d) $$\frac{\sqrt{yz}}{x+2}+\frac{\sqrt{zx}}{y+2}+\frac{\sqrt{xy}}{z+2} \ge 1.$$

2.84. Let a, b, c be nonnegative real numbers, no two of which are zero, and let

$$x = \frac{2a}{b+c}, \quad y = \frac{2b}{c+a}, \quad z = \frac{2c}{a+b}.$$

Prove that

$$\sqrt{1+24x} + \sqrt{1+24y} + \sqrt{1+24z} \ge 15.$$

2.85. If a, b, c are positive real numbers, then

$$\sqrt{\frac{7a}{a+3b+3c}} + \sqrt{\frac{7b}{b+3c+3a}} + \sqrt{\frac{7c}{c+3a+3b}} \le 3.$$

2.86. If a, b, c are positive real numbers such that $a+b+c=3$, then

$$\sqrt[3]{a^2(b^2+c^2)} + \sqrt[3]{b^2(c^2+a^2)} + \sqrt[3]{c^2(a^2+b^2)} \le 3\sqrt[3]{2}.$$

2.87. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \ge \frac{1}{a+b+c} + \frac{2}{\sqrt{ab+bc+ca}}.$$

2.88. If $a, b \ge 1$, then

$$\frac{1}{\sqrt{3ab+1}} + \frac{1}{2} \ge \frac{1}{\sqrt{3a+1}} + \frac{1}{\sqrt{3b+1}}.$$

2.89. Let a, b, c be positive real numbers such that $a+b+c=3$. If $k \ge \frac{1}{\sqrt{2}}$, then

$$(abc)^k(a^2+b^2+c^2) \le 3.$$

2.90. If $a, b, c \in [0, 4]$ and $ab+bc+ca=4$, then

$$\sqrt{a+b} + \sqrt{b+c} + \sqrt{c+a} \le 3 + \sqrt{5}.$$

2.91. Let a, b, c be positive real numbers such that $a \le b \le c$ and $a^4bc \ge 1$, and let

$$F(a,b,c) = \sqrt{\frac{a^2+b^2+c^2}{3}} - \frac{a+b+c}{3}.$$

Then,

$$F(a,b,c) \ge F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

2.92. Let a, b, c be positive real numbers such that $a \le b \le c$ and $a^2(b+c) \ge 2$, and let

$$F(a,b,c) = \sqrt{\frac{a^2+b^2+c^2}{3}} - \frac{a+b+c}{3}.$$

Then,

$$F(a,b,c) \ge F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

2.93. Let a, b, c be positive real numbers such that $a \le b \le c$ and $a^4(b^2+c^2) \ge 2$, and let

$$F(a,b,c) = \sqrt{\frac{a^2+b^2+c^2}{3}} - \frac{a+b+c}{3}.$$

Then,

$$F(a,b,c) \ge F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

2.94. Let a, b, c be positive real numbers such that $a \ge b \ge c$ and $a^4b^7c^7 \ge 1$, and let

$$F(a,b,c) = \sqrt[3]{abc} - \frac{3}{\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}}.$$

Then,

$$F(a,b,c) \ge F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

2.95. Let a, b, c be positive real numbers such that $a = \min\{a,b,c\}$ and $a^4bc \ge 1$, and let

$$F(a,b,c) = \sqrt[3]{abc} - \frac{3}{\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}}.$$

Prove that

$$F(a,b,c) \ge F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

2.96. Let a, b, c be positive real numbers such that $a \le b \le c$ and $a^4bc \ge 1$, and let

$$F(a,b,c) = \frac{a+b+c}{3} - \sqrt{\frac{ab+bc+ca}{3}}.$$

Then,

$$F(a,b,c) \ge F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

2.97. Let a, b, c be positive real numbers such that $a \ge b \ge c$ and $a^2b^5c^5 \ge 1$, and let

$$F(a,b,c) = \sqrt{\frac{ab+bc+ca}{3}} - \sqrt[3]{abc}.$$

Prove that

$$F(a,b,c) \ge F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

2.98. Let a, b, c be positive real numbers such that $\min\{ab, bc, ca\} \ge 1$, and let

$$F(a,b,c) = \sqrt{\frac{3}{ab+bc+ca}} - \frac{3}{a+b+c}.$$

Prove that

$$F(a,b,c) \le F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

2.99. Let a, b, c, d be positive real numbers such that $ab \ge 1$ and $cd \ge 1$, and let

$$F(a,b,c,d) = \sqrt[4]{abcd} - \frac{4}{\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}+\dfrac{1}{d}}.$$

Then,

$$F(a,b,c,d) \ge F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}\right).$$

2.100. Let a, b, c, d be nonnegative real numbers such that $a^2+b^2+c^2+d^2 = 1$. Prove that

$$\sqrt{1-a}+\sqrt{1-b}+\sqrt{1-c}+\sqrt{1-d} \ge \sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}.$$

2.101. Let a, b, c, d be positive real numbers, and let

$$A = (a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right), \qquad B = (a^2+b^2+c^2+d^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}\right).$$

Prove that

(a) $$\sqrt{B-12} \le A-14;$$

(b) $$2\sqrt{B} \ge A-8.$$

2.102. Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1+a_2+\cdots+a_n = 1$. Prove that

$$\sqrt{3a_1+1}+\sqrt{3a_2+1}+\cdots+\sqrt{3a_n+1} \ge n+1.$$

2.103. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1a_2\cdots a_n = 1$. Prove that

$$\frac{1}{\sqrt{1+(n^2-1)a_1}}+\frac{1}{\sqrt{1+(n^2-1)a_2}}+\cdots+\frac{1}{\sqrt{1+(n^2-1)a_n}} \ge 1.$$

2.104. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1a_2\cdots a_n = 1$. Prove that

$$\sum_{i=1}^{n}\frac{1}{1+\sqrt{1+4n(n-1)a_i}} \ge \frac{1}{2}.$$

2.105. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1a_2\cdots a_n = 1$, then

$$a_1+a_2+\cdots+a_n \ge n-1+\sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}}.$$

2.106. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1a_2\cdots a_n = 1$, then

$$\sqrt{(n-1)(a_1^2+a_2^2+\cdots+a_n^2)}+n-\sqrt{n(n-1)} \ge a_1+a_2+\cdots+a_n.$$

2.107. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1a_2\cdots a_n \ge 1$. If $k > 1$, then

$$\sum\frac{a_1^k}{a_1^k+a_2+\cdots+a_n} \ge 1.$$

2.108. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 a_2 \cdots a_n \ge 1$. If

$$\frac{-2}{n-2} \le k < 1,$$

then

$$\sum \frac{a_1^k}{a_1^k + a_2 + \cdots + a_n} \le 1.$$

2.109. Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n \ge n$. If $1 < k \le n+1$, then

$$\sum \frac{a_1}{a_1^k + a_2 + \cdots + a_n} \le 1.$$

2.110. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 a_2 \cdots a_n \ge 1$. If $k > 1$, then

$$\sum \frac{a_1}{a_1^k + a_2 + \cdots + a_n} \le 1.$$

2.111. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 a_2 \cdots a_n \ge 1$. If

$$-1 - \frac{2}{n-2} \le k < 1,$$

then

$$\sum \frac{a_1}{a_1^k + a_2 + \cdots + a_n} \ge 1.$$

2.112. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 a_2 \cdots a_n = 1$. If $k \ge 0$, then

$$\sum \frac{1}{a_1^k + a_2 + \cdots + a_n} \le 1.$$

2.113. Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n \le n$. If $0 \le k < 1$, then

$$\frac{1}{a_1^k + a_2 + \cdots + a_n} + \frac{1}{a_1 + a_2^k + \cdots + a_n} + \cdots + \frac{1}{a_1 + a_2 + \cdots + a_n^k} \ge 1.$$

2.114. Let $a_1, a_2, \ldots, a_n$ be positive real numbers. If $k > 1$, then

$$\sum \frac{a_2^k + a_3^k + \cdots + a_n^k}{a_2 + a_3 + \cdots + a_n} \le \frac{n(a_1^k + a_2^k + \cdots + a_n^k)}{a_1 + a_2 + \cdots + a_n}.$$

2.115. Let f be a convex function on the closed interval $[a,b]$, and let $a_1,a_2,\dots,a_n\in[a,b]$ such that

$$a_1+a_2+\cdots+a_n=pa+qb,$$

where $p,q\ge 0$ such that $p+q=n$. Prove that

$$f(a_1)+f(a_2)+\cdots+f(a_n)\le pf(a)+qf(b).$$

2.116. If a,b,c are positive real numbers such that

$$a\ge 1\ge b\ge c,\qquad abc=1,$$

then

$$\frac{1}{\sqrt{3a+1}}+\frac{1}{\sqrt{3b+1}}+\frac{1}{\sqrt{3c+1}}\ge\frac{3}{2}.$$

2.117. If $a_1,a_2,\dots,a_n$ are nonnegative real numbers such that

$$a_1\le 1\le a_2\le\cdots\le a_n,\qquad a_1+a_2+\cdots+a_n=n,$$

then

$$\sqrt{\frac{3a_1}{4-a_1}}+\sqrt{\frac{3a_2}{4-a_2}}+\cdots+\sqrt{\frac{3a_n}{4-a_n}}\le n.$$

2.118. If a,b,c are positive real numbers and

$$x=\frac{2a}{b+c},\qquad y=\frac{2b}{c+a},\qquad z=\frac{2c}{a+b},$$

then

$$\frac{1}{\sqrt{5x+4}}+\frac{1}{\sqrt{5y+4}}+\frac{1}{\sqrt{5z+4}}\ge 1.$$

2.119. If a,b,c are nonnegative real numbers such that $ab+bc+ca=3$, then

$$\sqrt{(a+3b)(a+3c)}+\sqrt{(b+3c)(b+3a)}+\sqrt{(c+3a)(c+3b)}\ge 12.$$

2.120. If a,b,c are nonnegative real numbers such that $ab+bc+ca=3$, then

$$\sqrt{\frac{2a+1}{3}}+\sqrt{\frac{2b+1}{3}}+\sqrt{\frac{2c+1}{3}}\ge 3.$$

2.121. If a, b, c are the lengths of the sides of a triangle such that $a+b+c=2$, then

$$\sqrt{\frac{a}{a^2+bc}}+\sqrt{\frac{b}{b^2+ca}}+\sqrt{\frac{c}{c^2+ab}} \ge 2.$$

2.122. For given $n \ge 3$, prove that $\frac{2n-1}{n-1}$ is the least positive value of the constant k such that

$$\sum_{cyclic} \sqrt{\frac{a_2+a_3+\cdots+a_n}{ka_1+a_2+a_3+\cdots+a_n}} \ge n\sqrt{\frac{n-1}{k+n-1}}$$

holds for any nonnegative real numbers $a_1, a_2, \ldots, a_n$ with $a_1+a_2+\cdots+a_n > 0$.

2.2 Solutions

P 2.1. *If a, b are nonnegative real numbers such that $a^2+b^2 \le 1+\frac{2}{\sqrt{3}}$, then*

$$\frac{a}{2a^2+1}+\frac{b}{2b^2+1} \le \frac{\sqrt{2(a^2+b^2)}}{a^2+b^2+1}.$$

(Vasile Cîrtoaje, 2012)

Solution. With

$$s=\frac{a^2+b^2}{2}, \quad p=ab, \quad 0 \le p \le s \le \frac{1}{2}+\frac{1}{\sqrt{3}},$$

the inequality becomes as follows:

$$\frac{(2p+1)\sqrt{2(s+p)}}{4p^2+4s+1} \le \frac{2\sqrt{s}}{2s+1},$$

$$\sqrt{\frac{2s}{s+p}}-1 \ge \frac{(2p+1)(2s+1)}{4p^2+4s+1}-1,$$

$$\frac{s-p}{(s+p)\left(\sqrt{\frac{2s}{s+p}}+1\right)} \ge \frac{2(s-p)(2p-1)}{4p^2+4s+1}.$$

Thus, we need to show that

$$\frac{1}{(s+p)\left(\sqrt{\frac{2s}{s+p}}+1\right)} \ge \frac{2(2p-1)}{4p^2+4s+1}.$$

Since $\sqrt{\frac{2s}{s+p}} \ge 1$, it suffices to show that

$$\frac{1}{(s+p)\left(\sqrt{\frac{2s}{s+p}}+\sqrt{\frac{2s}{s+p}}\right)} \ge \frac{2(2p-1)}{4p^2+4s+1},$$

which is equivalent to

$$4p^2+4s+1 \ge 4(2p-1)\sqrt{2s(s+p)}.$$

For the nontrivial case $2p-1>0$, which involves $2s-1>0$, since $2\sqrt{2s(s+p)} \le 2s+(s+p)$, it suffices to show that

$$4p^2+4s+1 \ge 2(2p-1)(3s+p),$$

that is

$$10s+1 \ge 2p(6s-1).$$

We have

$$10s+1-2p(6s-1)\ge 10s+1-2s(6s-1)=1+12s-12s^2\ge 0.$$

The equality holds for $a=b$.

□

P 2.2. *If a,b,c are real numbers, then*

$$\sum\sqrt{a^2-ab+b^2}\le\sqrt{6(a^2+b^2+c^2)-3(ab+bc+ca)}.$$

Solution. By squaring, the inequality becomes as follows:

$$2(ab+bc+ca)+2\sum\sqrt{(a^2-ab+b^2)(a^2-ac+c^2)}\le 4(a^2+b^2+c^2),$$

$$\sum\left(\sqrt{a^2-ab+b^2}-\sqrt{a^2-ac+c^2}\right)^2\ge 0.$$

The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 2.3. *If a,b,c are positive real numbers, then*

$$a\sqrt{b+c}+b\sqrt{c+a}+c\sqrt{a+b}\ge\frac{2bc}{\sqrt{b+c}}+\frac{2ca}{\sqrt{c+a}}+\frac{2ab}{\sqrt{a+b}}.$$

(Lorian Saceanu, 2015)

Solution. Use the SOS method. Write the inequality as follows:

$$\sum a\sqrt{b+c}-\sum\frac{2bc}{\sqrt{b+c}}\ge 0,$$

$$\sum\frac{a(b+c)-2bc}{\sqrt{b+c}}\ge 0,$$

$$\sum\frac{b(a-c)}{\sqrt{b+c}}+\sum\frac{c(a-b)}{\sqrt{b+c}}\ge 0,$$

$$\sum\frac{c(b-a)}{\sqrt{c+a}}+\sum\frac{c(a-b)}{\sqrt{b+c}}\ge 0,$$

$$\sum c(a-b)\left(\frac{1}{\sqrt{b+c}}-\frac{1}{\sqrt{c+a}}\right)\ge 0,$$

$$\sum\frac{c(a-b)^2}{\sqrt{(b+c)(c+a)}\left(\sqrt{b+c}+\sqrt{c+a}\right)}\ge 0.$$

The equality holds for $a=b=c$.

□

P 2.4. *If a, b, c are nonnegative real numbers, then*

$$\sqrt{a^2-ab+b^2}+\sqrt{b^2-bc+c^2}+\sqrt{c^2-ca+a^2} \le 3\sqrt{\frac{a^2+b^2+c^2}{2}}.$$

Solution *(by Nguyen Van Quy).* Assume that $c = \min\{a, b, c\}$. Since

$$b^2-bc+c^2 \le b^2$$

and

$$c^2-ca+a^2 \le a^2,$$

it suffices to show that

$$\sqrt{a^2-ab+b^2}+b+a \le 3\sqrt{\frac{a^2+b^2+c^2}{2}}.$$

Using the Cauchy-Schwarz inequality, we have

$$\sqrt{a^2-ab+b^2}+a+b \le \sqrt{\left[(a^2-ab+b^2)+\frac{(a+b)^2}{k}\right](1+k)}$$

$$= \sqrt{\frac{(1+k)[(1+k)(a^2+b^2)+(2-k)ab]}{k}}, \qquad k > 0.$$

Choosing $k = 2$, we get

$$\sqrt{a^2-ab+b^2}+a+b \le 3\sqrt{\frac{a^2+b^2}{2}} \le 3\sqrt{\frac{a^2+b^2+c^2}{2}} = 3.$$

The equality holds for $a = b$ and $c = 0$ (or any cyclic permutation).

□

P 2.5. *If a, b, c are nonnegative real numbers, then*

$$\sqrt{a^2+b^2-\frac{2}{3}ab}+\sqrt{b^2+c^2-\frac{2}{3}bc}+\sqrt{c^2+a^2-\frac{2}{3}ca} \ge 2\sqrt{a^2+b^2+c^2}.$$

(Vasile Cîrtoaje, 2012)

First Solution. By squaring, the inequality becomes

$$2\sum \sqrt{(3a^2+3b^2-2ab)(3a^2+3c^2-2ac)} \ge 6(a^2+b^2+c^2)+2(ab+bc+ca),$$

$$6(a^2+b^2+c^2-ab-bc-ca) \ge \sum \left(\sqrt{3a^2+3b^2-2ab}-\sqrt{3a^2+3c^2-2ac}\right)^2,$$

$$3\sum(b-c)^2 \ge \sum \frac{(b-c)^2(3b+3c-2a)^2}{\left(\sqrt{3a^2+3b^2-2ab}+\sqrt{3a^2+3c^2-2ac}\right)^2},$$

$$\sum(b-c)^2\left[1-\frac{(3b+3c-2a)^2}{\left(\sqrt{9a^2+9b^2-6ab}+\sqrt{9a^2+9c^2-6ac}\right)^2}\right].$$

Since

$$\sqrt{9a^2+9b^2-6ab}=\sqrt{(3b-a)^2+8a^2}\ge|3b-a|,$$

$$\sqrt{9a^2+9c^2-6ac}=\sqrt{(3c-a)^2+8a^2}\ge|3c-a|,$$

it suffices to show that

$$\sum(b-c)^2\left[1-\left(\frac{|3b+3c-2a|}{|3b-a|+|3c-a|}\right)^2\right]\ge 0.$$

This is true since

$$|3b+3c-2a|=|(3b-a)+(3c-a)|\le|3b-a|+|3c-a|.$$

The equality holds for $a=b=c$, and also for $b=c=0$ (or any cyclic permutation).

Second Solution. Assume that $a\ge b\ge c$. Write the inequality as

$$\sqrt{(a+b)^2+2(a-b)^2}+\sqrt{(b+c)^2+2(b-c)^2}+\sqrt{(a+c)^2+2(a-c)^2}\ge$$

$$\ge 2\sqrt{3(a^2+b^2+c^2)}.$$

By Minkowski's inequality, it suffices to show that

$$\sqrt{[(a+b)+(b+c)+(a+c)]^2+2[(a-b)+(b-c)+(a-c)]^2}\ge 2\sqrt{3(a^2+b^2+c^2)},$$

which is equivalent to

$$\sqrt{(a+b+c)^2+2(a-c)^2}\ge\sqrt{3(a^2+b^2+c^2)}.$$

By squaring, the inequality turns into

$$(a-b)(b-c)\ge 0.$$

□

P 2.6. *If a,b,c are nonnegative real numbers, then*

$$\sum\sqrt{a^2+ab+b^2}\ge\sqrt{4(a^2+b^2+c^2)+5(ab+bc+ca)}.$$

(Vasile Cîrtoaje, 2009)

First Solution. By squaring, the inequality becomes

$$\sum \sqrt{(a^2+ab+b^2)(a^2+ac+c^2)} \ge (a+b+c)^2.$$

Using the Cauchy-Schwarz inequality, we get

$$\sum \sqrt{(a^2+ab+b^2)(a^2+ac+c^2)} = \sum \sqrt{\left[\left(a+\frac{b}{2}\right)^2+\frac{3b^2}{4}\right]\left[\left(a+\frac{c}{2}\right)^2+\frac{3c^2}{4}\right]}$$

$$\ge \sum \left[\left(a+\frac{b}{2}\right)\left(a+\frac{c}{2}\right)+\frac{3bc}{4}\right] = (a+b+c)^2.$$

The equality holds for $a=b=c$, and also for $b=c=0$ (or any cyclic permutation).

Second Solution. Assume that $a \ge b \ge c$. By Minkowski's inequality, we get

$$2\sum \sqrt{a^2+ab+b^2} = \sum \sqrt{3(a+b)^2+(a-b)^2}$$

$$\ge \sqrt{3[(a+b)+(b+c)+(c+a)]^2+[(a-b)+(b-c)+(a-c)]^2}$$
$$= 2\sqrt{3(a+b+c)^2+(a-c)^2}.$$

Therefore, it suffices to show that

$$3(a+b+c)^2+(a-c)^2 \ge 4(a^2+b^2+c^2)+5(ab+bc+ca),$$

which is equivalent to the obvious inequality

$$(a-b)(b-c) \ge 0.$$

Remark. Similarly, we can prove the following generalization:

- *Let a,b,c be nonnegative real numbers. If $|k| \le 2$, then*

$$\sum \sqrt{a^2+kab+b^2} \ge \sqrt{4(a^2+b^2+c^2)+(3k+2)(ab+bc+ca)},$$

with equality for $a=b=c$, and also for $b=c=0$ (or any cyclic permutation).

For $k=-2/3$ and $k=1$, we get the inequalities in P 2.5 and P 2.6, respectively. For $k=-1$ and $k=0$, we get the inequalities

$$\sum \sqrt{a^2-ab+b^2} \ge \sqrt{4(a^2+b^2+c^2)-ab-bc-ca},$$

$$\sum \sqrt{a^2+b^2} \ge \sqrt{4(a^2+b^2+c^2)+2(ab+bc+ca)}.$$

□

P 2.7. *If a, b, c are positive real numbers, then*

$$\sum \sqrt{a^2+ab+b^2} \le \sqrt{5(a^2+b^2+c^2)+4(ab+bc+ca)}.$$

(Michael Rozenberg, 2008)

First Solution *(by Vo Quoc Ba Can).* Using the Cauchy-Schwarz inequality, we have

$$\left(\sum \sqrt{b^2+bc+c^2}\right)^2 \le \left[\sum(b+c)\right]\left(\sum \frac{b^2+bc+c^2}{b+c}\right)$$

$$= 2(a+b+c)\left(\sum \frac{b^2+bc+c^2}{b+c}\right) = 2\sum\left(1+\frac{a}{b+c}\right)(b^2+bc+c^2)$$

$$= 4(a^2+b^2+c^2)+2(ab+bc+ca)+\sum \frac{2a(b^2+bc+c^2)}{b+c}$$

$$= 4(a^2+b^2+c^2)+2(ab+bc+ca)+\sum 2a\left(b+c-\frac{bc}{b+c}\right)$$

$$= 4(a^2+b^2+c^2)+6(ab+bc+ca)-2abc\sum \frac{1}{b+c}.$$

Thus, it suffices to prove that

$$4(a^2+b^2+c^2)+6(ab+bc+ca)-2abc\sum \frac{1}{b+c} \le 5(a^2+b^2+c^2)+4(ab+bc+ca),$$

which is equivalent to Schur's inequality

$$2(ab+bc+ca) \le a^2+b^2+c^2+2abc\sum \frac{1}{b+c}.$$

We can prove this inequality by writing it as follows:

$$(a+b+c)^2 \le 2\sum a\left(a+\frac{bc}{b+c}\right),$$

$$(a+b+c)^2 \le 2(ab+bc+ca)\sum \frac{a}{b+c},$$

$$(a+b+c)^2 \le \left[\sum a(b+c)\right]\sum \frac{a}{b+c}.$$

Clearly, the last inequality follows from the Cauchy-Schwarz inequality. The equality holds for $a = b = c$.

Second Solution. Use the SOS method. Let us denote

$$A = \sqrt{b^2+bc+c^2}, \quad B = \sqrt{c^2+ca+a^2}, \quad C = \sqrt{a^2+ab+b^2}.$$

Without loss of generality, assume that $a \ge b \ge c$. By squaring, the inequality becomes

$$2\sum BC \le 3\sum a^2 + 3\sum ab,$$

$$\sum a^2 - \sum ab \le \sum (B-C)^2,$$

$$\sum (b-c)^2 \le 2(a+b+c)^2 \sum \frac{(b-c)^2}{(B+C)^2}.$$

Since

$$(B+C)^2 \le 2(B^2+C^2) = 2(2a^2+b^2+c^2+ca+ab),$$

it suffices to show that

$$\sum (b-c)^2 \le (a+b+c)^2 \sum \frac{(b-c)^2}{2a^2+b^2+c^2+ca+ab},$$

which is equivalent to

$$\sum (b-c)^2 S_a \ge 0,$$

where

$$S_a = \frac{-a^2+ab+2bc+ca}{2a^2+b^2+c^2+ca+ab},$$

$$S_b = \frac{-b^2+bc+2ca+ab}{2b^2+c^2+a^2+ab+bc} \ge 0,$$

$$S_c = \frac{-c^2+ca+2ab+bc}{2c^2+a^2+b^2+bc+ca} \ge 0.$$

Since

$$\sum (b-c)^2 S_a \ge (b-c)^2 S_a + (a-c)^2 S_b \ge (b-c)^2 S_a + \frac{a^2}{b^2}(b-c)^2 S_b$$

$$\ge (b-c)^2 S_a + \frac{a}{b}(b-c)^2 S_b = a(b-c)^2\left(\frac{S_a}{a}+\frac{S_b}{b}\right),$$

we only need to prove that

$$\frac{S_a}{a}+\frac{S_b}{b} \ge 0,$$

which is equivalent to

$$\frac{-b^2+bc+2ca+ab}{b(2b^2+c^2+a^2+ab+bc)} \ge \frac{a^2-ab-2bc-ca}{a(2a^2+b^2+c^2+ca+ab)}.$$

Consider the nontrivial case where $a^2-ab-2bc-ca \ge 0$. Since

$$(2a^2+b^2+c^2+ca+ab)-(2b^2+c^2+a^2+ab+bc) = (a-b)(a+b+c) \ge 0,$$

it suffices to show that

$$\frac{-b^2+bc+2ca+ab}{b} \ge \frac{a^2-ab-2bc-ca}{a}.$$

Indeed,

$$a(-b^2+bc+2ca+ab) - b(a^2-ab-2bc-ca) = 2c(a^2+ab+b^2) > 0.$$

□

P 2.8. *If a, b, c are nonnegative real numbers, then*

$$\sum \sqrt{a^2+ab+b^2} \le 2\sqrt{a^2+b^2+c^2} + \sqrt{ab+bc+ca}.$$

(Vasile Cîrtoaje, 2010)

First Solution *(by Nguyen Van Quy).* Assume that $a = \max\{a, b, c\}$. Since

$$\sqrt{a^2+ab+b^2} + \sqrt{c^2+ca+a^2} \le \sqrt{2[(a^2+ab+b^2)+(c^2+ca+a^2)]},$$

it suffices to show that

$$2\sqrt{A} + \sqrt{b^2+bc+c^2} \le 2\sqrt{X} + \sqrt{Y},$$

where

$$A = a^2 + \frac{1}{2}(b^2+c^2+ab+ac), \quad X = a^2+b^2+c^2, \quad Y = ab+bc+ca.$$

Write the desired inequality as follows:

$$2(\sqrt{A} - \sqrt{X}) \le \sqrt{Y} - \sqrt{b^2+bc+c^2},$$

$$\frac{2(A-X)}{\sqrt{A}+\sqrt{X}} \le \frac{Y-(b^2+bc+c^2)}{\sqrt{Y}+\sqrt{b^2+bc+c^2}},$$

$$\frac{b(a-b)+c(a-c)}{\sqrt{A}+\sqrt{X}} \le \frac{b(a-b)+c(a-c)}{\sqrt{Y}+\sqrt{b^2+bc+c^2}}.$$

Since $b(a-b)+c(a-c) \ge 0$, we only need to show that

$$\sqrt{A}+\sqrt{X} \ge \sqrt{Y}+\sqrt{b^2+bc+c^2}.$$

This inequality is true because $X \ge Y$ and

$$\sqrt{A} \ge \sqrt{b^2+bc+c^2}.$$

Indeed,

$$2(A-b^2-bc-c^2) = 2a^2+(b+c)a-(b+c)^2 = (2a-b-c)(a+b+c) \ge 0.$$

The equality holds for $a = b = c$, and also for $b = c = 0$ (or any cyclic permutation).

Second Solution. In the first solution of P 2.7, we have shown that

$$\left(\sum \sqrt{b^2+bc+c^2}\right)^2 \le 4(a^2+b^2+c^2)+6(ab+bc+ca)-2abc\sum \frac{1}{b+c}.$$

Thus, it suffices to prove that

$$4(a^2+b^2+c^2)+6(ab+bc+ca)-2abc\sum \frac{1}{b+c} \le \left(2\sqrt{a^2+b^2+c^2}+\sqrt{ab+bc+ca}\right)^2,$$

which is equivalent to

$$2abc\sum \frac{1}{b+c} + 4\sqrt{(a^2+b^2+c^2)(ab+bc+ca)} \geq 5(ab+bc+ca).$$

Since

$$\sum \frac{1}{b+c} \geq \frac{9}{\sum(b+c)} = \frac{9}{2(a+b+c)},$$

it is enough to prove that

$$\frac{9abc}{a+b+c} + 4\sqrt{(a^2+b^2+c^2)(ab+bc+ca)} \geq 5(ab+bc+ca),$$

which can be written as

$$\frac{9abc}{p} + 4\sqrt{q(p^2-2q)} \geq 5q,$$

where

$$p = a+b+c, \quad q = ab+bc+ca.$$

For $p^2 \geq 4q$, this inequality is true because $4\sqrt{q(p^2-2q)} \geq 5q$. Consider further

$$3q \leq p^2 \leq 4q.$$

By Schur's inequality of third degree, we have

$$\frac{9abc}{p} \geq 4q - p^2.$$

Therefore, it suffices to show that

$$(4q-p^2) + 4\sqrt{q(p^2-2q)} \geq 5q,$$

which is

$$4\sqrt{q(p^2-2q)} \geq p^2 + q.$$

Indeed,

$$16q(p^2-2q) - (p^2+q)^2 = (p^2-3q)(11q-p^2) \geq 0.$$

Third Solution. Let us denote

$$A = \sqrt{b^2+bc+c^2}, \quad B = \sqrt{c^2+ca+a^2}, \quad C = \sqrt{a^2+ab+b^2},$$

$$X = \sqrt{a^2+b^2+c^2}, \quad Y = \sqrt{ab+bc+ca}.$$

By squaring, the inequality becomes

$$2\sum BC \leq 2\sum a^2 + 4XY,$$

$$\sum (B-C)^2 \geq 2(X-Y)^2,$$

$$2(a+b+c)^2\sum\frac{(b-c)^2}{(B+C)^2}\ge\frac{\left[\sum(b-c)^2\right]^2}{(X+Y)^2}.$$

Since

$$B+C\le(c+a)+(a+b)=2a+b+c,$$

it suffices to show that

$$2(a+b+c)^2\sum\frac{(b-c)^2}{(2a+b+c)^2}\ge\frac{\left[\sum(b-c)^2\right]^2}{(X+Y)^2}.$$

According to the Cauchy-Schwarz inequality, we have

$$\sum\frac{(b-c)^2}{(2a+b+c)^2}\ge\frac{\left[\sum(b-c)^2\right]^2}{\sum(b-c)^2(2a+b+c)^2}.$$

Therefore, it is enough to prove that

$$\frac{2(a+b+c)^2}{\sum(b-c)^2(2a+b+c)^2}\ge\frac{1}{(X+Y)^2},$$

which is

$$(a+b+c)^2(X+Y)^2\ge\frac{1}{2}\sum(b-c)^2(2a+b+c)^2.$$

We see that

$$(a+b+c)^2(X+Y)^2\ge\left(\sum a^2+2\sum ab\right)\left(\sum a^2+\sum ab\right)$$

$$=\left(\sum a^2\right)^2+3\left(\sum ab\right)\left(\sum a^2\right)+2\left(\sum ab\right)^2$$

$$\ge\sum a^4+3\sum ab(a^2+b^2)+4\sum a^2b^2$$

and

$$\sum(b-c)^2(2a+b+c)^2=\sum(b-c)^2[4a^2+4a(b+c)+(b+c)^2]$$

$$=4\sum a^2(b-c)^2+4\sum a(b-c)(b^2-c^2)+\sum(b^2-c^2)^2$$

$$\le 8\sum a^2b^2+4\sum a(b^3+c^3)+2\sum a^4.$$

Thus, it suffices to show that

$$\sum a^4+3\sum ab(a^2+b^2)+4\sum a^2b^2\ge 4\sum a^2b^2+2\sum a(b^3+c^3)+\sum a^4,$$

which is equivalent to the obvious inequality

$$\sum ab(a^2+b^2)\ge 0.$$

□

P 2.9. *If a, b, c are nonnegative real numbers, then*

$$\sqrt{a^2+2bc}+\sqrt{b^2+2ca}+\sqrt{c^2+2ab} \le \sqrt{a^2+b^2+c^2}+2\sqrt{ab+bc+ca}.$$

(Vasile Cîrtoaje and Nguyen Van Quy, 1989)

Solution *(by Nguyen Van Quy).* Let

$$X=\sqrt{a^2+b^2+c^2}, \quad Y=\sqrt{ab+bc+ca}.$$

Consider the nontrivial case when no two of a, b, c are zero and write the inequality as

$$\sum\left(X-\sqrt{a^2+2bc}\right) \ge 2(X-Y),$$

$$\sum \frac{(b-c)^2}{X+\sqrt{a^2+2bc}} \ge \frac{\sum(b-c)^2}{X+Y}.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{(b-c)^2}{X+\sqrt{a^2+2bc}} \ge \frac{\left[\sum(b-c)^2\right]^2}{\sum(b-c)^2\left(X+\sqrt{a^2+2bc}\ \right)}.$$

Therefore, it suffices to show that

$$\frac{\sum(b-c)^2}{\sum(b-c)^2\left(X+\sqrt{a^2+2bc}\ \right)} \ge \frac{1}{X+Y},$$

which is equivalent to

$$\sum(b-c)^2\left(Y-\sqrt{a^2+2bc}\right) \ge 0.$$

From

$$\left(Y-\sqrt{a^2+2bc}\right)^2 \ge 0.$$

we get

$$Y-\sqrt{a^2+2bc} \ge \frac{Y^2-(a^2+2bc)}{2Y} = \frac{(a-b)(c-a)}{2Y}.$$

Thus,

$$\sum(b-c)^2\left(Y-\sqrt{a^2+2bc}\right) \ge \sum \frac{(b-c)^2(a-b)(c-a)}{2Y}$$

$$= \frac{(a-b)(b-c)(c-a)}{2Y}\sum(b-c) = 0.$$

The equality holds for $a=b$, or $b=c$, or $c=a$.

□

P 2.10. *If a, b, c are nonnegative real numbers, then*

$$\frac{1}{\sqrt{a^2+2bc}}+\frac{1}{\sqrt{b^2+2ca}}+\frac{1}{\sqrt{c^2+2ab}} \ge \frac{1}{\sqrt{a^2+b^2+c^2}}+\frac{2}{\sqrt{ab+bc+ca}}.$$

(Vasile Cîrtoaje, 1989)

Solution . Let

$$X=\sqrt{a^2+b^2+c^2}, \quad Y=\sqrt{ab+bc+ca}.$$

Consider the nontrivial case when $Y > 0$ and write the inequality as

$$\sum \left(\frac{1}{\sqrt{a^2+2bc}}-\frac{1}{X}\right) \ge 2\left(\frac{1}{Y}-\frac{1}{X}\right),$$

$$\sum \frac{(b-c)^2}{\sqrt{a^2+2bc}\left(X+\sqrt{a^2+2bc}\right)} \ge \frac{\sum(b-c)^2}{Y(X+Y)}.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{(b-c)^2}{\sqrt{a^2+2bc}\left(X+\sqrt{a^2+2bc}\right)} \ge \frac{\left[\sum(b-c)^2\right]^2}{\sum(b-c)^2\sqrt{a^2+2bc}\left(X+\sqrt{a^2+2bc}\ \right)}.$$

Therefore, it suffices to show that

$$\frac{\sum(b-c)^2}{\sum(b-c)^2\sqrt{a^2+2bc}\left(X+\sqrt{a^2+2bc}\ \right)} \ge \frac{1}{Y(X+Y)},$$

which is equivalent to

$$\sum(b-c)^2[XY-X\sqrt{a^2+2bc}+(a-b)(c-a)] \ge 0.$$

Since

$$\sum(b-c)^2(a-b)(c-a)=(a-b)(b-c)(c-a)\sum(b-c)=0,$$

we can write the inequality as

$$\sum(b-c)^2\left(Y-\sqrt{a^2+2bc}\right) \ge 0.$$

We have proved this inequality at the preceding problem P 2.9. The equality holds for $a=b$, or $b=c$, or $c=a$.

□

P 2.11. *If a, b, c are positive real numbers, then*

$$\sqrt{2a^2+bc}+\sqrt{2b^2+ca}+\sqrt{2c^2+ab} \le 2\sqrt{a^2+b^2+c^2}+\sqrt{ab+bc+ca}.$$

Solution. We will apply Lemma below for

$$X = 2a^2 + bc, \quad Y = 2b^2 + ca, \quad Z = 2c^2 + ab$$

and

$$A = B = a^2 + b^2 + c^2, \quad C = ab + bc + ca.$$

We have

$$X + Y + Z = A + B + C, \qquad A = B \ge C.$$

Without loss of generality, assume that

$$a \ge b \ge c,$$

which involves

$$X \ge Y \ge Z.$$

By Lemma below, it suffices to show that

$$\max\{X, Y, Z\} \ge A, \qquad \min\{X, Y, Z\} \le C.$$

Indeed, we have

$$\max\{X, Y, Z\} - A = X - A = (a^2 - b^2) + c(b - c) \ge 0,$$

$$\min\{X, Y, Z\} - C = Z - C = c(2c - a - b) \le 0.$$

Equality holds for $a = b = c$.

Lemma. *If X, Y, Z and A, B, C are positive real numbers such that*

$$X + Y + Z = A + B + C,$$

$$\max\{X, Y, Z\} \ge \max\{A, B, C\}, \qquad \min\{X, Y, Z\} \le \min\{A, B, C\},$$

then

$$\sqrt{X} + \sqrt{Y} + \sqrt{Z} \le \sqrt{A} + \sqrt{B} + \sqrt{C}.$$

Proof. On the assumption that $X \ge Y \ge Z$ and $A \ge B \ge C$, we have

$$X \ge A, \qquad Z \le C,$$

hence

$$\sqrt{X} + \sqrt{Y} + \sqrt{Z} - \sqrt{A} - \sqrt{B} - \sqrt{C} = (\sqrt{X} - \sqrt{A}\,) + (\sqrt{Y} - \sqrt{B}\,) + (\sqrt{Z} - \sqrt{C}\,)$$

$$\le \frac{X - A}{2\sqrt{A}} + \frac{Y - B}{2\sqrt{B}} + \frac{Z - C}{2\sqrt{C}} \le \frac{X - A}{2\sqrt{B}} + \frac{Y - B}{2\sqrt{B}} + \frac{Z - C}{2\sqrt{C}}$$

$$= \frac{C - Z}{2\sqrt{B}} + \frac{Z - C}{2\sqrt{C}} = (C - Z)\left(\frac{1}{2\sqrt{B}} - \frac{1}{2\sqrt{C}}\right) \le 0.$$

Remark. This Lemma is a particular case of Karamata's inequality.

□

P 2.12. *Let a,b,c be nonnegative real numbers such that $a+b+c=3$. If $k=\sqrt{3}-1$, then*

$$\sum \sqrt{a(a+kb)(a+kc)} \le 3\sqrt{3}.$$

Solution. By the Cauchy-Schwarz inequality, we have

$$\sum \sqrt{a(a+kb)(a+kc)} \le \sqrt{\left(\sum a\right)\left[\sum (a+kb)(a+kc)\right]}.$$

Thus, it suffices to show that

$$\sqrt{\sum (a+kb)(a+kc)} \le a+b+c,$$

which is an identity. The equality holds for $a=b=c=1$, and also for $a=3$ and $b=c=0$ (or any cyclic permutation).

□

P 2.13. *If a,b,c are nonnegative real numbers such that $a+b+c=3$, then*

$$\sum \sqrt{a(2a+b)(2a+c)} \ge 9.$$

Solution. Write the inequality as follows:

$$\sum \left[\sqrt{a(2a+b)(2a+c)} - a\sqrt{3(a+b+c)}\right] \ge 0,$$

$$\sum (a-b)(a-c)E_a \ge 0,$$

where

$$E_a = \frac{\sqrt{a}}{\sqrt{(2a+b)(2a+c)} + \sqrt{3a(a+b+c)}}.$$

Assume that $a \ge b \ge c$. Since $(c-a)(c-b)E_c \ge 0$, it suffices to show that

$$(a-c)E_a \ge (b-c)E_b,$$

which is equivalent to

$$(a-b)\sqrt{3ab(a+b+c)} + (a-c)\sqrt{a(2b+c)(2b+a)} \ge (b-c)\sqrt{b(2a+b)(2a+c)}.$$

This is true if

$$(a-c)\sqrt{a(2b+c)(2b+a)} \ge (b-c)\sqrt{b(2a+b)(2a+c)}.$$

For the nontrivial case $b > c$, we have

$$\frac{a-c}{b-c} \geq \frac{a}{b} \geq \frac{\sqrt{a}}{\sqrt{b}}.$$

Therefore, it is enough to show that

$$a^2(2b+c)(2b+a) \geq b^2(2a+b)(2a+c).$$

Write this inequality as

$$a^2(2ab+2bc+ca) \geq b^2(2ab+bc+2ca).$$

It is true if

$$a(2ab+2bc+ca) \geq b(2ab+bc+2ca).$$

Indeed,

$$a(2ab+2bc+ca) - b(2ab+bc+2ca) = (a-b)(2ab+bc+ca) \geq 0.$$

The equality holds for $a = b = c = 1$, and also for $a = b = 3/2$ and $c = 0$ (or any cyclic permutation).

□

P 2.14. *Let a, b, c be nonnegative real numbers such that $a + b + c = 3$. Prove that*

$$\sqrt{b^2+c^2+a(b+c)} + \sqrt{c^2+a^2+b(c+a)} + \sqrt{a^2+b^2+c(a+b)} \geq 6.$$

Solution. Denote

$$A = b^2+c^2+a(b+c), \quad B = c^2+a^2+b(c+a), \quad C = a^2+b^2+c(a+b),$$

and write the inequality in the homogeneous form

$$\sqrt{A} + \sqrt{B} + \sqrt{C} \geq 2(a+b+c).$$

Further, we use the SOS method.

First Solution. By squaring, the inequality becomes

$$2\sum \sqrt{BC} \geq 2\sum a^2 + 6\sum bc,$$

$$\sum (b-c)^2 \geq \sum \left(\sqrt{B} - \sqrt{C}\right)^2,$$

$$\sum (b-c)^2 S_a \geq 0,$$

where

$$S_a = 1 - \frac{(b+c-a)^2}{(\sqrt{B}+\sqrt{C})^2}.$$

Since

$$S_a \ge 1 - \frac{(b+c-a)^2}{B+C} = \frac{a(a+3b+3c)}{B+C} \ge 0, \quad S_b \ge 0, \quad S_c \ge 0,$$

the conclusion follows. The equality holds for $a = b = c = 1$, and also for $a = 3$ and $b = c = 0$ (or any cyclic permutation).

Second Solution. Write the desired inequality as follows:

$$\sum(\sqrt{A} - b - c) \ge 0,$$

$$\sum \frac{c(a-b) + b(a-c)}{\sqrt{A} + b + c} \ge 0,$$

$$\sum \frac{c(a-b)}{\sqrt{A} + b + c} + \sum \frac{c(b-a)}{\sqrt{B} + c + a} \ge 0,$$

$$\sum \frac{c(a-b)[a-b-(\sqrt{A}-\sqrt{B})]}{(\sqrt{A}+b+c)(\sqrt{B}+c+a)} \ge 0.$$

It suffices to show that

$$(a-b)[a-b+(\sqrt{B}-\sqrt{A})] \ge 0.$$

Indeed,

$$(a-b)[a-b+(\sqrt{B}-\sqrt{A})] = (a-b)^2\left(1 + \frac{a+b-c}{\sqrt{B}+\sqrt{A}}\right) \ge 0,$$

because, for the nontrivial case $a + b - c < 0$, we have

$$1 + \frac{a+b-c}{\sqrt{B}+\sqrt{A}} > 1 + \frac{a+b-c}{c+c} > 0.$$

Generalization. *Let a, b, c be nonnegative real numbers. If $0 < k \le \frac{16}{9}$, then*

$$\sum \sqrt{(b+c)^2 + k(ab - 2bc + ca)} \ge 2(a+b+c).$$

Notice that if $k = \frac{16}{9}$, then the equality holds for $a = b = c = 1$, for $a = 0$ and $b = c$ (or any cyclic permutation), and for $b = c = 0$ (or any cyclic permutation).

□

P 2.15. *Let a, b, c be nonnegative real numbers such that $a + b + c = 3$. Prove that*

(a) $$\sqrt{a(3a^2+abc)} + \sqrt{b(3b^2+abc)} + \sqrt{c(3c^2+abc)} \ge 6;$$

(b) $$\sqrt{3a^2+abc} + \sqrt{3b^2+abc} + \sqrt{3c^2+abc} \ge 3\sqrt{3+abc}.$$

(Lorian Saceanu, 2015)

Solution. (a) Write the inequality in the homogeneous form

$$3\sum a\sqrt{(a+b)(a+c)} \ge 2(a+b+c)^2.$$

First Solution. Use the SOS method. Write the inequality as

$$\sum a^2 - \sum ab \ge \frac{3}{2}\sum a\left(\sqrt{a+b}-\sqrt{a+c}\right)^2,$$

$$\sum (b-c)^2 \ge 3\sum \frac{a(b-c)^2}{\left(\sqrt{a+b}+\sqrt{a+c}\right)^2},$$

$$\sum (b-c)^2 S_a \ge 0,$$

where

$$S_a = 1 - \frac{3a}{\left(\sqrt{a+b}+\sqrt{a+c}\right)^2}.$$

Since

$$S_a \ge 1 - \frac{3a}{\left(\sqrt{a}+\sqrt{a}\right)^2} > 0, \quad S_b > 0, \quad S_c > 0,$$

the inequality is true. The equality holds for $a = b = c = 1$.

Second Solution. By Hölder's inequality, we have

$$\left[\sum a\sqrt{(a+b)(a+c)}\right]^2 \ge \frac{(\sum a)^3}{\sum \frac{a}{(a+b)(a+c)}} = \frac{27}{\sum \frac{a}{(a+b)(a+c)}}.$$

Therefore, it suffices to show that

$$\sum \frac{a}{(a+b)(a+c)} \le \frac{3}{4}.$$

This inequality has the homogeneous form

$$\sum \frac{a}{(a+b)(a+c)} \le \frac{9}{4(a+b+c)},$$

which is equivalent to the obvious inequality

$$\sum a(b-c)^2 \ge 0.$$

(b) By squaring, the inequality becomes

$$3\sum a^2 + 2\sum \sqrt{(3b^2+abc)(3c^2+abc)} \ge 27 + 6abc.$$

According to the Cauchy-Schwarz inequality, we have

$$\sqrt{(3b^2+abc)(3c^2+abc)} \ge 3bc + abc.$$

Therefore, it suffices to show that

$$3\sum a^2 + 6\sum bc + 6abc \ge 27 + 6abc,$$

which is an identity. The equality holds for $a = b = c = 1$, and also for $a = 0$, or $b = 0$, or $c = 0$.

□

P 2.16. *Let a, b, c be positive real numbers such that $ab + bc + ca = 3$. Prove that*

$$a\sqrt{(a+2b)(a+2c)} + b\sqrt{(b+2c)(b+2a)} + c\sqrt{(c+2a)(c+2b)} \ge 9.$$

First Solution. Use the SOS method. Write the inequality as follows:

$$\sum a\sqrt{(a+2b)(a+2c)} \ge 3(ab+bc+ca),$$

$$\sum a^2 - \sum ab \ge \frac{1}{2}\sum a\left(\sqrt{a+2b} - \sqrt{a+2c}\right)^2,$$

$$\sum (b-c)^2 \ge 4\sum \frac{a(b-c)^2}{\left(\sqrt{a+2b}+\sqrt{a+2c}\right)^2},$$

$$\sum (b-c)^2 S_a \ge 0,$$

where

$$S_a = 1 - \frac{4a}{\left(\sqrt{a+2b}+\sqrt{a+2c}\right)^2}.$$

Since

$$S_a > 1 - \frac{4a}{(\sqrt{a}+\sqrt{a})^2} = 0, \quad S_b > 0, \quad S_c > 0,$$

the inequality is true. The equality holds for $a = b = c = 1$.

Second Solution. We use the AM-GM inequality to get

$$\sum a\sqrt{(a+2b)(a+2c)} = \sum \frac{2a(a+2b)(a+2c)}{2\sqrt{(a+2b)(a+2c)}} \ge \sum \frac{2a(a+2b)(a+2c)}{(a+2b)+(a+2c)}$$

$$= \frac{1}{a+b+c}\sum a(a+2b)(a+2c).$$

Thus, it suffices to show that

$$\sum a(a+2b)(a+2c) \ge 9(a+b+c).$$

Write this inequality in the homogeneous form

$$\sum a(a+2b)(a+2c) \ge 3(a+b+c)(ab+bc+ca),$$

which is equivalent to Schur's inequality of degree three

$$a^3 + b^3 + c^3 + 3abc \ge ab(a+b) + bc(b+c) + ca(c+a).$$

□

P 2.17. *Let a, b, c be nonnegative real numbers such that $a+b+c=1$. Prove that*

$$\sqrt{a+(b-c)^2}+\sqrt{b+(c-a)^2}+\sqrt{c+(a-b)^2} \geq \sqrt{3}.$$

(Phan Thanh Nam, 2007)

Solution. By squaring, the inequality becomes

$$\sum \sqrt{[a+(b-c)^2][b+(c-a)^2]} \geq 3(ab+bc+ca).$$

Applying the Cauchy-Schwarz inequality, it suffices to show that

$$\sum \sqrt{ab}+\sum (b-c)(a-c) \geq 3(ab+bc+ca).$$

This is equivalent to the homogeneous inequality

$$\left(\sum a\right)\left(\sum \sqrt{ab}\right)+\sum a^2 \geq 4(ab+bc+ca).$$

Making the substitution $x=\sqrt{a}$, $y=\sqrt{b}$, $z=\sqrt{c}$, the inequality turns into

$$\left(\sum x^2\right)\left(\sum xy\right)+\sum x^4 \geq 4\sum x^2y^2,$$

which is equivalent to

$$\sum x^4+\sum xy(x^2+y^2)+xyz\sum x \geq 4\sum x^2y^2.$$

Since

$$4\sum x^2y^2 \leq 2\sum xy(x^2+y^2),$$

it suffices to show that

$$\sum x^4+xyz\sum x \geq \sum xy(x^2+y^2),$$

which is just Schur's inequality of degree four. The equality holds for $a=b=c=\frac{1}{3}$, and for $a=0$ and $b=c=\frac{1}{2}$ (or any cyclic permutation).

□

P 2.18. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\sqrt{\frac{a(b+c)}{a^2+bc}}+\sqrt{\frac{b(c+a)}{b^2+ca}}+\sqrt{\frac{c(a+b)}{c^2+ab}} \geq 2.$$

(Vasile Cîrtoaje, 2006)

Solution. Using the AM-GM inequality gives

$$\sqrt{\frac{a(b+c)}{a^2+bc}} = \frac{a(b+c)}{\sqrt{(a^2+bc)(ab+ac)}} \ge \frac{2a(b+c)}{(a^2+bc)+(ab+ac)} = \frac{2a(b+c)}{(a+b)(a+c)}.$$

Therefore, it suffices to show that

$$\frac{a(b+c)}{(a+b)(a+c)} + \frac{b(c+a)}{(b+c)(b+a)} + \frac{c(a+b)}{(c+a)(c+b)} \ge 1,$$

which is equivalent to

$$a(b+c)^2 + b(c+a)^2 + c(a+b)^2 \ge (a+b)(b+c)(c+a),$$

$$4abc \ge 0.$$

The equality holds for $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 2.19. *Let a, b, c be positive real numbers such that $abc = 1$. Prove that*

$$\frac{1}{\sqrt[3]{a^2+25a+1}} + \frac{1}{\sqrt[3]{b^2+25b+1}} + \frac{1}{\sqrt[3]{c^2+25c+1}} \ge 1.$$

Solution. Replacing a, b, c by a^3, b^3, c^3, respectively, we need to show that $abc = 1$ yields

$$\frac{1}{\sqrt[3]{a^6+25a^3+1}} + \frac{1}{\sqrt[3]{b^6+25b^3+1}} + \frac{1}{\sqrt[3]{c^6+25c^3+1}} \ge 1.$$

We first show that

$$\frac{1}{\sqrt[3]{a^6+25a^3+1}} \ge \frac{1}{a^2+a+1}.$$

This is equivalent to

$$(a^2+a+1)^3 \ge a^6+25a^3+1,$$

which is true since

$$(a^2+a+1)^3 - (a^6+25a^3+1) = 3a(a-1)^2(a^2+4a+1) \ge 0.$$

Therefore, it suffices to prove that

$$\frac{1}{a^2+a+1} + \frac{1}{b^2+b+1} + \frac{1}{b^2+b+1} \ge 1.$$

Putting

$$a = \frac{yz}{x^2}, \quad b = \frac{zx}{y^2}, \quad c = \frac{xy}{z^2}, \qquad x, y, z > 0$$

we need to show that

$$\sum \frac{x^4}{x^4+x^2yz+y^2z^2} \ge 1.$$

Indeed, the Cauchy-Schwarz inequality gives

$$\sum \frac{x^4}{x^4+x^2yz+y^2z^2} \ge \frac{\left(\sum x^2\right)^2}{\sum(x^4+x^2yz+y^2z^2)} = \frac{\sum x^4+2\sum y^2z^2}{\sum x^4+xyz\sum x+\sum y^2z^2} \ge 1.$$

The equality holds for $a=b=c=1$.

□

P 2.20. *If a,b,c are nonnegative real numbers, then*

$$\sqrt{a^2+bc}+\sqrt{b^2+ca}+\sqrt{c^2+ab} \le \frac{3}{2}(a+b+c).$$

(Pham Kim Hung, 2005)

Solution. Without loss of generality, assume that $a \ge b \ge c$. Since the equality occurs for $a=b$ and $c=0$, we use the inequalities

$$\sqrt{a^2+bc} \le a+\frac{c}{2}$$

and

$$\sqrt{b^2+ca}+\sqrt{c^2+ab} \le \sqrt{2(b^2+ca)+2(c^2+ab)}.$$

Thus, it suffices to prove that

$$\sqrt{2(b^2+ca)+2(c^2+ab)} \le \frac{a+3b+2c}{2}.$$

By squaring, this inequality becomes

$$a^2+b^2-4c^2-2ab+12bc-4ca \ge 0,$$

$$(a-b-2c)^2+8c(b-c) \ge 0.$$

The equality holds for $a=b$ and $c=0$ (or any cyclic permutation).

□

P 2.21. *If a,b,c are nonnegative real numbers, then*

$$\sqrt{a^2+9bc}+\sqrt{b^2+9ca}+\sqrt{c^2+9ab} \ge 5\sqrt{ab+bc+ca}.$$

(Vasile Cîrtoaje, 2012)

Solution *(by Nguyen Van Quy).* Assume that

$$c = \min\{a, b, c\}.$$

Since the equality occurs for $a = b$ and $c = 0$, we use the inequality

$$\sqrt{c^2 + 9ab} \ge 3\sqrt{ab}.$$

On the other hand, by Minkowski's inequality, we have

$$\sqrt{a^2 + 9bc} + \sqrt{b^2 + 9ca} \ge \sqrt{(a + b)^2 + 9c\left(\sqrt{a} + \sqrt{b}\right)^2}.$$

Therefore, it suffices to show that

$$\sqrt{(a + b)^2 + 9c\left(\sqrt{a} + \sqrt{b}\right)^2} \ge 5\sqrt{ab + bc + ca} - 3\sqrt{ab}.$$

By squaring, this inequality becomes

$$(a + b)^2 + 18c\sqrt{ab} + 30\sqrt{ab(ab + bc + ca)} \ge 34ab + 16c(a + b).$$

Since

$$ab(ab + bc + ca) - \left[ab + \frac{c(a + b)}{3}\right]^2 = \frac{c(a + b)(3ab - ac - bc)}{9} \ge 0,$$

it suffices to show that $f(c) \ge 0$ for $0 \le c \le \sqrt{ab}$, where

$$\begin{aligned} f(c) &= (a + b)^2 + 18c\sqrt{ab} + [30ab + 10c(a + b)] - 34ab - 16c(a + b) \\ &= (a + b)^2 - 4ab + 6c(3\sqrt{ab} - a - b). \end{aligned}$$

Since $f(c)$ is a linear function, we only need to prove that $f(0) \ge 0$ and $f(\sqrt{ab}) \ge 0$. We have

$$f(0) = (a - b)^2 \ge 0,$$

$$\begin{aligned} f(\sqrt{ab}) &= (a + b)^2 + 14ab - 6(a + b)\sqrt{ab} \ge (a + b)^2 + 9ab - 6(a + b)\sqrt{ab} \\ &= \left(a + b - 3\sqrt{ab}\right)^2 \ge 0. \end{aligned}$$

The equality holds for $a = b$ and $c = 0$ (or any cyclic permutation).

□

P 2.22. *If a, b, c are nonnegative real numbers, then*

$$\sum \sqrt{(a^2 + 4bc)(b^2 + 4ca)} \ge 5(ab + ac + bc).$$

(Vasile Cîrtoaje, 2012)

Solution. Assume that

$$a \ge b \ge c.$$

First Solution (by *Michael Rozenberg*). Use the SOS method. For $b = c = 0$, the inequality is trivial. Consider further that $b > 0$ and write the inequality as follows:

$$\sum \left[\sqrt{(b^2+4ca)(c^2+4ab)} - (bc+2ab+2ac)\right] \ge 0,$$

$$\sum \frac{(b^2+4ca)(c^2+4ab)-(bc+2ab+2ac)^2}{\sqrt{(b^2+4ca)(c^2+4ab)}+bc+2a(b+c)} \ge 0,$$

$$\sum (b-c)^2 S_a \ge 0,$$

where

$$S_a = \frac{a(b+c-a)}{A}, \quad A = \sqrt{(b^2+4ca)(c^2+4ab)}+bc+2a(b+c),$$

$$S_b = \frac{b(c+a-b)}{B}, \quad B = \sqrt{(c^2+4ab)(a^2+4bc)}+ca+2b(c+a),$$

$$S_c = \frac{c(a+b-c)}{C}, \quad C = \sqrt{(a^2+4bc)(b^2+4ac)}+ab+2c(a+b).$$

Since $S_b \ge 0$ and $S_c \ge 0$, we have

$$\sum (b-c)^2 S_a \ge (b-c)^2 S_a + (a-c)^2 S_b \ge (b-c)^2 S_a + \frac{a^2}{b^2}(b-c)^2 S_b$$

$$= \frac{a}{b}(b-c)^2\left(\frac{bS_a}{a}+\frac{aS_b}{b}\right).$$

Thus, it suffices to prove that

$$\frac{bS_a}{a}+\frac{aS_b}{b} \ge 0,$$

which is equivalent to

$$\frac{b(b+c-a)}{A}+\frac{a(c+a-b)}{B} \ge 0.$$

Since

$$\frac{b(b+c-a)}{A}+\frac{a(c+a-b)}{B} \ge \frac{b(b-a)}{A}+\frac{a(a-b)}{B} = \frac{(a-b)(aA-bB)}{AB},$$

it is enough to show that

$$aA - bB \ge 0.$$

Indeed,

$$aA - bB = \sqrt{c^2+4ab}\left[a\sqrt{b^2+4ca} - b\sqrt{a^2+4bc}\right] + 2(a-b)(ab+bc+ca)$$

$$= \frac{4c(a^3-b^3)\sqrt{c^2+4ab}}{a\sqrt{b^2+4ca}+b\sqrt{a^2+4bc}} + 2(a-b)(ab+bc+ca) \ge 0.$$

The equality holds for $a=b=c$, and also for $a=b$ and $c=0$ (or any cyclic permutation).

Second Solution (by *Nguyen Van Quy*). Write the inequality as

$$\left(\sqrt{a^2+4bc}+\sqrt{b^2+4ca}+\sqrt{c^2+4ab}\right)^2 \ge a^2+b^2+c^2+14(ab+bc+ca),$$

$$\sqrt{a^2+4bc}+\sqrt{b^2+4ca}+\sqrt{c^2+4ab} \ge \sqrt{a^2+b^2+c^2+14(ab+bc+ca)}.$$

For $t=2c$, the inequality (b) in Lemma below becomes

$$\sqrt{a^2+4bc}+\sqrt{b^2+4ca} \ge \sqrt{(a+b)^2+8(a+b)c}.$$

Thus, it suffices to show that

$$\sqrt{(a+b)^2+8(a+b)c}+\sqrt{c^2+4ab} \ge \sqrt{a^2+b^2+c^2+14(ab+bc+ca)}.$$

By squaring, this inequality becomes

$$\sqrt{[(a+b)^2+8(a+b)c]\,(c^2+4ab)} \ge 4ab+3(a+b)c,$$

$$2(a+b)c^3-2(a+b)^2c^2+2ab(a+b)c+ab(a+b)^2-4a^2b^2 \ge 0,$$

$$2(a+b)(a-c)(b-c)c+ab(a-b)^2 \ge 0.$$

Lemma. *Let a,b and t be nonnegative numbers such that*

$$t \le 2(a+b).$$

Then,

(a) $$\sqrt{(a^2+2bt)(b^2+2at)} \ge ab+(a+b)t;$$

(b) $$\sqrt{a^2+2bt}+\sqrt{b^2+2at} \ge \sqrt{(a+b)^2+4(a+b)t}.$$

Proof. (a) By squaring, the inequality becomes

$$(a-b)^2t[2(a+b)-t] \ge 0,$$

which is clearly true.

(b) By squaring, this inequality turns into the inequality in (a).

□

P 2.23. *If a,b,c are nonnegative real numbers, then*

$$\sum \sqrt{(a^2+9bc)(b^2+9ca)} \ge 7(ab+ac+bc).$$

(Vasile Cîrtoaje, 2012)

Solution *(by Nguyen Van Quy).* We see that the equality holds for $a = b$ and $c = 0$. Without loss of generality, assume that

$$c = \min\{a, b, c\}.$$

For $t = 4c$, the inequality (a) in Lemma from the preceding P 2.22 becomes

$$\sqrt{(a^2 + 8bc)(b^2 + 8ca)} \ge ab + 4(a + b)c.$$

Thus, we have

$$\sqrt{(a^2 + 9bc)(b^2 + 9ca)} \ge ab + 4(a + b)c$$

and

$$\begin{aligned}\sqrt{c^2 + 9ab}\left(\sqrt{a^2 + 9bc} + \sqrt{b^2 + 9ca}\right) &\ge 3\sqrt{ab} \cdot 2\sqrt[4]{(a^2 + 9bc)(b^2 + 9ca)}\\ &\ge 6\sqrt{ab} \cdot \sqrt{ab + 4(a + b)c} = 3\sqrt{4a^2b^2 + 16abc(a + b)}\\ &\ge 3\sqrt{4a^2b^2 + 4abc(a + b) + c^2(a + b)^2} = 3(2ab + bc + ca).\end{aligned}$$

Therefore,

$$\begin{aligned}\sum \sqrt{(a^2 + 9bc)(b^2 + 9ca)} &\ge (ab + 4bc + 4ca) + 3(2ab + bc + ca)\\ &= 7(ab + bc + ca).\end{aligned}$$

The equality holds for $a = b$ and $c = 0$ (or any cyclic permutation).

□

P 2.24. *If a, b, c are nonnegative real numbers, then*

$$\sqrt{(a^2 + b^2)(b^2 + c^2)} + \sqrt{(b^2 + c^2)(c^2 + a^2)} + \sqrt{(c^2 + a^2)(a^2 + b^2)} \le (a + b + c)^2.$$

(Vasile Cîrtoaje, 2007)

Solution. Without loss of generality, assume that

$$a = \min\{a, b, c\}.$$

Let us denote

$$y = \frac{a}{2} + b, \qquad z = \frac{a}{2} + c.$$

Since

$$a^2 + b^2 \le y^2, \quad b^2 + c^2 \le y^2 + z^2, \quad c^2 + a^2 \le z^2,$$

it suffices to prove that

$$yz + (y + z)\sqrt{y^2 + z^2} \le (y + z)^2.$$

This is true since

$$y^2 + yz + z^2 - (y + z)\sqrt{y^2 + z^2} = \frac{y^2z^2}{y^2 + yz + z^2 + (y + z)\sqrt{y^2 + z^2}} \ge 0.$$

The equality holds for $a = b = 0$ (or any cyclic permutation).

□

P 2.25. *If a, b, c are nonnegative real numbers, then*

$$\sum \sqrt{(a^2+ab+b^2)(b^2+bc+c^2)} \ge (a+b+c)^2.$$

Solution. By the Cauchy-Schwarz inequality, we have

$$(a^2+ab+b^2)(a^2+ac+c^2) = \left[\left(a+\frac{b}{2}\right)^2 + \frac{3b^2}{4}\right]\left[\left(a+\frac{c}{2}\right)^2 + \frac{3c^2}{4}\right]$$

$$\ge \left(a+\frac{b}{2}\right)\left(a+\frac{c}{2}\right) + \frac{3bc}{4} = a^2 + \frac{a(b+c)}{2} + bc.$$

Then,

$$\sum \sqrt{(a^2+ab+b^2)(a^2+ac+c^2)} \ge \sum \left[a^2 + \frac{a(b+c)}{2} + bc\right] = (a+b+c)^2.$$

The equality holds for $a = b = c$, and also for $b = c = 0$ (or any cyclic permutation).

□

P 2.26. *If a, b, c are nonnegative real numbers, then*

$$\sum \sqrt{(a^2+7ab+b^2)(b^2+7bc+c^2)} \ge 7(ab+ac+bc).$$

(Vasile Cîrtoaje, 2012)

First Solution. Without loss of generality, assume that

$$c = \min\{a, b, c\}.$$

We see that the equality holds for $a = b$ and $c = 0$. Since

$$\sqrt{(a^2+7ac+c^2)(b^2+7bc+c^2)} \ge (a+2c)(b+2c) \ge ab + 2c(a+b),$$

it suffices to show that

$$\sqrt{a^2+7ab+b^2}\left(\sqrt{a^2+7ac} + \sqrt{b^2+7bc}\right) \ge 6ab + 5c(a+b).$$

By Minkowski's inequality, we have

$$\sqrt{a^2+7ac} + \sqrt{b^2+7bc} \ge \sqrt{(a+b)^2 + 7c\left(\sqrt{a}+\sqrt{b}\right)^2}$$

$$\ge \sqrt{(a+b)^2 + 7c(a+b) + \frac{28abc}{a+b}}.$$

Therefore, it suffices to show that

$$(a^2+7ab+b^2)\left[(a+b)^2+7c(a+b)+\frac{28abc}{a+b}\right] \ge (6ab+5bc+5ca)^2.$$

Due to homogeneity, we may assume that $a+b=1$. Let us denote $d=ab$, $d \le \frac{1}{4}$. Since

$$c \le \frac{2ab}{a+b} = 2d,$$

we need to show that $f(c) \ge 0$ for $0 \le c \le 2d \le \frac{1}{2}$, where

$$f(c) = (1+5d)(1+7c+28cd)-(6d+5c)^2.$$

Since $f(c)$ is concave, it suffices to show that $f(0) \ge 0$ and $f(2d) \ge 0$. Indeed,

$$f(0) = 1+5d-36d^2 = (1-4d)(1+9d) \ge 0$$

and

$$f(2d) = (1+5d)(1+14d+56d^2)-256d^2 \ge (1+4d)(1+14d+56d^2)-256d^2$$
$$= (1-4d)(1+22d-56d^2) \ge d(1-4d)(22-56d) \ge 0.$$

The equality holds for $a=b$ and $c=0$ (or any cyclic permutation).

Second Solution. We will use the inequality

$$\sqrt{x^2+7xy+y^2} \ge x+y+\frac{2xy}{x+y}, \quad x,y \ge 0,$$

which, by squaring, reduces to

$$xy(x-y)^2 \ge 0.$$

We have

$$\sum \sqrt{(a^2+7ab+b^2)(a^2+7ac+c^2)} \ge \sum \left(a+b+\frac{2ab}{a+b}\right)\left(a+c+\frac{2ac}{a+c}\right)$$
$$\ge \sum a^2 + 3\sum ab + \sum \frac{2a^2b}{a+b} + \sum \frac{2a^2c}{a+c} + \sum \frac{2abc}{a+b}.$$

Since

$$\sum \frac{2a^2b}{a+b} + \sum \frac{2a^2c}{a+c} = \sum \frac{2a^2b}{a+b} + \sum \frac{2b^2a}{b+a} = 2\sum ab$$

and

$$\sum \frac{2abc}{a+b} \ge \frac{18abc}{\sum(a+b)} = \frac{9abc}{a+b+c},$$

it suffices to show that

$$\sum a^2 + \frac{9abc}{a+b+c} \ge 2\sum ab,$$

which is just Schur's inequality of degree three.

□

P 2.27. *If a,b,c are nonnegative real numbers, then*

$$\sum\sqrt{\left(a^2+\frac{7}{9}ab+b^2\right)\left(b^2+\frac{7}{9}bc+c^2\right)}\le\frac{13}{12}(a+b+c)^2.$$

(Vasile Cîrtoaje, 2012)

Solution *(by Nguyen Van Quy).* Without loss of generality, assume that

$$c=\min\{a,b,c\}.$$

It is easy to see that the equality holds for $a=b=1$ and $c=0$. By the AM-GM inequality, the following inequality holds for any $k>0$:

$$12\sqrt{a^2+\frac{7}{9}ab+b^2}\left(\sqrt{a^2+\frac{7}{9}ac+c^2}+\sqrt{b^2+\frac{7}{9}bc+c^2}\right)\le$$

$$\le\frac{36}{k}\left(a^2+\frac{7}{9}ab+b^2\right)+k\left(\sqrt{a^2+\frac{7}{9}ac+c^2}+\sqrt{b^2+\frac{7}{9}bc+c^2}\right)^2.$$

We can use this inequality to prove the original inequality only if

$$\frac{36}{k}\left(a^2+\frac{7}{9}ab+b^2\right)=k\left(\sqrt{a^2+\frac{7}{9}ac+c^2}+\sqrt{b^2+\frac{7}{9}bc+c^2}\right)^2$$

for $a=b=1$ and $c=0$. This condition if satisfied for $k=5$. Therefore, it suffices to show that

$$12\sqrt{\left(a^2+\frac{7}{9}ac+c^2\right)\left(b^2+\frac{7}{9}bc+c^2\right)}+\frac{36}{5}\left(a^2+\frac{7}{9}ab+b^2\right)+$$

$$+5\left(\sqrt{a^2+\frac{7}{9}ac+c^2}+\sqrt{b^2+\frac{7}{9}bc+c^2}\right)^2\le 13(a+b+c)^2.$$

which is equivalent to

$$22\sqrt{\left(a^2+\frac{7}{9}ac+c^2\right)\left(b^2+\frac{7}{9}bc+c^2\right)}\le\frac{4(a+b)^2+94ab}{5}+3c^2+\frac{199c(a+b)}{9}.$$

Since

$$\begin{aligned}2\sqrt{\left(a^2+\frac{7}{9}ac+c^2\right)\left(b^2+\frac{7}{9}bc+c^2\right)}&\le 2\sqrt{\left(a^2+\frac{16}{9}ac\right)\left(b^2+\frac{16}{9}bc\right)}\\&=2\sqrt{a\left(b+\frac{16}{9}c\right)\cdot b\left(a+\frac{16}{9}c\right)}\\&\le a\left(b+\frac{16}{9}c\right)+b\left(a+\frac{16}{9}c\right)\\&=2ab+\frac{16c(a+b)}{9},\end{aligned}$$

we only need to prove that

$$22\left[ab+\frac{8c(a+b)}{9}\right] \le \frac{4(a^2+b^2)+102ab}{5}+3c^2+\frac{199c(a+b)}{9}.$$

This reduces to the obvious inequality

$$\frac{4(a-b)^2}{5}+\frac{23c(a+b)}{9}+3c^2 \ge 0.$$

Thus, the proof is completed. The equality holds for $a=b$ and $c=0$ (or any cyclic permutation).

□

P 2.28. *If a,b,c are nonnegative real numbers, then*

$$\sum \sqrt{\left(a^2+\frac{1}{3}ab+b^2\right)\left(b^2+\frac{1}{3}bc+c^2\right)} \le \frac{61}{60}(a+b+c)^2.$$

(Vasile Cîrtoaje, 2012)

Solution *(by Nguyen Van Quy).* Without loss of generality, assume that

$$c=\min\{a,b,c\}.$$

It is easy to see that the equality holds for $c=0$ and $11(a^2+b^2)=38ab$. By the AM-GM inequality, the following inequality holds for any $k>0$:

$$60\sqrt{a^2+\frac{1}{3}ab+b^2}\left(\sqrt{a^2+\frac{1}{3}ac+c^2}+\sqrt{b^2+\frac{1}{3}bc+c^2}\right) \le$$

$$\le \frac{36}{k}\left(a^2+\frac{1}{3}ab+b^2\right)+25k\left(\sqrt{a^2+\frac{1}{3}ac+c^2}+\sqrt{b^2+\frac{1}{3}bc+c^2}\right)^2.$$

We can use this inequality to prove the original inequality only if the equality

$$\frac{36}{k}\left(a^2+\frac{1}{3}ab+b^2\right)=25k\left(\sqrt{a^2+\frac{1}{3}ac+c^2}+\sqrt{b^2+\frac{1}{3}bc+c^2}\right)^2$$

holds for $c=0$ and $11(a^2+b^2)=38ab$. This necessary condition if satisfied for $k=1$. Therefore, it suffices to show that

$$60\sqrt{\left(a^2+\frac{1}{3}ab+b^2\right)\left(b^2+\frac{1}{3}bc+c^2\right)}+36\left(a^2+\frac{1}{3}ab+b^2\right)+$$

$$+25\left(\sqrt{a^2+\frac{1}{3}ac+c^2}+\sqrt{b^2+\frac{1}{3}bc+c^2}\right)^2 \le 61(a+b+c)^2,$$

which is equivalent to

$$10\sqrt{\left(a^2+\frac{1}{3}ac+c^2\right)\left(b^2+\frac{1}{3}bc+c^2\right)} \le 10ab+c^2+\frac{31c(a+b)}{3}.$$

Since

$$\begin{aligned}
2\sqrt{\left(a^2+\frac{1}{3}ac+c^2\right)\left(b^2+\frac{1}{3}bc+c^2\right)} &\le 2\sqrt{\left(a^2+\frac{4}{3}ac\right)\left(b^2+\frac{4}{3}bc\right)}\\
&= 2\sqrt{a\left(b+\frac{4}{3}c\right)\cdot b\left(a+\frac{4}{3}c\right)}\\
&\le a\left(b+\frac{4}{3}c\right)+b\left(a+\frac{4}{3}c\right)\\
&= 2ab+\frac{4c(a+b)}{3},
\end{aligned}$$

we only need to prove that

$$10\left[ab+\frac{2c(a+b)}{3}\right] \le 10ab+c^2+\frac{31c(a+b)}{3}.$$

This reduces to the obvious inequality

$$3c^2+11c(a+b) \ge 0.$$

Thus, the proof is completed. The equality holds for $11(a^2+b^2)=38ab$ and $c=0$ (or any cyclic permutation).

□

P 2.29. *If a,b,c are nonnegative real numbers, then*

$$\frac{a}{\sqrt{4b^2+bc+4c^2}}+\frac{b}{\sqrt{4c^2+ca+4a^2}}+\frac{c}{\sqrt{4a^2+ab+4b^2}} \ge 1.$$

(Pham Kim Hung, 2006)

Solution. By Hölder's inequality, we have

$$\left(\sum\frac{a}{\sqrt{4b^2+bc+4c^2}}\right)^2 \ge \frac{\left(\sum a\right)^3}{\sum a(4b^2+bc+4c^2)} = \frac{\sum a^3+3\sum ab(a+b)+6abc}{4\sum ab(a+b)+3abc}.$$

Thus, it suffices to show that

$$\sum a^3 + 3abc \geq \sum ab(a+b),$$

which is Schur's inequality of degree three. The equality holds for $a = b = c$, and also for $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 2.30. *If a, b, c are nonnegative real numbers, then*

$$\frac{a}{\sqrt{b^2+bc+c^2}} + \frac{b}{\sqrt{c^2+ca+a^2}} + \frac{c}{\sqrt{a^2+ab+b^2}} \geq \frac{a+b+c}{\sqrt{ab+bc+ca}}.$$

Solution. By Hölder's inequality, we have

$$\left(\sum \frac{a}{\sqrt{b^2+bc+c^2}}\right)^2 \geq \frac{\left(\sum a\right)^3}{\sum a(b^2+bc+c^2)} = \frac{\left(\sum a\right)^2}{\sum ab},$$

from which the desired inequality follows. The equality holds for $a = b = c$, and also for $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 2.31. *If a, b, c are nonnegative real numbers, then*

$$\frac{a}{\sqrt{a^2+2bc}} + \frac{b}{\sqrt{b^2+2ca}} + \frac{c}{\sqrt{c^2+2ab}} \leq \frac{a+b+c}{\sqrt{ab+bc+ca}}.$$

(Ho Phu Thai, 2007)

Solution. Without loss of generality, assume that

$$a \geq b \geq c.$$

First Solution. Since

$$\frac{c}{\sqrt{c^2+2ab}} \leq \frac{c}{\sqrt{ab+bc+ca}},$$

it suffices to show that

$$\frac{a}{\sqrt{a^2+2bc}} + \frac{b}{\sqrt{b^2+2ca}} \leq \frac{a+b}{\sqrt{ab+bc+ca}},$$

which is equivalent to

$$\frac{a(\sqrt{a^2+2bc} - \sqrt{ab+bc+ca})}{\sqrt{a^2+2bc}} \geq \frac{b(\sqrt{ab+bc+ca} - \sqrt{b^2+2ca})}{\sqrt{b^2+2ca}}.$$

Since
$$\sqrt{a^2+2bc}-\sqrt{ab+bc+ca}\ge 0$$
and
$$\frac{a}{\sqrt{a^2+2bc}}\ge\frac{b}{\sqrt{b^2+2ca}},$$
it suffices to show that
$$\sqrt{a^2+2bc}-\sqrt{ab+bc+ca}\ge\sqrt{ab+bc+ca}-\sqrt{b^2+2ca},$$
which is equivalent to
$$\sqrt{a^2+2bc}+\sqrt{b^2+2ca}\ge 2\sqrt{ab+bc+ca}.$$
Using the AM-GM inequality, it suffices to show that
$$(a^2+2bc)(b^2+2ca)\ge(ab+bc+ca)^2,$$
which is equivalent to the obvious inequality
$$c(a-b)^2(2a+2b-c)\ge 0.$$
The equality holds for $a=b=c$, and also for $a=b$ and $c=0$ (or any cyclic permutation).

Second Solution. By the Cauchy-Schwarz inequality, we have
$$\left(\sum\frac{a}{\sqrt{a^2+2bc}}\right)^2\le\left(\sum a\right)\left(\sum\frac{a}{a^2+2bc}\right).$$
Thus, it suffices to prove that
$$\sum\frac{a}{a^2+2bc}\le\frac{a+b+c}{ab+bc+ca}.$$
This is equivalent to
$$\sum a\left(\frac{1}{ab+bc+ca}-\frac{1}{a^2+2bc}\right)\ge 0,$$
$$\sum\frac{a(a-b)(a-c)}{a^2+2bc}\ge 0.$$
We have
$$\sum\frac{a(a-b)(a-c)}{a^2+2bc}\ge\frac{a(a-b)(a-c)}{a^2+2bc}+\frac{b(b-c)(b-a)}{b^2+2ca}$$
$$=\frac{c(a-b)^2[2a(a-c)+2b(b-c)+3ab]}{(a^2+2bc)(b^2+2ca)}\ge 0.$$

□

P 2.32. *If a, b, c are nonnegative real numbers, then*

$$a^3 + b^3 + c^3 + 3abc \geq a^2\sqrt{a^2 + 3bc} + b^2\sqrt{b^2 + 3ca} + c^2\sqrt{c^2 + 3ab}.$$

(Vo Quoc Ba Can, 2008)

Solution. For $a = 0$, the inequality is an identity. Consider further that $a, b, c > 0$, and write the inequality as follows:

$$\sum a^2(\sqrt{a^2 + 3bc} - a) \leq 3abc,$$

$$\sum \frac{3a^2bc}{\sqrt{a^2 + 3bc} + a} \leq 3abc,$$

$$\sum \frac{1}{\sqrt{1 + 3bc/a^2} + 1} \leq 1.$$

Using the notation

$$x = \frac{1}{\sqrt{1 + 3bc/a^2} + 1}, \quad y = \frac{1}{\sqrt{1 + 3ca/b^2} + 1}, \quad z = \frac{1}{\sqrt{1 + 3ab/c^2} + 1},$$

implies

$$\frac{bc}{a^2} = \frac{1 - 2x}{3x^2}, \quad \frac{ca}{b^2} = \frac{1 - 2y}{3y^2}, \quad \frac{ab}{c^2} = \frac{1 - 2z}{3z^2}, \quad 0 < x, y, z < \frac{1}{2},$$

$$(1 - 2x)(1 - 2y)(1 - 2z) = 27x^2y^2z^2.$$

We need to prove that

$$x + y + z \leq 1$$

for $0 < x, y, z < \frac{1}{2}$ such that $(1 - 2x)(1 - 2y)(1 - 2z) = 27x^2y^2z^2$. To do it, we will use the contradiction method. Thus, assume that

$$x + y + z > 1, \qquad 0 < x, y, z < \frac{1}{2},$$

and show that

$$(1 - 2x)(1 - 2y)(1 - 2z) < 27x^2y^2z^2.$$

We have

$$\begin{aligned}
(1 - 2x)(1 - 2y)(1 - 2z) &< (x + y + z - 2x)(x + y + z - 2y)(x + y + z - 2z) \\
&< (y + z - x)(z + x - y)(x + y - z)(x + y + z)^3 \\
&\leq 3(y + z - x)(z + x - y)(x + y - z)(x + y + z)(x^2 + y^2 + z^2) \\
&= 3(2x^2y^2 + 2y^2z^2 + 2z^2x^2 - x^4 - y^4 - z^4)(x^2 + y^2 + z^2).
\end{aligned}$$

Therefore, it suffices to show that

$$(2x^2y^2 + 2y^2z^2 + 2z^2x^2 - x^4 - y^4 - z^4)(x^2 + y^2 + z^2) \leq 9x^2y^2z^2,$$

which is equivalent to

$$x^6 + y^6 + z^5 + 3x^2y^2z^2 \geq \sum y^2z^2(y^2 + z^2).$$

Clearly, this is just Schur's inequality of degree three applied to x^2, y^2, z^2. So, the proof is completed. The equality holds for $a = b = c$, and also for $a = 0$ or $b = 0$ or $c = 0$.

□

P 2.33. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{a}{\sqrt{4a^2 + 5bc}} + \frac{b}{\sqrt{4b^2 + 5ca}} + \frac{c}{\sqrt{4c^2 + 5ab}} \leq 1.$$

(Vasile Cîrtoaje, 2004)

First Solution *(by Vo Quoc Ba Can).* If one of a, b, c is zero, then the desired inequality is an equality. Consider next that $a, b, c > 0$ and denote

$$x = \frac{a}{\sqrt{4a^2 + 5bc}}, \quad y = \frac{b}{\sqrt{4b^2 + 5ca}}, \quad z = \frac{c}{\sqrt{4c^2 + 5ab}}, \quad x, y, z \in \left(0, \frac{1}{2}\right).$$

We have

$$\frac{bc}{a^2} = \frac{1 - 4x^2}{5x^2}, \quad \frac{ca}{b^2} = \frac{1 - 4y^2}{5y^2}, \quad \frac{ab}{c^2} = \frac{1 - 4z^2}{5z^2},$$

and

$$(1 - 4x^2)(1 - 4y^2)(1 - 4z^2) = 125x^2y^2z^2.$$

We use the contradiction method. For the sake of contradiction, assume that $x + y + z > 1$. Using the AM-GM inequality and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
x^2y^2z^2 &= \frac{1}{125}\prod(1 - 4x^2) < \frac{1}{125}\prod[(x + y + z)^2 - 4x^2] \\
&= \frac{1}{125}\prod(3x + y + z) \cdot \prod(y + z - x) \\
&\leq \left(\frac{x + y + z}{3}\right)^3 \prod(y + z - x) \\
&\leq \frac{1}{9}(x^2 + y^2 + z^2)(x + y + z)\prod(y + z - x) \\
&= \frac{1}{9}(x^2 + y^2 + z^2)[2(x^2y^2 + y^2z^2 + z^2x^2) - x^4 - y^4 - z^4],
\end{aligned}$$

hence

$$9x^2y^2z^2 < (x^2 + y^2 + z^2)[2(x^2y^2 + y^2z^2 + z^2x^2) - x^4 - y^4 - z^4],$$

$$x^6 + y^6 + z^6 + 3x^2y^2z^2 < \sum x^2y^2(x^2 + y^2).$$

The last inequality contradicts Schur's inequality

$$x^6 + y^6 + z^6 + 3x^2y^2z^2 \ge \sum x^2y^2(x^2 + y^2).$$

Thus, the proof is completed. The equality holds for $a = b = c$, and also for $a = 0$ or $b = 0$ or $c = 0$.

Second Solution. Use the mixing variables method. In the nontrivial case when $a, b, c > 0$, setting $x = \dfrac{bc}{a^2}$, $y = \dfrac{ca}{b^2}$ and $z = \dfrac{ab}{c^2}$ (that implies $xyz = 1$), the desired inequality becomes $E(x, y, z) \le 1$, where

$$E(x, y, z) = \frac{1}{\sqrt{4 + 5x}} + \frac{1}{\sqrt{4 + 5y}} + \frac{1}{\sqrt{4 + 5z}}.$$

Without loss of generality, we may assume that

$$x \ge y \ge z, \qquad x \ge 1, \quad yz \le 1.$$

We will prove that

$$E(x, y, z) \le E(x, \sqrt{yz}, \sqrt{yz}) \le 1.$$

The left inequality has the form

$$\frac{1}{\sqrt{4 + 5y}} + \frac{1}{\sqrt{4 + 5z}} \le \frac{1}{\sqrt{4 + 5\sqrt{yz}}}.$$

For the nontrivial case $y \ne z$, consider $y > z$ and denote

$$s = \frac{y + z}{2}, \quad p = \sqrt{yz},$$

$$q = \sqrt{(4 + 5y)(4 + 5z)}.$$

We have $s > p$, $p \le 1$ and

$$q = \sqrt{16 + 40s + 25p^2} > \sqrt{16 + 40p + 25p^2} = 4 + 5p.$$

By squaring, the desired inequality becomes in succession as follows:

$$\frac{1}{4 + 5y} + \frac{1}{4 + 5z} + \frac{2}{q} \le \frac{4}{4 + 5p},$$

$$\frac{1}{4 + 5y} + \frac{1}{4 + 5z} - \frac{2}{4 + 5p} \le \frac{2}{4 + 5p} - \frac{2}{q},$$

$$\frac{8 + 10s}{q^2} - \frac{2}{4 + 5p} \le \frac{2(q - 4 - 5p)}{q(4 + 5p)},$$

$$\frac{(s - p)(5p - 4)}{q^2(4 + 5p)} \le \frac{8(s - p)}{q(4 + 5p)(q + 4 + 5p)},$$

$$\frac{5p-4}{q} \le \frac{8}{q+4+5p},$$

$$25p^2 - 16 \le (12-5p)q.$$

The last inequality is true since

$$(12-5p)q - 25p^2 + 16 > (12-5p)(4+5p) - 25p^2 + 16$$
$$= 2(8-5p)(4+5p) > 0.$$

In order to prove the right inequality, namely

$$\frac{1}{\sqrt{4+5x}} + \frac{2}{\sqrt{4+5\sqrt{yz}}} \le 1,$$

let us denote

$$\sqrt{4+5\sqrt{yz}} = 3t, \qquad t \in (2/3, 1].$$

Since

$$x = \frac{1}{yz} = \frac{25}{(9t^2-4)^2},$$

the inequality becomes

$$\frac{9t^2-4}{3\sqrt{36t^4-32t^2+21}} + \frac{2}{3t} \le 1,$$

$$(2-3t)\left(\sqrt{36t^4-32t^2+21} - 3t^2 - 2t\right) \le 0.$$

Since $2 - 3t < 0$, we still have to show that

$$\sqrt{36t^4-32t^2+21} \ge 3t^2 + 2t.$$

Indeed, we have

$$36t^4 - 32t^2 + 21 - (3t^2+2t)^2 = 3(t-1)^2(9t^2+14t+7) \ge 0.$$

□

P 2.34. *Let a, b, c be nonnegative real numbers. Prove that*

$$a\sqrt{4a^2+5bc} + b\sqrt{4b^2+5ca} + c\sqrt{4c^2+5ab} \ge (a+b+c)^2.$$

(Vasile Cîrtoaje, 2004)

First Solution. Write the inequality as

$$\sum a\left(\sqrt{4a^2+5bc} - 2a\right) \ge 2(ab+bc+ca) - a^2 - b^2 - c^2,$$

$$5abc\sum \frac{1}{\sqrt{4a^2+5bc}+2a} \ge 2(ab+bc+ca)-a^2-b^2-c^2.$$

Writing Schur's inequality

$$a^3+b^3+c^3+3abc \ge \sum ab(a^2+b^2)$$

in the form

$$\frac{9abc}{a+b+c} \ge 2(ab+bc+ca)-a^2-b^2-c^2,$$

it suffices to prove that

$$\sum \frac{5}{\sqrt{4a^2+5bc}+2a} \ge \frac{9}{a+b+c}.$$

Let $p=a+b+c$ and $q=ab+bc+ca$. By the AM-GM inequality, we have

$$\sqrt{4a^2+5bc} = \frac{2\sqrt{(16a^2+20bc)(3b+3c)^2}}{12(b+c)} \le \frac{(16a^2+20bc)+(3b+3c)^2}{12(b+c)}$$

$$\le \frac{16a^2+16bc+10(b+c)^2}{12(b+c)} = \frac{8a^2+5b^2+5c^2+18bc}{6(b+c)},$$

hence

$$\sum \frac{5}{\sqrt{4a^2+5bc}+2a} \ge \sum \frac{5}{\dfrac{8a^2+5b^2+5c^2+18bc}{6(b+c)}+2a}$$

$$= \sum \frac{30(b+c)}{8a^2+5b^2+5c^2+12ab+18bc+12ac} = \sum \frac{30(b+c)}{5p^2+2q+3a^2+6bc}.$$

Thus, it suffices to show that

$$\sum \frac{30(b+c)}{5p^2+2q+3a^2+6bc} \ge \frac{9}{p}.$$

By the Cauchy-Schwarz inequality, we get

$$\sum \frac{30(b+c)}{5p^2+2q+3a^2+6bc} \ge \frac{30\left[\sum(b+c)\right]^2}{\sum(b+c)(5p^2+2q+3a^2+6bc)}$$

$$= \frac{120p^2}{10p^3+4pq+9\sum bc(b+c)} = \frac{120p^2}{10p^3+13pq-27abc}.$$

Therefore, it is enough to show that

$$\frac{120p^2}{10p^3+13pq-27abc} \ge \frac{9}{p},$$

which is equivalent to

$$10p^3+81abc \ge 39pq.$$

From Schur's inequality $p^3 + 9abc \geq 4pq$ and the known inequality $pq \geq 9abc$, we have

$$10p^3 + 81abc - 39pq = 10(p^3 + 9abc - 4pq) + pq - 9abc \geq 0.$$

This completes the proof. The equality holds for $a = b = c$, and also for $a = 0$ and $b = c$ (or any cyclic permutation).

Second Solution. By the Cauchy-Schwarz inequality, we have

$$\left(\sum a\sqrt{4a^2 + 5bc}\right)\left(\sum \frac{a}{\sqrt{4a^2 + 5bc}}\right) \geq (a + b + c)^2.$$

From this inequality and the inequality in P 2.33, namely

$$\sum \frac{a}{\sqrt{4a^2 + 5bc}} \leq 1,$$

the desired inequality follows.

Remark. Using the same way as in the second solution, we can prove the following inequalities for $a, b, c > 0$ satisfying $abc = 1$:

$$a\sqrt{4a^2 + 5} + b\sqrt{4b^2 + 5} + c\sqrt{4c^2 + 5} \geq (a + b + c)^2;$$

$$\sqrt{4a^4 + 5} + \sqrt{4b^4 + 5} + \sqrt{4c^4 + 5} \geq (a + b + c)^2.$$

The first inequality is a consequence of the the Cauchy-Schwarz inequality

$$\left(\sum a\sqrt{4a^2 + 5}\right)\left(\sum \frac{a}{\sqrt{4a^2 + 5}}\right) \geq (a + b + c)^2$$

and the inequality

$$\sum \frac{a}{\sqrt{4a^2 + 5}} \leq 1, \qquad abc = 1,$$

which follows from the inequality in P 2.33 by replacing bc/a^2, ca/b^2, ab/c^2 with $1/a^2$, $1/b^2$, $1/c^2$, respectively.

The second inequality is a consequence of the the Cauchy-Schwarz inequality

$$\left(\sum \sqrt{4a^4 + 5}\right)\left(\sum \frac{a^2}{\sqrt{4a^4 + 5}}\right) \geq (a + b + c)^2$$

and the inequality

$$\sum \frac{a^2}{\sqrt{4a^4 + 5}} \leq 1, \qquad abc = 1,$$

which follows from the inequality in P 2.33 by replacing bc/a^2, ca/b^2, ab/c^2 with $1/a^4$, $1/b^4$, $1/c^4$, respectively.

□

P 2.35. *Let a, b, c be nonnegative real numbers. Prove that*

$$a\sqrt{a^2+3bc}+b\sqrt{b^2+3ca}+c\sqrt{c^2+3ab} \ge 2(ab+bc+ca).$$

(Vasile Cîrtoaje, 2005)

First Solution *(by Vo Quoc Ba Can).* Using the AM-GM inequality yields

$$\begin{aligned}\sum a\sqrt{a^2+3bc} &= \sum \frac{a(b+c)(a^2+3bc)}{\sqrt{(b+c)^2(a^2+3bc)}}\\ &\ge \sum \frac{2a(b+c)(a^2+3bc)}{(b+c)^2+(a^2+3bc)}.\end{aligned}$$

Thus, it suffices to prove that

$$\sum \frac{2a(b+c)(a^2+3bc)}{a^2+b^2+c^2+5bc} \ge \sum a(b+c).$$

We will use the SOS method. Write the inequality as follows:

$$\sum \frac{a(b+c)(a^2-b^2-c^2+bc)}{a^2+b^2+c^2+5bc} \ge 0,$$

$$\sum \frac{a^3(b+c)-a(b^3+c^3)}{a^2+b^2+c^2+5bc} \ge 0,$$

$$\sum \frac{ab(a^2-b^2)-ac(c^2-a^2)}{a^2+b^2+c^2+5bc} \ge 0,$$

$$\sum \frac{ab(a^2-b^2)}{a^2+b^2+c^2+5bc} - \sum \frac{ba(a^2-b^2)}{b^2+c^2+a^2+5ca} \ge 0,$$

$$\sum \frac{5abc(a+b)(a-b)^2}{(a^2+b^2+c^2+5bc)(a^2+b^2+c^2+5ac)} \ge 0.$$

The equality holds $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

Second Solution. Write the inequality as

$$\sum \left(a\sqrt{a^2+3bc}-a^2\right) \ge 2(ab+bc+ca)-a^2-b^2-c^2.$$

Due to homogeneity, we may assume that $a+b+c=3$. By the AM-GM inequality, we have

$$\begin{aligned}a\sqrt{a^2+3bc}-a^2 = \frac{3abc}{\sqrt{a^2+3bc}+a} &= \frac{12abc}{2\sqrt{4(a^2+3bc)}+4a}\\ &\ge \frac{12abc}{4+a^2+3bc+4a}.\end{aligned}$$

Thus, it suffices to show that

$$12abc\sum \frac{1}{4+a^2+3bc+4a} \ge 2(ab+bc+ca)-a^2-b^2-c^2.$$

On the other hand, by Schur's inequality of degree three, we have

$$\frac{9abc}{a+b+c} \ge 2(ab+bc+ca) - a^2 - b^2 - c^2.$$

Therefore, it is enough to prove that

$$\sum \frac{1}{4+a^2+3bc+4a} \ge \frac{3}{4(a+b+c)}.$$

By the AM-HM inequality, we have

$$\begin{aligned}\sum \frac{1}{4+a^2+3bc+4a} &\ge \frac{9}{\sum(4+a^2+3bc+4a)} = \frac{9}{24+\sum a^2+3\sum ab} \\ &= \frac{27}{8\left(\sum a\right)^2+3\sum a^2+9\sum ab} \\ &= \frac{9\sum a}{11\left(\sum a\right)^2+3\sum ab} \ge \frac{3}{4\sum a}.\end{aligned}$$

□

P 2.36. *Let a, b, c be nonnegative real numbers. Prove that*

$$a\sqrt{a^2+8bc} + b\sqrt{b^2+8ca} + c\sqrt{c^2+8ab} \le (a+b+c)^2.$$

Solution. Multiplying by $a+b+c$, the inequality becomes

$$\sum a\sqrt{(a+b+c)^2(a^2+8bc)} \le (a+b+c)^3.$$

Since

$$2\sqrt{(a+b+c)^2(a^2+8bc)} \le (a+b+c)^2 + (a^2+8bc),$$

it suffices to show that

$$\sum a[(a+b+c)^2 + (a^2+8bc)] \le 2(a+b+c)^3,$$

which can be written as

$$a^3+b^3+c^3+24abc \le (a+b+c)^3.$$

This inequality is equivalent to

$$a(b-c)^2 + b(c-a)^2 + c(a-b)^2 \ge 0.$$

The equality holds for $a=b=c$, and also for $b=c=0$ (or any cyclic permutation).

□

P 2.37. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{a^2+2bc}{\sqrt{b^2+bc+c^2}}+\frac{b^2+2ca}{\sqrt{c^2+ca+a^2}}+\frac{c^2+2ab}{\sqrt{a^2+ab+b^2}} \ge 3\sqrt{ab+bc+ca}.$$

(Michael Rozenberg and Marius Stanean, 2011)

Solution. By the AM-GM inequality, we have

$$\begin{aligned}\sum \frac{a^2+2bc}{\sqrt{b^2+bc+c^2}} &= \sum \frac{2(a^2+2bc)\sqrt{ab+bc+ca}}{2\sqrt{(b^2+bc+c^2)(ab+bc+ca)}} \\ &\ge \sqrt{ab+bc+ca}\sum \frac{2(a^2+2bc)}{(b^2+bc+c^2)+(ab+bc+ca)} \\ &= \sqrt{ab+bc+ca}\sum \frac{2(a^2+2bc)}{(b+c)(a+b+c)}.\end{aligned}$$

Thus, it suffices to show that

$$\frac{a^2+2bc}{b+c}+\frac{b^2+2ca}{c+a}+\frac{c^2+2ab}{a+b} \ge \frac{3}{2}(a+b+c).$$

This inequality is equivalent to

$$a^4+b^4+c^4+abc(a+b+c) \ge \frac{1}{2}\sum ab(a+b)^2.$$

We can prove this inequality by summing Schur's inequality of fourth degree

$$a^4+b^4+c^4+abc(a+b+c) \ge \sum ab(a^2+b^2)$$

and the obvious inequality

$$\sum ab(a^2+b^2) \ge \frac{1}{2}\sum ab(a+b)^2.$$

The equality holds for $a = b = c$.

□

P 2.38. *Let a, b, c be nonnegative real numbers, no two of which are zero. If $k \ge 1$, then*

$$\frac{a^{k+1}}{2a^2+bc}+\frac{b^{k+1}}{2b^2+ca}+\frac{c^{k+1}}{2c^2+ab} \le \frac{a^k+b^k+c^k}{a+b+c}.$$

(Vasile Cîrtoaje and Vo Quoc Ba Can, 2011)

Solution. Write the inequality as follows:

$$\sum \left(\frac{a^k}{a+b+c} - \frac{a^{k+1}}{2a^2+bc} \right) \ge 0,$$

$$\sum \frac{a^k(a-b)(a-c)}{2a^2+bc} \ge 0.$$

Assume that $a \ge b \ge c$. Since $(c-a)(c-b) \ge 0$, it suffices to show that

$$\frac{a^k(a-b)(a-c)}{2a^2+bc} + \frac{b^k(b-a)(b-c)}{2b^2+ca} \ge 0.$$

This is true if

$$\frac{a^k(a-c)}{2a^2+bc} - \frac{b^k(b-c)}{2b^2+ca} \ge 0,$$

which is equivalent to

$$a^k(a-c)(2b^2+ca) \ge b^k(b-c)(2a^2+bc).$$

Since $a^k/b^k \ge a/b$, it remains to show that

$$a(a-c)(2b^2+ca) \ge b(b-c)(2a^2+bc),$$

which is equivalent to the obvious inequality

$$(a-b)c[a^2+3ab+b^2-c(a+b)] \ge 0.$$

The equality holds for $a=b=c$, and also for $a=b$ and $c=0$ (or any cyclic permutation).

□

P 2.39. *If a,b,c are positive real numbers, then*

(a)
$$\frac{a^2-bc}{\sqrt{3a^2+2bc}} + \frac{b^2-ca}{\sqrt{3b^2+2ca}} + \frac{c^2-ab}{\sqrt{3c^2+2ab}} \ge 0;$$

(b)
$$\frac{a^2-bc}{\sqrt{8a^2+(b+c)^2}} + \frac{b^2-ca}{\sqrt{8b^2+(c+a)^2}} + \frac{c^2-ab}{\sqrt{8c^2+(a+b)^2}} \ge 0.$$

(Vasile Cîrtoaje, 2006)

Solution. (a) Use the SOS technique. Let

$$A = \sqrt{3a^2+2bc}, \quad B = \sqrt{3b^2+2ca}, \quad C = \sqrt{3c^2+2ab}.$$

We have

$$2\sum \frac{a^2-bc}{A} = \sum \frac{(a-b)(a+c)+(a-c)(a+b)}{A}$$
$$= \sum \frac{(a-b)(a+c)}{A} + \sum \frac{(b-a)(b+c)}{B}$$
$$= \sum (a-b)\left(\frac{a+c}{A} - \frac{b+c}{B}\right)$$
$$= \sum \frac{a-b}{AB} \cdot \frac{(a+c)^2B^2-(b+c)^2A^2}{(a+c)B+(b+c)A},$$

hence

$$2\sum \frac{a^2-bc}{A} = \sum \frac{c(a-b)^2}{AB} \cdot \frac{2(a-b)^2+c(a+b+2c)}{(a+c)B+(b+c)A} \ge 0.$$

The equality holds for $a=b=c$.

(b) Let

$$A=\sqrt{8a^2+(b+c)^2}, \quad B=\sqrt{8b^2+(c+a)^2}, \quad C=\sqrt{8c^2+(a+b)^2b}.$$

As we have shown before,

$$2\sum \frac{a^2-bc}{A} = \sum \frac{a-b}{AB} \cdot \frac{(a+c)^2B^2-(b+c)^2A^2}{(a+c)B+(b+c)A},$$

hence

$$2\sum \frac{a^2-bc}{A} = \sum \frac{(a-b)^2}{AB} \cdot \frac{C_1}{(a+c)B+(b+c)A} \ge 0,$$

since

$$C_1 = [(a+c)+(b+c)][(a+c)^2+(b+c)^2] - 8ac(b+c) - 8bc(a+c)$$
$$\ge [(a+c)+(b+c)](4ac+4bc) - 8ac(b+c) - 8bc(a+c)$$
$$= 4c(a-b)^2 \ge 0.$$

The equality holds for $a=b=c$.

□

P 2.40. *Let a,b,c be positive real numbers. If $0 \le k \le 1+2\sqrt{2}$, then*

$$\frac{a^2-bc}{\sqrt{ka^2+b^2+c^2}} + \frac{b^2-ca}{\sqrt{kb^2+c^2+a^2}} + \frac{c^2-ab}{\sqrt{kc^2+a^2+b^2}} \ge 0.$$

Solution. Use the SOS method. Let

$$A = \sqrt{ka^2 + b^2 + c^2}, \quad B = \sqrt{kb^2 + c^2 + a^2}, \quad C = \sqrt{kc^2 + a^2 + b^2}.$$

As we have shown at the preceding problem,

$$2\sum \frac{a^2 - bc}{A} = \sum \frac{a-b}{AB} \cdot \frac{(a+c)^2B^2 - (b+c)^2A^2}{(a+c)B + (b+c)A};$$

therefore

$$2\sum \frac{a^2 - bc}{A} = \sum \frac{(a-b)^2}{AB} \cdot \frac{C_1}{(a+c)B + (b+c)A},$$

where

$$C_1 = (a^2 + b^2 + c^2)(a + b + 2c) - (k-1)c(2ab + bc + ca).$$

It suffices to show that $C_1 \ge 0$. Putting $a + b = 2x$, we have $a^2 + b^2 \ge 2x^2$, $ab \le x^2$, hence

$$\begin{aligned} C_1 &\ge (a^2 + b^2 + c^2)(a + b + 2c) - 2\sqrt{2}\, c(2ab + bc + ca) \\ &\ge (2x^2 + c^2)(2x + 2c) - 2\sqrt{2}\, c(2x^2 + 2cx) \\ &= 2(x+c)(x\sqrt{2} - c)^2 \ge 0. \end{aligned}$$

The equality holds for $a = b = c$.

□

P 2.41. *If a, b, c are nonnegative real numbers, then*

$$(a^2 - bc)\sqrt{b+c} + (b^2 - ca)\sqrt{c+a} + (c^2 - ab)\sqrt{a+b} \ge 0.$$

First Solution. Let us denote

$$x = \sqrt{\frac{b+c}{2}}, \quad y = \sqrt{\frac{c+a}{2}}, \quad z = \sqrt{\frac{a+b}{2}},$$

hence

$$a = y^2 + z^2 - x^2, \quad b = z^2 + x^2 - y^2, \quad c = x^2 + y^2 - z^2.$$

The inequality turns into

$$xy(x^3 + y^3) + yz(y^3 + z^3) + zx(z^3 + x^3) \ge x^2y^2(x+y) + y^2z^2(y+z) + z^2x^2(z+x),$$

which is equivalent to the obvious inequality

$$xy(x+y)(x-y)^2 + yz(y+z)(y-z)^2 + zx(z+x)(z-x)^2 \ge 0.$$

The equality holds for $a = b = c$, and also for $b = c = 0$ (or any cyclic permutation).

Second Solution. Use the SOS technique. Write the inequality as

$$A(a^2-bc)+B(b^2-ca)+C(c^2-ab)\ge 0,$$

where

$$A=\sqrt{b+c},\quad B=\sqrt{c+a},\quad C=\sqrt{a+b}.$$

We have

$$\begin{aligned}2\sum A(a^2-bc) &= \sum A[(a-b)(a+c)+(a-c)(a+b)]\\ &= \sum A(a-b)(a+c)+\sum B(b-a)(b+c)\\ &= \sum (a-b)[A(a+c)-B(b+c)]\\ &= \sum (a-b)\cdot\frac{A^2(a+c)^2-B^2(b+c)^2}{A(a+c)+B(b+c)}\\ &= \sum \frac{(a-b)^2(a+c)(b+c)}{A(a+c)+B(b+c)}\ge 0.\end{aligned}$$

□

P 2.42. *If a,b,c are nonnegative real numbers, then*

$$(a^2-bc)\sqrt{a^2+4bc}+(b^2-ca)\sqrt{b^2+4ca}+(c^2-ab)\sqrt{c^2+4ab}\ge 0.$$

(Vasile Cîrtoaje, 2005)

Solution. If two of a,b,c are zero, then the inequality is clearly true. Otherwise, write the inequality as

$$AX+BY+CZ\ge 0,$$

where

$$A=\frac{\sqrt{a^2+4bc}}{b+c},\quad B=\frac{\sqrt{b^2+4ca}}{c+a},\quad C=\frac{\sqrt{c^2+4ab}}{a+b},$$

$$X=(a^2-bc)(b+c),\quad Y=(b^2-bc)(b+c),\quad Z=(c^2-ab)(a+b).$$

Without loss of generality, assume that

$$a\ge b\ge c.$$

We have

$$X\ge 0,\quad Z\le 0,\quad X+Y+Z=0.$$

In addition,

$$X-Y=ab(a-b)+2(a^2-b^2)c+(a-b)c^2\ge 0$$

and

$$A^2-B^2=\frac{a^4-b^4+2(a^3-c^3)c+(a^2-c^2)c^2+4abc(a-b)-4(a-b)c^3}{(b+c)^2(c+a)^2}$$

$$\geq \frac{4abc(a-b)-4(a-b)c^3}{(b+c)^2(c+a)^2} = \frac{4c(a-b)(ab-c^2)}{(b+c)^2(c+a)^2} \geq 0.$$

Since

$$\begin{aligned} 2(AX+BY+CZ) &= (A-B)(X-Y)+(A+B)(X+Y)+2CZ \\ &= (A-B)(X-Y)-(A+B-2C)Z, \end{aligned}$$

it suffices to show that

$$A+B-2C \geq 0.$$

This is true if $AB \geq C^2$. Using the Cauchy-Schwarz inequality gives

$$AB \geq \frac{ab+4c\sqrt{ab}}{(b+c)(c+a)} \geq \frac{ab+2c\sqrt{ab}+2c^2}{(b+c)(c+a)}.$$

Thus, it is enough to show that

$$(a+b)^2(ab+2c\sqrt{ab}+2c^2) \geq (b+c)(c+a)(c^2+4ab).$$

Write this inequality as

$$ab(a-b)^2+2c\sqrt{ab}(a+b)\left(\sqrt{a}-\sqrt{b}\right)^2+c^2[2(a+b)^2-5ab-c(a+b)-c^2] \geq 0.$$

It is true since

$$2(a+b)^2-5ab-c(a+b)-c^2 = a(2a-b-c)+b(b-c)+b^2-c^2 \geq 0.$$

The equality holds for $a=b=c$, and also for $a=b$ and $c=0$ (or any cyclic permutation).

□

P 2.43. *If a,b,c are nonnegative real numbers, then*

$$\sqrt{\frac{a^3}{a^3+(b+c)^3}}+\sqrt{\frac{b^3}{b^3+(c+a)^3}}+\sqrt{\frac{c^3}{c^3+(a+b)^3}} \geq 1.$$

Solution. For $a=0$, the inequality reduces to the obvious inequality

$$\sqrt{b^3}+\sqrt{c^3} \geq \sqrt{b^3+c^3}.$$

For $a,b,c>0$, write the inequality as

$$\sum \sqrt{\frac{1}{1+\left(\frac{b+c}{a}\right)^3}} \geq 1.$$

For any $x \ge 0$, we have

$$\sqrt{1+x^3} = \sqrt{(1+x)(1-x+x^2)} \le \frac{(1+x)+(1-x+x^2)}{2} = 1+\frac{1}{2}x^2.$$

Therefore, we get

$$\sum \sqrt{\frac{1}{1+\left(\frac{b+c}{a}\right)^3}} \ge \sum \frac{1}{1+\frac{1}{2}\left(\frac{b+c}{a}\right)^2}$$

$$\ge \sum \frac{1}{1+\frac{b^2+c^2}{a^2}} = \sum \frac{a^2}{a^2+b^2+c^2} = 1.$$

The equality holds for $a=b=c$, and also for $b=c=0$ (or any cyclic permutation).

□

P 2.44. *If a, b, c are positive real numbers, then*

$$\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} \ge 1+\sqrt{1+\sqrt{(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}}.$$

(Vasile Cîrtoaje, 2002)

Solution. Using the Cauchy-Schwarz inequality, we have

$$\left(\sum a\right)\left(\sum \frac{1}{a}\right) = \sqrt{\left(\sum a^2+2\sum bc\right)\left(\sum \frac{1}{a^2}+2\sum \frac{1}{bc}\right)}$$

$$\ge \sqrt{\left(\sum a^2\right)\left(\sum \frac{1}{a^2}\right)}+2\sqrt{\left(\sum bc\right)\left(\sum \frac{1}{bc}\right)}$$

$$= \sqrt{\left(\sum a^2\right)\left(\sum \frac{1}{a^2}\right)}+2\sqrt{\left(\sum a\right)\left(\sum \frac{1}{a}\right)},$$

hence

$$\left[\sqrt{\left(\sum a\right)\left(\sum \frac{1}{a}\right)}-1\right]^2 \ge 1+\sqrt{\left(\sum a^2\right)\left(\sum \frac{1}{a^2}\right)},$$

$$\sqrt{\left(\sum a\right)\left(\sum \frac{1}{a}\right)}-1 \ge \sqrt{1+\sqrt{\left(\sum a^2\right)\left(\sum \frac{1}{a^2}\right)}}.$$

The equality holds if and only if

$$\left(\sum a^2\right)\left(\sum \frac{1}{bc}\right) = \left(\sum \frac{1}{a^2}\right)\left(\sum bc\right),$$

which is equivalent to

$$(a^2 - bc)(b^2 - ca)(c^2 - ab) = 0.$$

Consequently, the equality occurs for $a^2 = bc$ or $b^2 = ca$ or $c^2 = ab$.

□

P 2.45. *If a, b, c are positive real numbers, then*

$$5 + \sqrt{2(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) - 2} \ge (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

(Vasile Cîrtoaje, 2004)

Solution. Let us denote

$$x = \frac{a}{b}+\frac{b}{c}+\frac{c}{a}, \quad y = \frac{b}{a}+\frac{c}{b}+\frac{a}{c}.$$

From

$$2(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) - 2 =$$
$$= 2\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right) + 2\left(\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}\right) + 4$$
$$= 2(x^2-2y) + 2(y^2-2x) + 4$$
$$= (x+y-2)^2 + (x-y)^2$$
$$\ge (x+y-2)^2$$

and

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = x+y+3,$$

we get

$$\sqrt{2(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) - 2} \ge x+y-2$$
$$= (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) - 5.$$

The equality occurs for $a = b$ or $b = c$ or $c = a$.

□

P 2.46. *If a, b, c are real numbers, then*

$$2(1+abc) + \sqrt{2(1+a^2)(1+b^2)(1+c^2)} \ge (1+a)(1+b)(1+c).$$

(Wolfgang Berndt, 2006)

First Solution. Denoting

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc,$$

the inequality becomes

$$\sqrt{2(p^2 + q^2 + r^2 - 2pr - 2q + 1)} \geq p + q - r - 1.$$

It suffices to show that

$$2(p^2 + q^2 + r^2 - 2pr - 2q + 1) \geq (p + q - r - 1)^2,$$

which is equivalent to

$$p^2 + q^2 + r^2 - 2pq + 2qr - 2pr + 2p - 2q - 2r + 1 \geq 0,$$

$$(p - q - r + 1)^2 \geq 0.$$

The equality holds for $p+1 = q+r$ and $q \geq 1$. The last condition follows from $p+q-r-1 \geq 0$.

Second Solution. Since

$$2(1 + a^2) = (1 + a)^2 + (1 - a)^2$$

and

$$(1 + b^2)(1 + c^2) = (b + c)^2 + (bc - 1)^2,$$

by the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\sqrt{2(1 + a^2)(1 + b^2)(1 + c^2)} &\geq (1 + a)(b + c) + (1 - a)(bc - 1) \\ &= (1 + a)(1 + b)(1 + c) - 2(1 + abc).\end{aligned}$$

□

P 2.47. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\sqrt{\frac{a^2 + bc}{b^2 + c^2}} + \sqrt{\frac{b^2 + ca}{c^2 + a^2}} + \sqrt{\frac{c^2 + ab}{a^2 + b^2}} \geq 2 + \frac{1}{\sqrt{2}}.$$

(Vo Quoc Ba Can, 2006)

Solution. Assume that

$$a \geq b \geq c.$$

It suffices to show that

$$\sqrt{\frac{a^2 + c^2}{b^2 + c^2}} + \sqrt{\frac{b^2 + c^2}{c^2 + a^2}} + \sqrt{\frac{ab}{a^2 + b^2}} \geq 2 + \frac{1}{\sqrt{2}}.$$

Let us denote

$$x = \sqrt{\frac{a^2+c^2}{b^2+c^2}}, \quad y = \sqrt{\frac{a}{b}}.$$

From

$$x^2 - y^2 = \frac{(a-b)(ab-c^2)}{b(b^2+c^2)} \ge 0,$$

it follows that

$$x \ge y \ge 1.$$

Also, from

$$x + \frac{1}{x} - \left(y + \frac{1}{y}\right) = \frac{(x-y)(xy-1)}{xy} \ge 0,$$

we have

$$\sqrt{\frac{a^2+c^2}{b^2+c^2}} + \sqrt{\frac{b^2+c^2}{c^2+a^2}} \ge \sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}}.$$

Therefore, it is enough to show that

$$\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} + \sqrt{\frac{ab}{a^2+b^2}} \ge 2 + \frac{1}{\sqrt{2}},$$

which is equivalent to

$$\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} - 2 \ge \frac{1}{\sqrt{2}} - \sqrt{\frac{ab}{a^2+b^2}},$$

$$\frac{(\sqrt{a}-\sqrt{b})^2}{\sqrt{ab}} \ge \frac{(a-b)^2}{\sqrt{2(a^2+v^2}\left(\sqrt{a^2+b^2}+\sqrt{2ab}\right)}.$$

Since $2\sqrt{ab} \le \sqrt{2(a^2+b^2)}$, it suffices to show that

$$2 \ge \frac{(\sqrt{a}+\sqrt{b}\ )^2}{\sqrt{a^2+b^2}+\sqrt{2ab}}.$$

Indeed,

$$2\left(\sqrt{a^2+b^2}+\sqrt{2ab}\right) > \sqrt{2(a^2+b^2)} + 2\sqrt{ab} \ge a+b+2\sqrt{ab} = \left(\sqrt{a}+\sqrt{b}\right)^2.$$

The equality holds for $a = b$ and $c = 0$ (or any cyclic permutation).

□

P 2.48. *If a, b, c are nonnegative real numbers, then*

$$\sqrt{a(2a+b+c)} + \sqrt{b(2b+c+a)} + \sqrt{c(2c+a+b)} \ge \sqrt{12(ab+bc+ca)}.$$

(Vasile Cîrtoaje, 2012)

Solution. By squaring, the inequality becomes

$$a^2+b^2+c^2+\sum\sqrt{bc(2b+c+a)(2c+a+b)}\geq 5(ab+bc+ca).$$

Using the Cauchy-Schwarz inequality yields

$$\sum\sqrt{bc(2b+c+a)(2c+a+b)}=\sum\sqrt{(2b^2+bc+ab)(2c^2+bc+ac)}$$

$$\geq\sum\left(2bc+bc+a\sqrt{bc}\right)=3(ab+bc+ca)+\sum a\sqrt{bc}.$$

Therefore, it suffices to show that

$$a^2+b^2+c^2+\sum a\sqrt{bc}\geq 2(ab+bc+ca).$$

We can get this inequality by summing Schur's inequality

$$a^2+b^2+c^2+\sum a\sqrt{bc}\geq\sum\sqrt{ab}(a+b)$$

and

$$\sum\sqrt{ab}\ (a+b)\geq 2(ab+bc+ca).$$

The last inequality is equivalent to the obvious inequality

$$\sum\sqrt{ab}\ \left(\sqrt{a}-\sqrt{b}\right)^2\geq 0.$$

The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 2.49. *Let a,b,c be nonnegative real numbers such that $a+b+c=3$. Prove that*

$$a\sqrt{(4a+5b)(4a+5c)}+b\sqrt{(4b+5c)(4b+5a)}+c\sqrt{(4c+5a)(4c+5b)}\geq 27.$$

(Vasile Cîrtoaje, 2010)

Solution. Use the SOS technique. Assume that

$$a\geq b\geq c,$$

consider the nontrivial case $b>0$, and write the inequality in the following equivalent homogeneous forms:

$$\sum a\sqrt{(4a+5b)(4a+5c)}\geq 3(a+b+c)^2,$$

$$2\left(\sum a^2-\sum ab\right)\geq\sum a\left(\sqrt{4a+5b}-\sqrt{4a+5c}\right)^2,$$

$$\sum(b-c)^2\geq\sum\frac{25a(b-c)^2}{\left(\sqrt{4a+5b}+\sqrt{4a+5c}\right)^2},$$

$$\sum (b-c)^2 S_a \ge 0,$$

where

$$S_a = 1 - \frac{25a}{\left(\sqrt{4a+5b}+\sqrt{4a+5c}\right)^2}.$$

Since

$$S_b = 1 - \frac{25b}{\left(\sqrt{4b+5c}+\sqrt{4b+5a}\right)^2} \ge 1 - \frac{25b}{\left(\sqrt{4b}+\sqrt{9b}\right)^2} = 0$$

and

$$S_c = 1 - \frac{25c}{\left(\sqrt{4c+5a}+\sqrt{4c+5b}\right)^2} \ge 1 - \frac{25c}{\left(\sqrt{9c}+\sqrt{9c}\right)^2} = 1 - \frac{25}{36} > 0,$$

we have

$$\sum (b-c)^2 S_a \ge (b-c)^2 S_a + (a-c)^2 S_b \ge (b-c)^2 S_a + \frac{a^2}{b^2}(b-c)^2 S_b$$

$$= \frac{a}{b}(b-c)^2 \left(\frac{b}{a}S_a + \frac{a}{b}S_b\right).$$

Thus, it suffices to prove that

$$\frac{b}{a}S_a + \frac{a}{b}S_b \ge 0.$$

We have

$$S_a \ge 1 - \frac{25a}{\left(\sqrt{4a+5b}+\sqrt{4a}\right)^2} = 1 - \frac{a\left(\sqrt{4a+5b}-\sqrt{4a}\right)^2}{b^2},$$

$$S_b \ge 1 - \frac{25b}{\left(\sqrt{4b}+\sqrt{4b+5a}\right)^2} = 1 - \frac{b\left(\sqrt{4b+5a}-\sqrt{4b}\right)^2}{a^2},$$

hence

$$\begin{aligned}\frac{b}{a}S_a + \frac{a}{b}S_b &\ge \frac{b}{a} - \frac{\left(\sqrt{4a+5b}-\sqrt{4a}\right)^2}{b} + \frac{a}{b} - \frac{\left(\sqrt{4b+5a}-\sqrt{4b}\right)^2}{a}\\ &= 4\left(\sqrt{\frac{4a^2}{b^2}+\frac{5a}{b}} + \sqrt{\frac{4b^2}{a^2}+\frac{5b}{a}}\right) - 7\left(\frac{a}{b}+\frac{b}{a}\right) - 10\\ &= 4\sqrt{4x^2+5x-8+2\sqrt{20x+41}} - 7x - 10,\end{aligned}$$

where

$$x = \frac{a}{b} + \frac{b}{a} \ge 2.$$

To end the proof, we only need to show that $x \ge 2$ yields

$$4\sqrt{4x^2+5x-8+2\sqrt{20x+41}} \ge 7x+10.$$

By squaring, this inequality becomes

$$15x^2 - 60x - 228 + 32\sqrt{20x+41} \ge 0.$$

Indeed,

$$15x^2 - 60x - 228 + 32\sqrt{20x+41} \ge 15x^2 - 60x - 228 + 32\sqrt{81} = 15(x-2)^2 \ge 0.$$

The equality holds for $a = b = c = 1$, and also for $a = b = \frac{3}{2}$ and $c = 0$ (or any cyclic permutation).

□

P 2.50. *Let a, b, c be nonnegative real numbers such that $ab + bc + ca = 3$. Prove that*

$$a\sqrt{(a+3b)(a+3c)} + b\sqrt{(b+3c)(b+3a)} + c\sqrt{(c+3a)(c+3b)} \ge 12.$$

(Vasile Cîrtoaje, 2010)

Solution. Use the SOS method. Assume that $a \ge b \ge c$ $(b > 0)$, and write the inequality as

$$\sum a\sqrt{(a+3b)(a+3c)} \ge 4(ab+bc+ca),$$

$$2(\sum a^2 - \sum ab) = \sum a\left(\sqrt{a+3b} - \sqrt{a+3c}\right)^2,$$

$$\sum (b-c)^2 \ge \sum \frac{9a(b-c)^2}{\left(\sqrt{a+3b} + \sqrt{a+3c}\right)^2},$$

$$\sum (b-c)^2 S_a \ge 0,$$

where

$$S_a = 1 - \frac{9a}{\left(\sqrt{a+3b} + \sqrt{a+3c}\right)^2}.$$

Since

$$S_b = 1 - \frac{9b}{\left(\sqrt{b+3c} + \sqrt{b+3a}\right)^2} \ge 1 - \frac{9b}{\left(\sqrt{b} + \sqrt{4b}\right)^2} = 0$$

and

$$S_c = 1 - \frac{9c}{\left(\sqrt{c+3a} + \sqrt{c+3b}\right)^2} \ge 1 - \frac{9c}{\left(\sqrt{4c} + \sqrt{4c}\right)^2} = 1 - \frac{9}{16} > 0,$$

we have

$$\sum (b-c)^2 S_a \ge (b-c)^2 S_a + (a-c)^2 S_b \ge (b-c)^2 S_a + \frac{a^2}{b^2}(b-c)^2 S_b$$

$$= \frac{a}{b}(b-c)^2\left(\frac{b}{a}S_a + \frac{a}{b}S_b\right).$$

Thus, it suffices to prove that

$$\frac{b}{a}S_a + \frac{a}{b}S_b \ge 0.$$

We have

$$S_a \ge 1 - \frac{9a}{\left(\sqrt{a+3b}+\sqrt{a}\right)^2} = 1 - \frac{a\left(\sqrt{a+3b}-\sqrt{a}\right)^2}{b^2},$$

$$S_b \ge 1 - \frac{9b}{\left(\sqrt{b}+\sqrt{b+3a}\right)^2} = 1 - \frac{b\left(\sqrt{b+3a}-\sqrt{b}\right)^2}{a^2},$$

hence

$$\begin{aligned}\frac{b}{a}S_a + \frac{a}{b}S_b &\ge \frac{b}{a} - \frac{\left(\sqrt{a+3b}-\sqrt{a}\right)^2}{b} + \frac{a}{b} - \frac{\left(\sqrt{b+3a}-\sqrt{b}\right)^2}{a} \\ &= 2\left(\sqrt{\frac{a^2}{b^2}+\frac{3a}{b}} + \sqrt{\frac{b^2}{a^2}+\frac{3b}{a}}\right) - \left(\frac{a}{b}+\frac{b}{a}\right) - 6 \\ &= 2\sqrt{x^2+3x-2+2\sqrt{3x+10}} - x - 6,\end{aligned}$$

where

$$x = \frac{a}{b} + \frac{b}{a} \ge 2.$$

To end the proof, it remains to show that

$$2\sqrt{x^2+35x-2+2\sqrt{3x+10}} \ge x+6$$

for $x \ge 2$. By squaring, this inequality becomes

$$3x^2 - 44 + 8\sqrt{3x+10} \ge 0.$$

Indeed,

$$3x^2 - 44 + 8\sqrt{3x+10} \ge 12 - 44 + 32 = 0.$$

The equality holds for $a = b = c = 1$, and also for $a = b = \sqrt{3}$ and $c = 0$ (or any cyclic permutation).

□

P 2.51. *Let a, b, c be nonnegative real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that*

$$\sqrt{2+7ab} + \sqrt{2+7bc} + \sqrt{2+7ca} \ge 3\sqrt{3(ab+bc+ca)}.$$

(Vasile Cîrtoaje, 2010)

Solution. Use the SOS method. Consider $a \ge b \ge c$. Since the inequality is trivial for $b = c = 0$, we may assume that $b > 0$. By squaring, the desired inequality becomes

$$6 + 2\sum \sqrt{(2+7ab)(2+7ac)} \ge 20(ab+bc+ca),$$

$$6(a^2+b^2+c^2-ab-bc-ca) \ge \sum \left(\sqrt{2+7ab} - \sqrt{2+7ac}\right)^2,$$

$$3\sum (b-c)^2 \ge \sum \frac{49a^2(b-c)^2}{\left(\sqrt{2+7ab}+\sqrt{2+7ac}\right)^2},$$

$$\sum (b-c)^2 S_a \ge 0,$$

where

$$S_a = 1 - \frac{49a^2}{\left(\sqrt{6+21ab}+\sqrt{6+21ac}\right)^2},$$

$$S_b = 1 - \frac{49b^2}{\left(\sqrt{6+21ab}+\sqrt{6+21bc}\right)^2},$$

$$S_c = 1 - \frac{49c^2}{\left(\sqrt{6+21ac}+\sqrt{6+21bc}\right)^2}.$$

Since $6 \ge 2(a^2+b^2) \ge 4ab$, we have

$$S_a \ge 1 - \frac{49a^2}{\left(\sqrt{4ab+21ab}+\sqrt{6}\right)^2} \ge 1 - \frac{49a^2}{\left(5\sqrt{ab}+2\sqrt{ab}\right)^2} = 1 - \frac{a}{b},$$

$$S_b \ge 1 - \frac{49b^2}{\left(\sqrt{4ab+21ab}+\sqrt{6}\right)^2} \ge 1 - \frac{49b^2}{\left(5\sqrt{ab}+2\sqrt{ab}\right)^2} = 1 - \frac{b}{a},$$

$$S_c \ge 1 - \frac{49c^2}{\left(\sqrt{4ab+21ac}+\sqrt{4ab+21bc}\right)^2} \ge 1 - \frac{49c^2}{(5c+5c)^2} = 1 - \frac{49}{100} > 0.$$

Therefore,

$$\begin{aligned}\sum (b-c)^2 S_a &\ge (b-c)^2 S_a + (c-a)^2 S_b \\ &\ge (b-c)^2\left(1-\frac{a}{b}\right) + (c-a)^2\left(1-\frac{b}{a}\right) \\ &= \frac{(a-b)^2(ab-c^2)}{ab} \ge 0.\end{aligned}$$

The equality holds for $a = b = c = 1$, and also for $a = b = \sqrt{3}$ and $c = 0$ (or any cyclic permutation).

□

P 2.52. *Let a, b, c be nonnegative real numbers such that $a^2+b^2+c^2=3$. Prove that*

$$\frac{a}{2a^2+1}+\frac{b}{2b^2+1}+\frac{c}{2c^2+1}\le 1.$$

(Vasile Cîrtoaje, 2006)

Solution. Assume that $a \le b \le c$ and denote

$$f(a,b,c)=\frac{a}{2a^2+1}+\frac{b}{2b^2+1}+\frac{c}{2c^2+1}.$$

We will show that

$$f(a,b,c)\le f(s,s,c)\le 1,$$

where

$$s=\sqrt{\frac{a^2+b^2}{2}}, \qquad s\le 1.$$

The inequality $f(a,b,c)\le f(s,s,c)$ follows from P 2.1. The inequality $f(s,s,c)\le 1$ is equivalent to

$$\frac{2s}{2s^2+1}+\frac{c}{2c^2+1}\le 1,$$

where

$$2s^2+c^2=3, \qquad 0\le s\le 1\le c.$$

Write the requested inequality as follows:

$$\frac{1}{3}-\frac{c}{2c^2+1}\ge\frac{2s}{2s^2+1}-\frac{2}{3},$$

$$\frac{(c-1)(2c-1)}{2c^2+1}\ge\frac{2(1-s)(2s-1)}{2s^2+1},$$

$$\frac{(c^2-1)(2c-1)}{(c+1)(2c^2+1)}\ge\frac{2(1-s^2)(2s-1)}{(1+s)(2s^2+1)}.$$

Since

$$c^2-1=2(1-s^2)\ge 0,$$

we only need to show that

$$\frac{2c-1}{(c+1)(2c^2+1)}\ge\frac{2s-1}{(s+1)(2s^2+1)},$$

which is equivalent to $(c-s)A\ge 0$, where

$$A=2(s+c)^2+2(s+c)+3-6sc-4sc(s+c).$$

Substituting

$$x=\frac{s+c}{2}, \quad y=\sqrt{sc}, \qquad x\ge y,$$

we need to show that $A(x,y) \ge 0$, where

$$A(x,y) = 8x^2 + 4x + 3 - 6y^2 - 8xy^2.$$

From

$$3 = 2s^2 + c^2 \ge 2\sqrt{2}sc = 2\sqrt{2}y^2,$$

we get

$$y \le \sqrt{\frac{3}{2\sqrt{2}}}.$$

We will show that

$$A(x,y) \ge A(y,y) \ge 0.$$

We have

$$A(x,y) - A(y,y) = 4(x-y)(2x+2y+1-2y^2) \ge 4(x-y)[2y(2-y)+1] \ge 0$$

and

$$A(y,y) = 3 + 4y + 2y^2 - 8y^3.$$

From

$$A(y,y) = y^3\left(\frac{3}{y^3} + \frac{4}{y^2} + \frac{2}{y} - 8\right),$$

it follows that it suffices to show that $A(y,y) \ge 0$ for $y = \sqrt{\frac{3}{2\sqrt{2}}}$. Indeed, we have

$$A(y,y) = 3 + 2y^2 - 4(2y^2-1)y = 3 + \frac{3}{\sqrt{2}} - 4\left(\frac{3}{\sqrt{2}} - 1\right)y$$

$$= \frac{3\sqrt{2} + 3 - 4(3-\sqrt{2})y}{\sqrt{2}} = \frac{B}{\sqrt{2}[3\sqrt{2} + 3 + 4(3-\sqrt{2})y]},$$

where

$$B = (3\sqrt{2}+3)^2 - 16(3-\sqrt{2})^2y^2 = 9(\sqrt{2}+1)^2 - 12\sqrt{2}(3-\sqrt{2})^2$$
$$= 57(3-2\sqrt{2}) > 0.$$

The equality holds for $a = b = c = 1$.
Remark. The following more general statement is also valid.

- *If a,b,c,d are nonnegative real numbers such that $a^2+b^2+c^2+d^2 = 4$, then*

$$\frac{a}{2a^2+1} + \frac{b}{2b^2+1} + \frac{c}{2c^2+1} + \frac{d}{2d^2+1} \le \frac{4}{3}.$$

□

P 2.53. *Let a, b, c be nonnegative real numbers such that $ab + bc + ca = 3$. Prove that*

(a) $$\sum \sqrt{a(b+c)(a^2+bc)} \ge 6;$$

(b) $$\sum a(b+c)\sqrt{a^2+2bc} \ge 6\sqrt{3};$$

(c) $$\sum a(b+c)\sqrt{(a+2b)(a+2c)} \ge 18.$$

(Vasile Cîrtoaje, 2010)

Solution. Assume that
$$a \ge b \ge c, \qquad b > 0.$$

(a) Write the inequality in the homogeneous form
$$\sum \sqrt{a(b+c)(a^2+bc)} \ge 2(ab+bc+ca).$$

First Solution. Write the homogeneous inequality as
$$\sum \sqrt{a(b+c)}\left[\sqrt{a^2+bc} - \sqrt{a(b+c)}\right] \ge 0,$$
$$\sum \frac{(a-b)(a-c)\sqrt{a(b+c)}}{\sqrt{a^2+bc}+\sqrt{a(b+c)}} \ge 0.$$

Since $(c-a)(c-b) \ge 0$, it suffices to show that
$$\frac{(a-b)(a-c)\sqrt{a(b+c)}}{\sqrt{a^2+bc}+\sqrt{a(b+c)}} + \frac{(b-c)(b-a)\sqrt{b(c+a)}}{\sqrt{b^2+ca}+\sqrt{b(c+a)}} \ge 0.$$

This is true if
$$\frac{(a-c)\sqrt{a(b+c)}}{\sqrt{a^2+bc}+\sqrt{a(b+c)}} \ge \frac{(b-c)\sqrt{b(c+a)}}{\sqrt{b^2+ca}+\sqrt{b(c+a)}}.$$

Since
$$\sqrt{a(b+c)} \ge \sqrt{b(c+a)},$$
it suffices to show that
$$\frac{a-c}{\sqrt{a^2+bc}+\sqrt{a(b+c)}} \ge \frac{b-c}{\sqrt{b^2+ca}+\sqrt{b(c+a)}}.$$

Moreover, since
$$\sqrt{a^2+bc} \ge \sqrt{a(b+c)}, \qquad \sqrt{b^2+ca} \le \sqrt{b(c+a)},$$
it is enough to show that
$$\frac{a-c}{\sqrt{a^2+bc}} \ge \frac{b-c}{\sqrt{b^2+ca}}.$$

Indeed, we have
$$(a-c)^2(b^2+ca) - (b-c)^2(a^2+bc) = (a-b)(a^2+b^2+c^2+3ab-3bc-3ca) \ge 0,$$

because

$$a^2+b^2+c^2+3ab-3bc-3ca=(a^2-bc)+(b-c)^2+3a(b-c)\geq 0.$$

The equality holds for $a=b=c=1$, and also for $a=b=\sqrt{3}$ and $c=0$ (or any cyclic permutation).

Second Solution. By squaring, the homogeneous inequality becomes

$$\sum a(b+c)(a^2+bc)+2\sum\sqrt{bc(a+b)(a+c)(b^2+ca)(c^2+ab)}\geq 4(ab+bc+ca)^2.$$

Since

$$(b^2+ca)(c^2+ab)-bc(a+b)(a+c)=a(b+c)(b-c)^2\geq 0,$$

it suffices to show that

$$\sum a(b+c)(a^2+bc)+2\sum bc(a+b)(a+c)\geq 4(ab+bc+ca)^2,$$

which is equivalent to

$$\sum bc(b-c)^2\geq 0.$$

(b) Write the inequality as

$$\sum a(b+c)\sqrt{a^2+2bc}\geq 2(ab+bc+ca)\sqrt{ab+bc+ca},$$

$$\sum a(b+c)\left[\sqrt{a^2+2bc}-\sqrt{ab+bc+ca}\right]\geq 0,$$

$$\sum\frac{a(b+c)(a-b)(a-c)}{\sqrt{a^2+2bc}+\sqrt{ab+bc+ca}}\geq 0.$$

Since $(c-a)(c-b)\geq 0$, it suffices to show that

$$\frac{a(b+c)(a-b)(a-c)}{\sqrt{a^2+2bc}+\sqrt{ab+bc+ca}}+\frac{b(c+a)(b-c)(b-a)}{\sqrt{b^2+2ca}+\sqrt{ab+bc+ca}}\geq 0.$$

This is true if

$$\frac{a(b+c)(a-c)}{\sqrt{a^2+2bc}+\sqrt{ab+bc+ca}}\geq\frac{b(c+a)(b-c)}{\sqrt{b^2+2ca}+\sqrt{ab+bc+ca}}.$$

Since

$$(b+c)(a-c)\geq(c+a)(b-c),$$

it suffices to show that

$$\frac{a}{\sqrt{a^2+2bc}+\sqrt{ab+bc+ca}}\geq\frac{b}{\sqrt{b^2+2ca}+\sqrt{ab+bc+ca}}.$$

Moreover, since

$$\sqrt{a^2+2bc}\geq\sqrt{ab+bc+ca},\qquad\sqrt{b^2+2ca}\leq\sqrt{ab+bc+ca},$$

it is enough to show that

$$\frac{a}{\sqrt{a^2+2bc}} \ge \frac{b}{\sqrt{b^2+2ca}}.$$

Indeed, we have

$$a^2(b^2+2ca) - b^2(a^2+2bc) = 2c(a^3-b^3) \ge 0.$$

The equality holds for $a=b=c=1$, and also for $a=b=\sqrt{3}$ and $c=0$ (or any cyclic permutation).

(c) Write the inequality as follows:

$$\sum a(b+c)\sqrt{(a+2b)(a+2c)} \ge 2(ab+bc+ca)\sqrt{3(ab+bc+ca)},$$

$$\sum a(b+c)\left[\sqrt{(a+2b)(a+2c)} - \sqrt{3(ab+bc+ca)}\right] \ge 0,$$

$$\sum \frac{a(b+c)(a-b)(a-c)}{\sqrt{(a+2b)(a+2c)} + \sqrt{3(ab+bc+ca)}} \ge 0.$$

Since $(c-a)(c-b) \ge 0$, it suffices to show that

$$\frac{a(b+c)(a-c)}{\sqrt{(a+2b)(a+2c)} + \sqrt{3(ab+bc+ca)}} \ge \frac{b(c+a)(b-c)}{\sqrt{(b+2c)(b+2a)} + \sqrt{3(ab+bc+ca)}}.$$

Since

$$(b+c)(a-c) \ge (c+a)(b-c),$$

it suffices to show that

$$\frac{a}{\sqrt{(a+2b)(a+2c)} + \sqrt{3(ab+bc+ca)}} \ge \frac{b}{\sqrt{(b+2c)(b+2a)} + \sqrt{3(ab+bc+ca)}}.$$

Moreover, since

$$\sqrt{(a+2b)(a+2c)} \ge \sqrt{3(ab+bc+ca)}, \qquad \sqrt{(b+2c)(b+2a)} \le \sqrt{3(ab+bc+ca)},$$

it is enough to show that

$$\frac{a}{\sqrt{(a+2b)(a+2c)}} \ge \frac{b}{\sqrt{(b+2c)(b+2a)}}.$$

This is true if

$$\frac{\sqrt{a}}{\sqrt{(a+2b)(a+2c)}} \ge \frac{\sqrt{b}}{\sqrt{(b+2c)(b+2a)}}.$$

Indeed, we have

$$a(b+2c)(b+2a) - b(a+2b)(a+2c) = (a-b)(ab+4bc+4ca) \ge 0.$$

The equality holds for $a=b=c=1$, and also for $a=b=\sqrt{3}$ and $c=0$ (or any cyclic permutation).

□

P 2.54. *Let a, b, c be nonnegative real numbers such that $ab + bc + ca = 3$. Prove that*

$$a\sqrt{bc+3} + b\sqrt{ca+3} + c\sqrt{ab+3} \ge 6.$$

(Vasile Cîrtoaje, 2010)

Solution. Use the SOS method. Denote

$$A = \sqrt{ab+2bc+ca}, \quad B = \sqrt{bc+2ca+ab}, \quad C = \sqrt{ca+2ab+bc},$$

and write the inequality as follows:

$$\sum aA \ge 2(ab+bc+ca),$$

$$\sum a\,(A-b-c) \ge 0,$$

$$\sum \frac{a(ab+ac-b^2-c^2)}{A+b+c} \ge 0,$$

$$\sum \frac{ab(a-b)+ac(a-c)}{A+b+c} \ge 0,$$

$$\sum \frac{ab(a-b)}{A+b+c} + \sum \frac{ba(b-a)}{B+c+a} \ge 0,$$

$$\sum ab(a-b)\left(\frac{1}{A+b+c} - \frac{1}{B+c+a}\right) \ge 0,$$

$$\sum ab(a+b+C)(a-b)(a-b+B-A) \ge 0,$$

$$\sum ab(a+b+C)(a-b)^2\left(1+\frac{c}{A+B}\right) \ge 0.$$

The equality holds for $a = b = c = 1$, and for $a = 0$ and $b = c = \sqrt{3}$ (or any cyclic permutation).

□

P 2.55. *Let a, b, c be nonnegative real numbers such that $a + b + c = 3$. Prove that*

(a)
$$\sum (b+c)\sqrt{b^2+c^2+7bc} \ge 18;$$

(b)
$$\sum (b+c)\sqrt{b^2+c^2+10bc} \le 12\sqrt{3}.$$

(Vasile Cîrtoaje, 2010)

Solution. Use the SOS technique.

(a) Write the inequality in the equivalent homogeneous forms

$$\sum (b+c)\sqrt{b^2+c^2+7bc} \ge 2(a+b+c)^2,$$

$$\sum \left[(b+c)\sqrt{b^2+c^2+7bc} - b^2 - c^2 - 4bc\right] \ge 0,$$

$$\sum \frac{(b+c)^2(b^2+c^2+7bc) - (b^2+c^2+4bc)^2}{(b+c)\sqrt{b^2+c^2+7bc} + b^2+c^2+4bc} \ge 0,$$

$$\sum \frac{bc(b-c)^2}{(b+c)\sqrt{b^2+c^2+7bc} + b^2+c^2+4bc} \ge 0.$$

The equality holds for $a = b = c = 1$, for $a = 0$ and $b = c = \dfrac{3}{2}$ (or any cyclic permutation), and for $a = 3$ and $b = c = 0$ (or any cyclic permutation).

(b) Write the inequality as follows:

$$\sum (b+c)\sqrt{3(b^2+c^2+10bc)} \le 4(a+b+c)^2,$$

$$\sum \left[2b^2 + 2c^2 + 8bc - (b+c)\sqrt{3(b^2+c^2+10bc)}\right] \ge 0,$$

$$\sum \frac{4(b^2+c^2+4bc)^2 - 3(b+c)^2(b^2+c^2+10bc)}{2b^2+2c^2+8bc+(b+c)\sqrt{3(b^2+c^2+10bc)}} \ge 0,$$

$$\sum \frac{(b-c)^4}{2b^2+2c^2+8bc+(b+c)\sqrt{3(b^2+c^2+10bc)}} \ge 0.$$

The equality holds for $a = b = c = 1$.

□

P 2.56. *Let a, b, c be nonnegative real numbers such then $a + b + c = 2$. Prove that*

$$\sqrt{a+4bc} + \sqrt{b+4ca} + \sqrt{c+4ab} \ge 4\sqrt{ab+bc+ca}.$$

(Vasile Cîrtoaje, 2012)

Solution. Without loss of generality, assume that

$$c = \min\{a, b, c\}.$$

Using Minkowski's inequality gives

$$\sqrt{a+4bc} + \sqrt{b+4ca} \ge \sqrt{\left(\sqrt{a}+\sqrt{b}\right)^2 + 4c\left(\sqrt{a}+\sqrt{b}\right)^2} = \left(\sqrt{a}+\sqrt{b}\right)\sqrt{1+4c}.$$

Therefore, it suffices to show that

$$\left(\sqrt{a}+\sqrt{b}\right)\sqrt{1+4c} \ge 4\sqrt{ab+bc+ca}-\sqrt{c+4ab}.$$

By squaring, this inequality becomes

$$\left(a+b+2\sqrt{ab}\right)(1+4c)+8\sqrt{(ab+bc+ca)(c+4ab)} \ge 16(ab+bc+ca)+c+4ab.$$

According to Lemma below, it suffices to show that

$$\left(a+b+2\sqrt{ab}\right)(1+4c)+8(2ab+bc+ca) \ge 16(ab+bc+ca)+c+4ab,$$

which is equivalent to

$$a+b-c+2\sqrt{ab}+8c\sqrt{ab} \ge 4(ab+bc+ca).$$

Write this inequality in the homogeneous form

$$(a+b+c)\left(a+b-c+2\sqrt{ab}\right)+16c\sqrt{ab} \ge 8(ab+bc+ca).$$

Due to homogeneity, we may assume that $a+b=1$. Let us denote

$$d=\sqrt{ab}, \qquad 0 \le d \le \frac{1}{2}.$$

We need to show that $f(c) \ge 0$ for $0 \le c \le d$, where

$$\begin{aligned} f(c) &= (1+c)(1-c+2d)+16cd-8d^2-8c \\ &= (1-2d)(1+4d)+2(9d-4)c-c^2. \end{aligned}$$

Since $f(c)$ is concave, it suffices to show that $f(0) \ge 0$ and $f(d) \ge 0$. Indeed,

$$f(0)=(1-2d)(1+4d) \ge 0,$$

$$f(d)=(3d-1)^2 \ge 0.$$

Thus, the proof is completed. The equality holds for $a=b=1$ and $c=0$ (or any cyclic permutation).

Lemma (*by Nguyen Van Quy*). *Let a,b,c be nonnegative real numbers such then*

$$c=\min\{a,b,c\}, \qquad a+b+c=2.$$

Then,

$$\sqrt{(ab+bc+ca)(c+4ab)} \ge 2ab+bc+ca.$$

Proof. By squaring, the inequality becomes

$$c[ab+bc+ca-c(a+b)^2] \ge 0.$$

We need to show that

$$(a+b+c)(ab+bc+ca)-2c(a+b)^2 \geq 0.$$

We have

$$\begin{aligned}(a+b+c)(ab+bc+ca)-2c(a+b)^2 &\geq (a+b)(b+c)(c+a)-2c(a+b)^2\\ &= (a+b)(a-c)(b-c) \geq 0.\end{aligned}$$

□

P 2.57. *If a,b,c are nonnegative real numbers, then*

$$\sqrt{a^2+b^2+7ab}+\sqrt{b^2+c^2+7bc}+\sqrt{c^2+a^2+7ca} \geq 5\sqrt{ab+bc+ca}.$$

(Vasile Cîrtoaje, 2012)

Solution *(by Nguyen Van Quy).* Assume that

$$c=\min\{a,b,c\}.$$

Using Minkowski's inequality yields

$$\sqrt{b^2+c^2+7bc}+\sqrt{a^2+c^2+7ca} \geq \sqrt{(a+b)^2+4c^2+7c\left(\sqrt{a}+\sqrt{b}\right)^2}.$$

Therefore, it suffices to show that

$$\sqrt{(a+b)^2+4c^2+7c\left(\sqrt{a}+\sqrt{b}\right)^2} \geq 5\sqrt{ab+bc+ca}-\sqrt{a^2+b^2+7ab}.$$

By squaring, this inequality becomes

$$2c^2+7c\sqrt{ab}+5\sqrt{(a^2+b^2+7ab)(ab+bc+ca)} \geq 15ab+9c(a+b).$$

Due to homogeneity, we may assume that $a+b=1$, which implies $c \leq \frac{1}{2}$. Let us denote $x=ab$. We need to show that $f(x) \geq 0$ for $c^2 \leq x \leq \frac{1}{4}$, where

$$f(x)=2c^2+7c\sqrt{x}+5\sqrt{(1+5x)(c+x)}-15x-9c.$$

Since

$$f''(x)=\frac{-7c}{4\sqrt{x^3}}-\frac{5(5c-1)^2}{4\sqrt{[5x^2+(5c+1)x+c]^3}}<0$$

$f(c)$ is concave. Thus, it suffices to show that $f(c^2) \geq 0$ and $f\left(\frac{1}{4}\right) \geq 0$.

Write the inequality $f(c^2) \geq 0$ as

$$5\sqrt{(1+5c^2)(c+c^2)} \geq 6c^2 + 9c.$$

By squaring, this inequality turns into

$$c(89c^3 + 17c^2 - 56c + 25) \geq 0,$$

which is true since

$$89c^3 + 17c^2 - 56c + 25 \geq 12c^2 - 56c + 25 = (1-2c)(25-6c) \geq 0.$$

Write the inequality $f\left(\frac{1}{4}\right) \geq 0$ as

$$8c^2 - 22c + 15\left(\sqrt{4c+1} - 1\right) \geq 0.$$

Making the substitution $t = \sqrt{4c+1}$, $t \geq 1$, the inequality becomes

$$(t-1)(t^3 + t^2 - 12t + 18) \geq 0.$$

It is true since

$$t^3 + t^2 - 12t + 18 \geq 2t^2 - 12t + 18 = 2(t-3)^2 \geq 0.$$

Thus, the proof is completed. The equality holds for $a = b$ and $c = 0$ (or any cyclic permutation).

□

P 2.58. *If a, b, c are nonnegative real numbers, then*

$$\sqrt{a^2+b^2+5ab} + \sqrt{b^2+c^2+5bc} + \sqrt{c^2+a^2+5ca} \geq \sqrt{21(ab+bc+ca)}.$$

(Nguyen Van Quy, 2012)

Solution. Without loss of generality, assume that $c = \min\{a, b, c\}$. Using Minkowski's inequality, we have

$$\sqrt{(a+c)^2+3ac} + \sqrt{(b+c)^2+3bc} \geq \sqrt{(a+b+2c)^2 + 3c\left(\sqrt{a}+\sqrt{b}\right)^2}.$$

Therefore, it suffices to show that

$$\sqrt{(a+b+2c)^2 + 3c\left(\sqrt{a}+\sqrt{b}\right)^2} \geq \sqrt{21(ab+bc+ca)} - \sqrt{a^2+b^2+5ab}.$$

By squaring, this inequality becomes

$$2c^2 + 3c\sqrt{ab} + \sqrt{21(a^2+b^2+5ab)(ab+bc+ca)} \geq 12ab + 7c(a+b).$$

Due to homogeneity, we may assume that $a+b=1$. Let us denote $x=ab$. We need to show that $f(x)\ge 0$ for $c^2\le x\le \frac{1}{4}$, where

$$f(x)=2c^2+3c\sqrt{x}+\sqrt{21(1+3x)(c+x)}-12x-7c.$$

Since

$$f''(x)=\frac{-3c}{4\sqrt{x^3}}-\frac{\sqrt{21}(3c-1)^2}{4\sqrt{[3x^2+(3c+1)x+c]^3}}<0$$

$f(c)$ is concave. Thus, it suffices to show that $f(c^2)\ge 0$ and $f\left(\frac{1}{4}\right)\ge 0$.

Write the inequality $f(c^2)\ge 0$ as

$$\sqrt{21(1+3c^2)(c+c^2)}\ge 7(c+c^2).$$

By squaring, this inequality turns into

$$c(c+1)(1-2c)(3-c)\ge 0,$$

which is clearly true.

Write the inequality $f\left(\frac{1}{4}\right)\ge 0$ as

$$8c^2-22c+7\sqrt{3(4c+1)}-12\ge 0.$$

Using the substitution $3t^2=4c+1$, $t\ge\frac{1}{\sqrt{3}}$, the inequality becomes

$$(t-1)^2(3t^2+6t-4)\ge 0.$$

This is true since

$$3t^2+6t-4\ge 1+2\sqrt{3}-4>0.$$

Thus, the proof is completed. The equality holds for $a=b=c$.

□

P 2.59. *Let a,b,c be nonnegative real numbers such that $ab+bc+ca=3$. Prove that*

$$a\sqrt{a^2+5}+b\sqrt{b^2+5}+c\sqrt{c^2+5}\ge\sqrt{\frac{2}{3}}\,(a+b+c)^2.$$

(Vasile Cîrtoaje, 2010)

Solution. Write the inequality in the homogeneous form

$$\sum a\sqrt{3a^2+5(ab+bc+ca)} \ge \sqrt{2}\ (a+b+c)^2.$$

Due to homogeneity, we may assume that

$$ab+bc+ca=1.$$

By squaring, the inequality becomes

$$\sum a^4+2\sum bc\sqrt{(3b^2+5)(3c^2+5)} \ge 12\sum a^2b^2+19abc\sum a+3\sum ab(a^2+b^2).$$

Applying Lemma below for $x=3b^2$, $y=3c^2$ and $d=5$, we have

$$2\sqrt{(3b^2+5)(3c^2+5)} \ge 3(b^2+c^2)+10-\frac{9}{20}(b^2-c^2)^2,$$

hence

$$2bc\sqrt{(3b^2+5)(3c^2+5)} \ge 3bc(b^2+c^2)+10bc-\frac{9}{20}bc(b^2-c^2)^2,$$

$$2\sum bc\sqrt{(3b^2+5)(3c^2+5)} \ge 3\sum bc(b^2+c^2)+10\left(\sum bc\right)^2-\frac{9}{20}\sum bc(b^2-c^2)^2$$

$$=10\sum a^2b^2+20abc\sum a+3\sum ab(a^2+b^2)-\frac{9}{20}\sum bc(b^2-c^2)^2.$$

Therefore, it suffices to show that

$$\sum a^4+10\sum a^2b^2+20abc\sum a+3\sum ab(a^2+b^2)-\frac{9}{20}\sum bc(b^2-c^2)^2 \ge$$

$$\ge 12\sum a^2b^2+19abc\sum a+3\sum ab(a^2+b^2),$$

which is equivalent to

$$\sum a^4-2\sum a^2b^2+abc\sum a-\frac{9}{20}\sum bc(b^2-c^2)^2 \ge 0.$$

To prove this inequality, we use the SOS method. Since

$$2\left(\sum a^4-2\sum a^2b^2+abc\sum a\right)=2\left(\sum a^4-\sum a^2b^2\right)-2\left(\sum a^2b^2-abc\sum a\right)$$

$$=\sum(b^2-c^2)^2-\sum a^2(b-c)^2,$$

we can write the inequality as

$$\sum(b-c)^2S_a \ge 0,$$

where

$$S_a=(b+c)^2-a^2-\frac{9}{10}bc(b+c)^2.$$

In addition, since

$$S_a \ge (b+c)^2 - a^2 - bc(b+c)^2 = (b+c)^2 - a^2 - \frac{bc(b+c)^2}{ab+bc+ca},$$

$$= \frac{a(b+c)^3 - a^2(ab+bc+ca)}{ab+bc+ca},$$

it is enough to show that

$$\sum (b-c)^2 E_a \ge 0,$$

where

$$E_a = a(b+c)^3 - a^2(ab+bc+ca).$$

Assume that

$$a \ge b \ge c, \qquad b > 0$$

Since

$$\begin{aligned} E_b &= b(c+a)^3 - b^2(ab+bc+ca) \ge b(c+a)^3 - b^2(c+a)(c+b) \\ &\ge b(c+a)^3 - b^2(c+a)^2 = b(c+a)^2(c+a-b) \ge 0, \\ E_c &= c(a+b)^3 - c^2(ab+bc+ca) \ge c(a+b)^3 - c^2(a+b)(b+c) \\ &\ge c(a+b)^3 - c^2(a+b)^2 = c(a+b)^2(a+b-c) \ge 0 \end{aligned}$$

and

$$\begin{aligned} \frac{E_a}{a^2} + \frac{E_b}{b^2} &= \frac{(b+c)^3}{a} + \frac{(c+a)^3}{b} - 2(ab+bc+ca) \\ &\ge \frac{b^3+2b^2c}{a} + \frac{a^3+2a^2c}{b} - 2(ab+bc+ca) \\ &= \frac{(a^2-b^2)^2 + 2c(a+b)(a-b)^2}{ab} \ge 0, \end{aligned}$$

we get

$$\sum (b-c)^2 E_a \ge (b-c)^2 E_a + (a-c)^2 E_b \ge a^2(b-c)^2 \left(\frac{E_a}{a^2} + \frac{E_b}{b^2} \right) \ge 0.$$

The equality holds for $a = b = c = 1$, and also for $a = b = \sqrt{3}$ and $c = 0$ (or any cyclic permutation).

Lemma. *If $x \ge 0$, $y \ge 0$ and $d > 0$, then*

$$2\sqrt{(x+d)(y+d)} \ge x + y + 2d - \frac{1}{4d}(x-y)^2.$$

Proof. We have

$$2\sqrt{(x+d)(y+d)} - 2d = \frac{2xy + 2d(x+y)}{\sqrt{(x+d)(y+d)} + d} \ge \frac{2xy + 2d(x+y)}{\frac{(x+d)+(y+d)}{2} + d}$$

$$= \frac{4xy + 4d(x+y)}{x+y+4d} = x + y - \frac{(x-y)^2}{x+y+4d} \ge x + y - \frac{(x-y)^2}{4d}.$$

□

P 2.60. *Let a, b, c be nonnegative real numbers such that $a^2+b^2+c^2=1$. Prove that*

$$a\sqrt{2+3bc}+b\sqrt{2+3ca}+c\sqrt{2+3ab}\geq(a+b+c)^2.$$

(Vasile Cîrtoaje, 2010)

Solution. Write the inequality as

$$\sum a\sqrt{2+3bc}\geq 1+2q,$$

where $q=ab+bc+ca$. By squaring, the inequality becomes

$$1+3abc\sum a+2\sum bc\sqrt{(2+3ab)(2+3ac)}\geq 4q+4q^2.$$

Applying Lemma from the preceding P 2.59 for $x=3ab$, $y=3ac$ and $d=2$, we have

$$2\sqrt{(2+3ab)(2+3ac)}\geq 3a(b+c)+4-\frac{9}{8}a^2(b-c)^2,$$

hence

$$2bc\sqrt{(2+3ab)(2+3ac)}\geq 3abc(b+c)+4-\frac{9}{8}a^2bc(b-c)^2,$$

$$2\sum bc\sqrt{(2+3ab)(2+3ac)}\geq 6abc\sum a+4q-\frac{9}{8}abc\sum a(b-c)^2.$$

Therefore, it suffices to show that

$$1+3abc\sum a+6abc\sum a+4q-\frac{9}{8}abc\sum a(b-c)^2\geq 4q+4q^2,$$

which is equivalent to

$$1+9abc\sum a-4q^2\geq\frac{9}{8}abc\sum a(b-c)^2.$$

Since

$$a^4+b^4+c^4=1-2(a^2b^2+b^2c^2+c^2a^2)=1-2q^2+4abc\sum a,$$

from Schur's inequality of fourth degree

$$a^4+b^4+c^4+2abc\sum a\geq\left(\sum a^2\right)\left(\sum ab\right),$$

we get

$$1\geq 2q^2+q-6abc\sum a.$$

Thus, it is enough to prove that

$$\left(2q^2+q-6abc\sum a\right)+9abc\sum a-4q^2\geq\frac{9}{8}abc\sum a(b-c)^2;$$

that is,

$$8\left(q-2q^2+3abc\sum a\right)\geq 9abc\sum a(b-c)^2.$$

Since

$$q - 2q^2 + 3abc\sum a = \left(\sum a^2\right)\left(\sum ab\right) - 2\left(\sum ab\right)^2 + 3abc\sum a$$
$$= \sum bc(b^2+c^2) - 2\sum b^2c^2 = \sum bc(b-c)^2,$$

we need to show that

$$\sum bc(8-9a^2)(b-c)^2 \ge 0.$$

Since

$$8 - 9a^2 = 8(b^2+c^2) - a^2 \ge b^2 + c^2 - a^2,$$

it suffices to prove the homogeneous inequality

$$\sum bc(b^2+c^2-a^2)(b-c)^2 \ge 0.$$

Assume that $a \ge b \ge c$. It is enough to show that

$$bc(b^2+c^2-a^2)(b-c)^2 + ca(c^2+a^2-b^2)(c-a)^2 \ge 0.$$

This is true if

$$a(c^2+a^2-b^2)(a-c)^2 \ge b(a^2-b^2-c^2)(b-c)^2.$$

For the nontrivial case $a^2 - b^2 - c^2 \ge 0$, this inequality follows from

$$a \ge b, \quad c^2+a^2-b^2 \ge a^2-b^2-c^2, \quad (a-c)^2 \ge (b-c)^2.$$

The equality holds for $a = b = c = \dfrac{1}{\sqrt{3}}$, and for $a = 0$ and $b = c = \dfrac{1}{\sqrt{2}}$ (or any cyclic permutation).

□

P 2.61. *Let a, b, c be nonnegative real numbers such that $a + b + c = 3$. Prove that*

(a)
$$a\sqrt{\frac{2a+bc}{3}} + b\sqrt{\frac{2b+ca}{3}} + c\sqrt{\frac{2c+ab}{3}} \ge 3;$$

(b)
$$a\sqrt{\frac{a(1+b+c)}{3}} + b\sqrt{\frac{b(1+c+a)}{3}} + c\sqrt{\frac{c(1+a+b)}{3}} \ge 3.$$

(Vasile Cîrtoaje, 2010)

Solution. (a) If two of a, b, c are zero, then the inequality is trivial. Otherwise, by Hölder's inequality, we have

$$\left(\sum a\sqrt{\frac{2a+bc}{3}}\right)^2 \ge \frac{\left(\sum a\right)^3}{\sum \dfrac{3a}{2a+bc}} = \frac{9}{\sum \dfrac{a}{2a+bc}}.$$

Therefore, it suffices to show that

$$\sum \frac{a}{2a+bc} \le 1.$$

Since

$$\frac{2a}{2a+bc} = 1 - \frac{bc}{2a+bc},$$

we can write this inequality as

$$\sum \frac{bc}{2a+bc} \ge 1.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{bc}{2a+bc} \ge \frac{(\sum bc)^2}{\sum bc(2a+bc)} = \frac{(\sum bc)^2}{2abc\sum a + \sum b^2c^2} = 1.$$

The equality holds for $a = b = c = 1$, and for $a = 0$ and $b = c = \frac{3}{2}$ (or any cyclic permutation).

(b) Write the inequality in the homogeneous form

$$\sum a\sqrt{a(a+4b+4c)} \ge (a+b+c)^2.$$

By squaring, the inequality becomes

$$\sum bc\sqrt{bc(b+4c+4a)(c+4a+4b)} \ge 3\sum b^2c^2 + 6abc\sum a.$$

Applying the Cauchy-Schwarz inequality, we have

$$\sqrt{(b+4c+4a)(c+4a+4b)} = \sqrt{(4a+b+c+3c)(4a+b+c+3b)}$$

$$\ge 4a+b+c+3\sqrt{bc},$$

hence

$$bc\sqrt{bc(b+4c+4a)(c+4a+4b)} \ge (4a+b+c)bc\sqrt{bc} + 3b^2c^2,$$

$$\sum bc\sqrt{bc(b+4c+4a)(c+4a+4b)} \ge \sum(4a+b+c)bc\sqrt{bc} + 3\sum b^2c^2.$$

Thus, it is enough to show that

$$\sum(4a+b+c)bc\sqrt{bc} \ge 6abc\sum a.$$

Replacing a, b, c by a^2, b^2, c^2, respectively, this inequality becomes

$$\sum(4a^2+b^2+c^2)b^3c^3 \ge 6a^2b^2c^2\sum a^2,$$

$$\left(\sum a^2\right)\left(\sum b^3c^3\right) + 3a^2b^2c^2\sum bc \ge 6a^2b^2c^2\sum a^2,$$

$$\left(\sum a^2\right)\left(\sum a^3b^3 - 3a^2b^2c^2\right) \ge 3a^2b^2c^2\left(\sum a^2 - \sum ab\right).$$

Use next the SOS method. Since

$$\sum a^3b^3 - 3a^2b^2c^2 = \left(\sum ab\right)\left(\sum a^2b^2 - abc\sum a\right) = \frac{1}{2}\left(\sum ab\right)\sum a^2(b-c)^2,$$

and

$$\sum a^2 - \sum ab = \frac{1}{2}\sum(b-c)^2,$$

we can write the inequality as

$$\sum(b-c)^2S_a \ge 0,$$

where

$$S_a = a^2\left(\sum a^2\right)\left(\sum ab\right) - 3a^2b^2c^2.$$

Assume that $a \ge b \ge c$. Since $S_a \ge S_b \ge 0$ and

$$\begin{aligned} S_b + S_c &= (b^2+c^2)\left(\sum a^2\right)\left(\sum ab\right) - 6a^2b^2c^2 \\ &\ge 2bc\left(\sum a^2\right)\left(\sum ab\right) - 6a^2b^2c^2 \\ &\ge 2bca^2\left(\sum ab\right) - 6a^2b^2c^2 = 2a^2bc(ab+ac-2bc) \ge 0, \end{aligned}$$

we get

$$\sum(b-c)^2S_a \ge (c-a)^2S_b + (a-b)^2S_c \ge (a-b)^2(S_b+S_c) \ge 0.$$

The equality holds for $a = b = c = 1$, and also for $a = 3$ and $b = c = 0$ (or any cyclic permutation).

□

P 2.62. *If a, b, c are nonnegative real numbers such that $a+b+c=3$, then*

$$\sqrt{8(a^2+bc)+9} + \sqrt{8(b^2+ca)+9} + \sqrt{8(c^2+ab)+9} \ge 15.$$

(Vasile Cîrtoaje, 2013)

Solution. Use the SOS technique. Let $q = ab+bc+ca$ and

$$A = (3a-b-c)^2 + 8q, \quad B = (3b-c-a)^2 + 8q, \quad C = (3c-a-b)^2 + 8q.$$

Since

$$\begin{gathered} 8(a^2+bc)+9 = 8(a^2+q)+9-8a(b+c) = 8(a^2+q)+9-8a(3-a) \\ = (4a-3)^2 + 8q = (3a-b-c)^2 + 8q = A, \end{gathered}$$

we can rewrite the inequality as follows:

$$\sum \sqrt{A} \ge 15,$$

$$\sum[\sqrt{A}-(3a+b+c)] \geq 0,$$

$$\sum \frac{2bc-ca-ab}{\sqrt{A}+3a+b+c} \geq 0,$$

$$\sum \left[\frac{b(c-a)}{\sqrt{A}+3a+b+c}+\frac{c(b-a)}{\sqrt{A}+3a+b+c}\right] \geq 0,$$

$$\sum \frac{c(a-b)}{\sqrt{B}+3b+c+a}+\sum \frac{c(b-a)}{\sqrt{A}+3a+b+c} \geq 0,$$

$$\sum c(a-b)(\sqrt{C}+3c+a+b)[\sqrt{A}-\sqrt{B}+2(a-b)] \geq 0,$$

$$\sum c(a-b)^2(\sqrt{C}+3c+a+b)\left[\frac{4(a+b-c)}{\sqrt{A}+\sqrt{B}}+1\right] \geq 0.$$

Without loss of generality, assume that $a \geq b \geq c$. Since $a+b-c>0$, it suffices to show that

$$b(a-c)^2(\sqrt{B}+3b+c+a)\left[\frac{4(c+a-b)}{\sqrt{A}+\sqrt{C}}+1\right] \geq$$
$$a(b-c)^2(\sqrt{A}+3a+b+c)\left[\frac{4(a-b-c)}{\sqrt{B}+\sqrt{C}}-1\right].$$

This inequality follows from the inequalities

$$b^2(a-c)^2 \geq a^2(b-c)^2,$$

$$a(\sqrt{B}+3b+c+a) \geq b(\sqrt{A}+3a+b+c),$$

$$\frac{4(c+a-b)}{\sqrt{A}+\sqrt{C}}+1 \geq \frac{4(a-b-c)}{\sqrt{B}+\sqrt{C}}-1.$$

Write the second inequality as

$$\frac{a^2B-b^2A}{a\sqrt{B}+b\sqrt{A}}+(a-b)(a+b+c) \geq 0.$$

Since

$$\begin{aligned}a^2B-b^2A &= (a-b)(a+b+c)(a^2+b^2-6ab+bc+ca)+8q(a^2-b^2)\\ &\geq (a-b)(a+b+c)(a^2+b^2-6ab) \geq -4ab(a-b)(a+b+c),\end{aligned}$$

it suffices to show that

$$\frac{-4ab}{a\sqrt{B}+b\sqrt{A}}+1 \geq 0.$$

Indeed, from $\sqrt{A}>\sqrt{8q} \geq 2\sqrt{ab}$ and $\sqrt{B} \geq \sqrt{8q} \geq 2\sqrt{ab}$, we get

$$a\sqrt{B}+b\sqrt{A}-4ab > 2(a+b)\sqrt{ab}-4ab = 2\sqrt{ab}(a+b-2\sqrt{ab}) \geq 0.$$

The third inequality holds if

$$1 \ge \frac{2(a-b-c)}{\sqrt{B}+\sqrt{C}}.$$

It suffices to show that $\sqrt{B} \ge a$ and $\sqrt{C} \ge a$. We have

$$B - a^2 = 8q - 2a(3b-c) + (3b-c)^2 \ge 8ab - 2a(3b-c) = 2a(b+c) \ge 0$$

and

$$C - a^2 = 8q - 2a(3c-b) + (3c-b)^2 \ge 8ab - 2a(3c-b) = 2a(5b-3c) \ge 0.$$

The equality holds for $a=b=c=1$, and also for $a=3$ and $b=c=0$ (or any cyclic permutation). □

P 2.63. *Let a, b, c be nonnegative real numbers such that $a+b+c=3$. If $k \ge \frac{9}{8}$, then*

$$\sqrt{a^2+bc+k} + \sqrt{b^2+ca+k} + \sqrt{c^2+ab+k} \ge 3\sqrt{2+k}.$$

Solution. We will show that

$$\sum \sqrt{8(a^2+bc+k)} \ge \sum \sqrt{(3a+b+c)^2+8k-9} \ge 6\sqrt{2(k+2)}.$$

The right inequality is equivalent to

$$\sum \sqrt{(2a+3)^2+8k-9} \ge 6\sqrt{2(k+2)},$$

which follows immediately from Jensen's inequality applied to the convex function $f : [0,\infty) \to \mathbb{R}$ defined by

$$f(x) = \sqrt{(2x+3)^2+8k-9}.$$

To prove the left inequality, we use the SOS method. By means of the substitutions

$$A_1 = 8(a^2+bc+k), \quad B_1 = 8(b^2+ca+k), \quad C_1 = 8(c^2+ab+k),$$

$$A_2 = (3a+b+c)^2+8k-9, \quad B_2 = (3b+c+a)^2+8k-9, \quad C_2 = (3c+a+b)^2+8k-9,$$

we can write the inequality as follows:

$$\frac{A_1-A_2}{\sqrt{A_1}+\sqrt{A_2}} + \frac{B_1-B_2}{\sqrt{B_1}+\sqrt{B_2}} + \frac{C_1-C_2}{\sqrt{C_1}+\sqrt{C_2}} \ge 0,$$

$$\frac{2bc-ca-ab}{\sqrt{A_1}+\sqrt{A_2}} + \frac{2ca-ab-bc}{\sqrt{B_1}+\sqrt{B_2}} + \frac{2ab-bc-ca}{\sqrt{C_1}+\sqrt{C_2}} \ge 0,$$

$$\sum \left[\frac{b(c-a)}{\sqrt{A_1}+\sqrt{A_2}} + \frac{c(b-a)}{\sqrt{A_1}+\sqrt{A_2}}\right] \ge 0,$$

$$\sum \frac{c(a-b)}{\sqrt{B_1}+\sqrt{B_2}} + \sum \frac{c(b-a)}{\sqrt{A_1}+\sqrt{A_2}} \ge 0,$$

$$\sum c(a-b)(\sqrt{C_1}+\sqrt{C_2})[(\sqrt{A_1}-\sqrt{B_1})+(\sqrt{A_2}-\sqrt{B_2})] \ge 0,$$

$$\sum c(a-b)^2(\sqrt{C_1}+\sqrt{C_2})\left[\frac{2(a+b-c)}{\sqrt{A_1}+\sqrt{B_1}} + \frac{2a+2b+c}{\sqrt{A_2}+\sqrt{B_2}}\right] \ge 0.$$

Without loss of generality, assume that $a \ge b \ge c$. Clearly, the desired inequality is true for $b+c \ge a$. Consider further the case $b+c < a$. Since $a+b-c > 0$, it suffices to show that

$$a(b-c)^2(\sqrt{A_1}+\sqrt{A_2})\left[\frac{2(b+c-a)}{\sqrt{B_1}+\sqrt{C_1}} + \frac{2b+2c+a}{\sqrt{B_2}+\sqrt{C_2}}\right] +$$
$$+b(a-c)^2(\sqrt{B_1}+\sqrt{B_2})\left[\frac{2(c+a-b)}{\sqrt{C_1}+\sqrt{A_1}} + \frac{2c+2a+b}{\sqrt{C_2}+\sqrt{AC_2}}\right] \ge 0.$$

Since

$$b^2(a-c)^2 \ge a^2(b-c)^2,$$

it suffices to show that

$$b(\sqrt{A_1}+\sqrt{A_2})\left[\frac{2(b+c-a)}{\sqrt{B_1}+\sqrt{C_1}} + \frac{2b+2c+a}{\sqrt{B_2}+\sqrt{C_2}}\right] +$$
$$+a(\sqrt{B_1}+\sqrt{B_2})\left[\frac{2(c+a-b)}{\sqrt{C_1}+\sqrt{A_1}} + \frac{2c+2a+b}{\sqrt{C_2}+\sqrt{A_2}}\right] \ge 0.$$

From

$$a^2B_1 - b^2A_1 = 8c(a^3-b^3) + 8k(a^2-b^2) \ge 0$$

and

$$a^2B_2 - b^2A_2 = (a-b)(a+b+c)(a^2+b^2+6ab+bc+ca) + (8k-9)(a^2-b^2) \ge 0,$$

we get $a\sqrt{B_1} \ge b\sqrt{A_1}$ and $a\sqrt{B_2} \ge b\sqrt{A_2}$, hence

$$a(\sqrt{B_1}+\sqrt{B_2}) \ge b(\sqrt{A_1}+\sqrt{A_2}).$$

Therefore, it is enough to show that

$$\frac{2(b+c-a)}{\sqrt{B_1}+\sqrt{C_1}} + \frac{2b+2c+a}{\sqrt{B_2}+\sqrt{C_2}} + \frac{2(c+a-b)}{\sqrt{C_1}+\sqrt{A_1}} + \frac{2c+2a+b}{\sqrt{C_2}+\sqrt{A_2}} \ge 0.$$

This is true if

$$\frac{2b}{\sqrt{B_1}+\sqrt{C_1}} + \frac{-2b}{\sqrt{C_1}+\sqrt{A_1}} \ge 0$$

and

$$\frac{-2a}{\sqrt{B_1}+\sqrt{C_1}} + \frac{2a}{\sqrt{C_1}+\sqrt{A_1}} + \frac{2a}{\sqrt{C_2}+\sqrt{A_2}} \ge 0.$$

The first inequality is true because $A_1 - B_1 = 8(a-b)(a+b-c) \geq 0$. The second inequality can be written as

$$\frac{1}{\sqrt{C_1}+\sqrt{A_1}} + \frac{1}{\sqrt{C_2}+\sqrt{A_2}} \geq \frac{1}{\sqrt{B_1}+\sqrt{C_1}}.$$

Since

$$\frac{1}{\sqrt{C_1}+\sqrt{A_1}} + \frac{1}{\sqrt{C_2}+\sqrt{A_2}} \geq \frac{4}{\sqrt{C_1}+\sqrt{A_1}+\sqrt{C_2}+\sqrt{A_2}},$$

it suffices to show that

$$4\sqrt{B_1} + 3\sqrt{C_1} \geq \sqrt{A_1} + \sqrt{A_2} + \sqrt{C_2}.$$

Taking account of

$$C_1 - C_2 = 4(2ab - bc - ca) \geq 0,$$
$$C_1 - B_1 = 8(b-c)(a-b-c) \geq 0,$$
$$A_2 - A_1 = 4(ab - 2bc + ca) \geq 0,$$

we have

$$\begin{aligned} 4\sqrt{B_1} + 3\sqrt{C_1} - \sqrt{A_1} - \sqrt{A_2} - \sqrt{C_2} &\geq 4\sqrt{B_1} + 2\sqrt{C_1} - \sqrt{A_1} - \sqrt{A_2} \\ &\geq 4\sqrt{B_1} + 2\sqrt{B_1} - \sqrt{A_2} - \sqrt{A_2} \\ &= 2(3\sqrt{B_1} - \sqrt{A_2}). \end{aligned}$$

In addition,

$$\begin{aligned} 9B_1 - A_2 &= 64k - 8a^2 + 72b^2 - 4ab + 68ac \\ &\geq 72 - 8a^2 + 72b^2 - 4ab + 68ac \\ &= 8(a+b+c)^2 - 8a^2 + 72b^2 - 4ab + 68ac \\ &= 4(20b^2 + 2c^2 + 3ab + 4bc + 21ac) \geq 0. \end{aligned}$$

Thus, the proof is completed. The equality holds for $a = b = c = 1$. If $k = 9/8$, then the equality holds also for $a = 3$ and $b = c = 0$ (or any cyclic permutation).

□

P 2.64. *If a, b, c are nonnegative real numbers such that $a+b+c = 3$, then*

$$\sqrt{a^3 + 2bc} + \sqrt{b^3 + 2ca} + \sqrt{c^3 + 2ab} \geq 3\sqrt{3}.$$

(Nguyen Van Quy, 2013)

Solution. Since

$$(a^3 + 2bc)(a + 2bc) \geq (a^2 + 2bc)^2,$$

it suffices to prove that

$$\sum \frac{a^2 + 2bc}{\sqrt{a + 2bc}} \geq 3\sqrt{3}.$$

By Hölder's inequality, we have

$$\left(\sum \frac{a^2+2bc}{\sqrt{a+2bc}}\right)^2 \sum (a^2+2bc)(a+2bc) \geq \left[\sum (a^2+2bc)\right]^3 = (a+b+c)^6.$$

Therefore, it suffices to show that

$$(a+b+c)^6 \geq 27\sum (a^2+2bc)(a+2bc).$$

which is equivalent to

$$(a+b+c)^4 \geq \sum (a^2+2bc)(a^2+6bc+ca+ab).$$

Indeed,

$$(a+b+c)^4 - \sum (a^2+2bc)(a^2+6bc+ca+ab) = 3\sum ab(a-b)^2 \geq 0.$$

The equality holds for $a=b=c=1$, and also for $a=3$ and $b=c=0$ (or any cyclic permutation).

□

P 2.65. *If a,b,c are positive real numbers, then*

$$\frac{\sqrt{a^2+bc}}{b+c} + \frac{\sqrt{b^2+ca}}{c+a} + \frac{\sqrt{c^2+ab}}{a+b} \geq \frac{3\sqrt{2}}{2}.$$

(Vasile Cîrtoaje, 2006)

Solution. According to the well-known inequality

$$(x+y+z)^2 \geq 3(xy+yz+zx), \quad x,y,z \geq 0,$$

it suffices to show that

$$\sum \frac{\sqrt{(b^2+ca)(c^2+ab)}}{(c+a)(a+b)} \geq \frac{3}{2}.$$

Replacing a,b,c by a^2,b^2,c^2, respectively, the inequality becomes

$$2\sum (b^2+c^2)\sqrt{(b^4+c^2a^2)(c^4+a^2b^2)} \geq 3(a^2+b^2)(b^2+c^2)(c^2+a^2).$$

Multiplying the Cauchy-Schwarz inequalities

$$\sqrt{(b^2+c^2)(b^4+c^2a^2)} \geq b^3+ac^2,$$

$$\sqrt{(c^2+b^2)(c^4+a^2b^2)} \geq c^3+ab^2,$$

we get

$$(b^2+c^2)\sqrt{(b^4+c^2a^2)(c^4+a^2b^2)} \ge (b^3+ac^2)(c^3+ab^2)$$
$$= b^3c^3 + a(b^5+c^5) + a^2b^2c^2.$$

Therefore, it suffices to show that

$$2\sum b^3c^3 + 2\sum a(b^5+c^5) + 6a^2b^2c^2 \ge 3(a^2+b^2)(b^2+c^2)(c^2+a^2).$$

This inequality is equivalent to

$$2\sum b^3c^3 + 2\sum bc(b^4+c^4) \ge 3\sum b^2c^2(b^2+c^2),$$

$$\sum bc[2b^2c^2 + 2(b^4+c^4) - 3bc(b^2+c^2)] \ge 0,$$

$$\sum bc(b-c)^2(2b^2+bc+2c^2) \ge 0.$$

The equality holds for $a=b=c$.

□

P 2.66. *If a,b,c are nonnegative real numbers, no two of which are zero,then*

$$\frac{\sqrt{bc+4a(b+c)}}{b+c} + \frac{\sqrt{ca+4b(c+a)}}{c+a} + \frac{\sqrt{ab+4c(a+b)}}{a+b} \ge \frac{9}{2}.$$

(Vasile Cîrtoaje, 2006)

Solution. Let us denote

$$A = 4ab+bc+4ca, \quad B = 4ab+4bc+ca, \quad C = ab+4bc+4ca.$$

By squaring, the inequality becomes

$$\sum \frac{A}{(b+c)^2} + 2\sum \frac{\sqrt{BC}}{(c+a)(a+b)} \ge \frac{81}{4}.$$

According to the known inequality Iran-1996, namely

$$\sum \frac{ab+bc+ca}{(b+c)^2} \ge \frac{9}{4}$$

(see Remark from the proof of P 1.72), we have

$$\sum \frac{A}{(b+c)^2} = \sum \frac{ab+bc+ca}{(b+c)^2} + 3\sum \frac{a}{b+c} \ge \frac{9}{4} + 3\sum \frac{a}{b+c}.$$

On the other hand, from Lemma below, we have

$$\sqrt{BC} \ge 2ab + 4bc + 2ca + \frac{2abc}{b+c},$$

$$\sqrt{BC} \ge \frac{2a(b^2+c^2)+4bc(b+c)+6abc}{b+c},$$

$$2\sum \frac{\sqrt{BC}}{(c+a)(a+b)} \ge \frac{4\sum a(b^2+c^2)+8\sum bc(b+c)+36abc}{(a+b)(b+c)c+a)},$$

$$2\sum \frac{\sqrt{BC}}{(c+a)(a+b)} \ge \frac{12\sum bc(b+c)+36abc}{(a+b)(b+c)c+a)}.$$

Thus, it suffices to show that

$$3\sum \frac{a}{b+c} + \frac{12\sum bc(b+c)+36abc}{(a+b)(b+c)c+a)} \ge 18.$$

This is equivalent to Schur's inequality of degree three

$$\sum a^3 + 3abc \ge \sum bc(b+c).$$

The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

Lemma. *If a,b,c are nonnegative real numbers, no two of which are zero, then*

$$\sqrt{(4ab+4bc+ca)(ab+4bc+4ca)} \ge 2ab+4bc+2ca+\frac{2abc}{b+c},$$

with equality for $b=c$, and also for $abc=0$.

Proof. We use the AM-GM inequality as follows:

$$\sqrt{(4ab+4bc+ca)(ab+4bc+4ca)} - 2ab - 4bc - 2ca =$$

$$= \frac{abc(9a+4b+4c)}{\sqrt{(4ab+4bc+ca)(ab+4bc+4ca)}+2ab+4bc+2ca}$$

$$\ge \frac{2abc(9a+4b+4c)}{(4ab+4bc+ca)+(ab+4bc+4ca)+4ab+8bc+4ca}$$

$$= \frac{2abc(9a+4b+4c)}{9ab+16bc+9ca}.$$

Thus, it suffices to show that

$$\frac{9a+4b+4c}{9ab+16bc+9ca} \ge \frac{1}{b+c}.$$

Indeed,

$$(9a+4b+4c)(b+c)-(9ab+16bc+9ca) = 4(b-c)^2 \ge 0.$$

□

P 2.67. *If a, b, c are nonnegative real numbers, no two of which are zero,then*

$$\frac{a\sqrt{a^2+3bc}}{b+c}+\frac{b\sqrt{b^2+3ca}}{c+a}+\frac{c\sqrt{c^2+3ab}}{a+b}\geq a+b+c.$$

(Cezar Lupu, 2006)

Solution. Using the AM-GM inequality, we have

$$\frac{a\sqrt{a^2+3bc}}{b+c}=\frac{2a(a^2+3bc)}{2\sqrt{(b+c)^2(a^2+3bc)}}\geq\frac{2a(a^2+3bc)}{(b+c)^2+(a^2+3bc)}=\frac{2a^3+6abc}{S+5bc},$$

where $S=a^2+b^2+c^2$. Thus, it suffices to show that

$$\sum\frac{2a^3+6abc}{S+5bc}\geq a+b+c.$$

Write this inequality as

$$\sum a\left(\frac{2a^2+6bc}{S+5bc}-1\right)\geq 0,$$

or, equivalently,

$$AX+BY+XZ\geq 0,$$

where

$$A=\frac{1}{S+5bc},\quad B=\frac{1}{S+5ca},\quad C=\frac{1}{S+5ab},$$

$$X=a^3+abc-a(b^2+c^2),\quad Y=b^3+abc-b(c^2+a^2),\quad Z=c^3+abc-c(a^2+b^2).$$

Without loss of generality, assume that $a\geq b\geq c$. We have

$$A\geq B\geq C,$$

$$X=a(a^2-b^2)+ac(b-c)\geq 0,\quad Z=c(c^2-b^2)+ac(b-a)\leq 0$$

and, according to Schur's inequality of third degree,

$$X+Y+Z=\sum a^3+3abc-\sum a(b^2+c^2)\geq 0.$$

Therefore,

$$AX+BY+CZ\geq BX+BY+BZ=B(X+Y+Z)\geq 0.$$

The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

Remark. We can also prove the inequality $AX+BY+XZ\geq 0$ by the SOS procedure. Write this inequality as follows:

$$\sum\frac{a(a^2+bc-b^2-c^2)}{S+5bc}\geq 0,$$

$$\sum\frac{a(a^2b+a^2c-b^3-c^3)}{(b+c)(S+5bc)}\geq 0,$$

$$\sum \frac{ab(a^2-b^2)+ac(a^2-c^2)}{(b+c)(S+5bc)} \ge 0,$$

$$\sum \frac{ab(a^2-b^2)}{(b+c)(S+5bc)} + \sum \frac{ba(b^2-a^2)}{(c+a)(S+5ca)} \ge 0,$$

$$\sum \frac{ab(a+b)(a-b)^2[S+5c(a+b+c)]}{(b+c)(c+a)(S+5bc)(S+5ca)} \ge 0.$$

□

P 2.68. *If a, b, c are nonnegative real numbers, no two of which are zero,then*

$$\sqrt{\frac{2a(b+c)}{(2b+c)(b+2c)}} + \sqrt{\frac{2b(c+a)}{(2c+a)(c+2a)}} + \sqrt{\frac{2c(a+b)}{(2a+b)(a+2b)}} \ge 2.$$

(Vasile Cîrtoaje, 2006)

Solution. Making the substitution

$$x = \sqrt{a}, \quad y = \sqrt{b}, \quad z = \sqrt{c},$$

the inequality becomes

$$\sum x\sqrt{\frac{2(y^2+z^2)}{(2y^2+z^2)(y^2+2z^2)}} \ge 2.$$

We claim that

$$\sqrt{\frac{2(y^2+z^2)}{(2y^2+z^2)(y^2+2z^2)}} \ge \frac{y+z}{y^2+yz+z^2}.$$

Indeed, be squaring and direct calculation, this inequality reduces to

$$y^2z^2(y-z)^2 \ge 0.$$

Thus, it suffices to show that

$$\sum \frac{x(y+z)}{y^2+yz+z^2} \ge 2,$$

which is just the inequality in P 1.69. The equality holds for $a = b = c$, and also for $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 2.69. *If a, b, c are nonnegative real numbers such that $ab + bc + ca = 3$, then*

$$\sqrt{\frac{bc}{3a^2+6}} + \sqrt{\frac{ca}{3b^2+6}} + \sqrt{\frac{ab}{3c^2+6}} \le 1 \le \sqrt{\frac{bc}{6a^2+3}} + \sqrt{\frac{ca}{6b^2+3}} + \sqrt{\frac{ab}{6c^2+3}}.$$

(Vasile Cîrtoaje, 2011)

Solution. By the Cauchy-Schwarz inequality, we have

$$\left(\sum\sqrt{\frac{bc}{3a^2+6}}\right)^2 \le \left(\sum\frac{1}{3a^2+6}\right)\left(\sum bc\right),$$

hence

$$\left(\sum\sqrt{\frac{bc}{3a^2+6}}\right)^2 \le \sum\frac{1}{a^2+2}.$$

Therefore, to prove the original left inequality, it suffices to show that

$$\sum\frac{1}{a^2+2} \le 1.$$

This inequality is equivalent to

$$\sum\frac{a^2}{a^2+2} \ge 1.$$

By the Cauchy-Schwarz inequality, we get

$$\sum\frac{a^2}{a^2+2} \ge \frac{(a+b+c)^2}{\sum(a^2+2)} = \frac{(a+b+c)^2}{\sum a^2+6} = 1.$$

The equality occurs for $a=b=c=1$.

To prove the original right inequality we apply Hölder's inequality as follows:

$$\left(\sum\sqrt{\frac{bc}{6a^2+3}}\right)^2\left[\sum b^2c^2(6a^2+3)\right] \ge \left(\sum bc\right)^3.$$

Thus, it suffices to show that

$$(ab+bc+ca)^3 \ge \sum b^2c^2(6a^2+ab+bc+ca),$$

which is equivalent to

$$(ab+bc+ca)\left[(ab+bc+ca)^2-\sum b^2c^2\right] \ge 18a^2b^2c^2,$$

$$2abc(ab+bc+ca)(a+b+c) \ge 18a^2b^2c^2,$$

$$2abc\sum a(b-c)^2 \ge 0.$$

The equality occurs for $a=b=c=1$, and for $a=0$ and $bc=3$ (or any cyclic permutation).

□

P 2.70. *Let a,b,c be nonnegative real numbers such that $ab+bc+ca=3$. If $k>1$, than*

$$a^k(b+c)+b^k(c+a)+c^k(a+b) \ge 6.$$

Solution. Let
$$E = a^k(b+c) + b^k(c+a) + c^k(a+b).$$
We consider two cases.

Case 1: $k \ge 2$. Applying Jensen's inequality to the convex function $f(x) = x^{k-1}$, $x \ge 0$, we get

$$\begin{aligned} E &= (ab+ac)a^{k-1} + (bc+ba)b^{k-1} + (ca+cb)c^{k-1} \\ &\ge 2(ab+bc+ca)\left[\frac{(ab+ac)a + (bc+ba)b + (ca+cb)c}{2(ab+bc+ca)}\right]^{k-1} \\ &= 6\left[\frac{a^2(b+c) + b^2(c+a) + c^2(a+b)}{6}\right]^{k-1}. \end{aligned}$$

Thus, it suffices to show that

$$a^2(b+c) + b^2(c+a) + c^2(a+b) \ge 6.$$

Write this inequality as

$$(ab+bc+ca)(a+b+c) - 3abc \ge 6,$$

$$a+b+c \ge 2 + abc.$$

It is true since
$$a+b+c \ge \sqrt{3(ab+bc+ca)} = 3$$
and
$$abc \le \left(\frac{a+b+c}{3}\right)^3 = 1.$$

Case 2: $1 < k < 2$. We have

$$E = a^{k-1}(3-bc) + b^{k-1}(3-ca) + c^{k-1}(3-ab)$$

$$= 3(a^{k-1} + b^{k-1} + c^{k-1}) - a^{k-1}b^{k-1}c^{k-1}\left[(ab)^{2-k} + (bc)^{2-k} + (ca)^{2-k}\right].$$

Since $0 < 2-k < 1$, $f(x) = x^{2-k}$ is concave for $x \ge 0$. Thus, by Jensen's inequality, we have

$$(ab)^{2-k} + (bc)^{2-k} + (ca)^{2-k} \le 3\left(\frac{ab+bc+ca}{3}\right)^{2-k} = 3,$$

hence
$$E \ge 3(a^{k-1} + b^{k-1} + c^{k-1}) - 3a^{k-1}b^{k-1}c^{k-1}.$$

Consequently, it suffices to show that

$$a^{k-1} + b^{k-1} + c^{k-1} \ge a^{k-1}b^{k-1}c^{k-1} + 2.$$

Due to symmetry, we may assume that

$$a \ge b \ge c,$$

which involves

$$ab \ge \frac{1}{3}(ab + bc + ca) \ge 1.$$

Let

$$x = \sqrt{a^{k-1}b^{k-1}}, \qquad x \ge 1.$$

From

$$2 \ge 3 - ab = bc + ca \ge 2c\sqrt{ab},$$

we get

$$c \le \frac{1}{\sqrt{ab}},$$

hence

$$c^{k-1} \le \frac{1}{x}.$$

Write the required inequality as

$$a^{k-1} + b^{k-1} - 2 \ge \left(a^{k-1}b^{k-1} - 1\right) c^{k-1}.$$

It suffices to show that

$$a^{k-1} + b^{k-1} - 2 \ge \frac{a^{k-1}b^{k-1} - 1}{x}.$$

Since

$$a^{k-1} + b^{k-1} \ge 2\sqrt{a^{k-1}b^{k-1}} = 2x,$$

we only need to prove that

$$2x - 2 \ge \frac{x^2 - 1}{x}.$$

Indeed,

$$2x - 2 - \frac{x^2 - 1}{x} = \frac{(x-1)^2}{x} \ge 0.$$

The equality holds for $a = b = c = 1$.

□

P 2.71. *Let a, b, c be nonnegative real numbers such that $a + b + c = 2$. If*

$$2 \le k \le 3,$$

than

$$a^k(b + c) + b^k(c + a) + c^k(a + b) \le 2.$$

Solution. Denote by $E_k(a,b,c)$ the left hand side of the inequality, assume that

$$a \le b \le c,$$

and show that

$$E_k(a,b,c) \le E_k(0,a+b,c) \le 2.$$

The left inequality is equivalent to

$$\frac{ab}{c}(a^{k-1}+b^{k-1}) \le (a+b)^k - a^k - b^k.$$

Clearly, it suffices to consider $c=b$, when the inequality becomes

$$2a^k + b^{k-1}(a+b) \le (a+b)^k.$$

Since $2a^k \le a^{k-1}(a+b)$, it remains to show that

$$a^{k-1}+b^{k-1} \le (a+b)^{k-1},$$

which is true since

$$\frac{a^{k-1}+b^{k-1}}{(a+b)^{k-1}} = \left(\frac{a}{a+b}\right)^{k-1} + \left(\frac{b}{a+b}\right)^{k-1} \le \frac{a}{a+b} + \frac{b}{a+b} = 1.$$

Using the notation $d=a+b$, we can write the right inequality $E_k(0,a+b,c) \le 2$ in the form

$$cd(c^{k-1}+d^{k-1}) \le 2,$$

where $c+d=2$. By the Power-Mean inequality , we have

$$\left(\frac{c^{k-1}+d^{k-1}}{2}\right)^{1/(k-1)} \le \left(\frac{c^2+d^2}{2}\right)^{1/2},$$

$$c^{k-1}+d^{k-1} \le 2\left(\frac{c^2+d^2}{2}\right)^{(k-1)/2}.$$

Thus, it suffices to show that

$$cd\left(\frac{c^2+d^2}{2}\right)^{(k-1)/2} \le 1,$$

which is equivalent to

$$cd(2-cd)^{(k-1)/2} \le 1.$$

Since $2-cd \ge 1$, we have

$$cd(2-cd)^{(k-1)/2} \le cd(2-cd) = 1-(1-cd)^2 \le 1.$$

The equality holds for $a=0$ and $b=c=1$ (or any cyclic permutation).

□

P 2.72. *Let a, b, c be nonnegative real numbers, no two of which are zero. If*

$$m > n \geq 0,$$

than
$$\frac{b^m + c^m}{b^n + c^n}(b + c - 2a) + \frac{c^m + a^m}{c^n + a^n}(c + a - 2b) + \frac{a^m + b^m}{a^n + b^n}(a + b - 2c) \geq 0.$$

(Vasile Cîrtoaje, 2006)

Solution. Write the inequality as

$$AX + BY + CZ \geq 0,$$

where
$$A = \frac{b^m + c^m}{b^n + c^n}, \quad B = \frac{c^m + a^m}{c^n + a^n}, \quad C = \frac{a^m + b^m}{a^n + b^n},$$
$$X = b + c - 2a, \quad Y = c + a - 2b, \quad Z = a + b - 2c, \quad X + Y + Z = 0.$$

Without loss of generality, assume that

$$a \leq b \leq c,$$

which involves $X \geq Y \geq Z$ and $X \geq 0$. Since

$$\begin{aligned} 2(AX + BY + CZ) &= (2A - B - C)X + (B + C)X + 2(BY + CZ) \\ &= (2A - B - C)X - (B + C)(Y + Z) + 2(BY + CZ) \\ &= (2A - B - C)X + (B - C)(Y - Z), \end{aligned}$$

it suffices to show that $B \geq C$ and $2A - B - C \geq 0$. The inequality $B \geq C$ can be written as
$$b^n c^n (c^{m-n} - b^{m-n}) + a^n (c^m - b^m) - a^m (c^n - b^n) \geq 0,$$
$$b^n c^n (c^{m-n} - b^{m-n}) + a^n [c^m - b^m - a^{m-n}(c^n - b^n)] \geq 0.$$

This is true since $c^{m-n} \geq b^{m-n}$ and

$$c^m - b^m - a^{m-n}(c^n - b^n) \geq c^m - b^m - b^{m-n}(c^n - b^n) = c^n (c^{m-n} - b^{m-n}) \geq 0.$$

The inequality $2A - B - C \geq 0$ follows from

$$2A \geq b^{m-n} + c^{m-n}, \quad b^{m-n} \geq C, \quad c^{m-n} \geq B.$$

Indeed, we have
$$2A - b^{m-n} - c^{m-n} = \frac{(b^n - c^n)(b^{m-n} - c^{m-n})}{b^n + c^n} \geq 0,$$
$$b^{m-n} - C = \frac{a^n (b^{m-n} - a^{m-n})}{a^n + b^n} \geq 0,$$
$$c^{m-n} - B = \frac{a^n (c^{m-n} - a^{m-n})}{c^n + a^n} \geq 0.$$

The equality holds for $a = b = c$, and also for $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 2.73. *Let a, b, c be positive real numbers such that $abc = 1$. Prove that*

$$\sqrt{a^2-a+1}+\sqrt{b^2-b+1}+\sqrt{c^2-c+1} \geq a+b+c.$$

(Vasile Cîrtoaje, 2012)

First Solution. Among $a-1$, $b-1$ and $c-1$ there are two with the same sign. Let $(b-1)(c-1) \geq 0$, that is,

$$t \leq \frac{1}{a}, \quad t = b+c-1.$$

By Minkowsky's inequality, we have

$$\sqrt{b^2-b+1}+\sqrt{c^2-c+1} = \sqrt{\left(b-\frac{1}{2}\right)^2+\frac{3}{4}}+\sqrt{\left(c-\frac{1}{2}\right)^2+\frac{3}{4}} \geq \sqrt{t^2+3}.$$

Thus, it suffices to show that

$$\sqrt{a^2-a+1}+\sqrt{t^2+3} \geq a+b+c,$$

which is equivalent to

$$\sqrt{a^2-a+1}+f(t) \geq a+1,$$

where

$$f(t) = \sqrt{t^2+3}-t.$$

Clearly, $f(t)$ is decreasing for $t \leq 0$. Since

$$f(t) = \frac{3}{\sqrt{t^2+3}+t},$$

$f(t)$ is also decreasing for $t \geq 0$. Then, $f(t) \geq f\left(\frac{1}{a}\right)$, and it suffices to show that

$$\sqrt{a^2-a+1}+f\left(\frac{1}{a}\right) \geq a+1,$$

which is equivalent to

$$\sqrt{a^2-a+1}+\sqrt{\frac{1}{a^2}+3} \geq a+\frac{1}{a}+1.$$

By squaring, this inequality becomes

$$2\sqrt{(a^2-a+1)\left(\frac{1}{a^2}+3\right)} \geq 3a+\frac{2}{a}-1.$$

Indeed, by the Cauchy-Schwarz inequality, we have

$$2\sqrt{(a^2-a+1)\left(\frac{1}{a^2}+3\right)} = \sqrt{[(2-a)^2+3a^2]\left(\frac{1}{a^2}+3\right)}$$

$$\geq \frac{2-a}{a}+3a = 3a+\frac{2}{a}-1.$$

The equality holds for $a = b = c$.

Second Solution. If the inequality

$$\sqrt{x^2 - x + 1} - x \geq \frac{1}{2}\left(\frac{3}{x^2 + x + 1} - 1\right)$$

holds for all $x > 0$, then it suffices to prove that

$$\frac{1}{a^2 + a + 1} + \frac{1}{b^2 + b + 1} + \frac{1}{c^2 + c + 1} \geq 1,$$

which is just the known inequality in P 1.45. The above inequality in x is equivalent to

$$\frac{1 - x}{\sqrt{x^2 - x + 1} + x} \geq \frac{(1 - x)(2 + x)}{2(x^2 + x + 1)},$$

$$(x - 1)\left[(x + 2)\sqrt{x^2 - x + 1} - x^2 - 2\right] \geq 0,$$

$$\frac{3x^2(x - 1)^2}{(x + 2)\sqrt{x^2 - x + 1} + x^2 + 2} \geq 0.$$

□

P 2.74. *Let a, b, c be positive real numbers such that $abc = 1$. Prove that*

$$\sqrt{16a^2 + 9} + \sqrt{16b^2 + 9} + \sqrt{16b^2 + 9} \geq 4(a + b + c) + 3.$$

(MEMO, 2012)

First Solution *(by Vo Quoc Ba Can).* Since

$$\sqrt{16a^2 + 9} - 4a = \frac{9}{\sqrt{16a^2 + 9} + 4a},$$

the inequality is equivalent to

$$\sum \frac{1}{\sqrt{16a^2 + 9} + 4a} \geq \frac{1}{3}.$$

By the AM-GM inequality, we have

$$2\sqrt{16a^2 + 9} \leq \frac{16a^2 + 9}{2a + 3} + 2a + 3,$$

$$2(\sqrt{16a^2 + 9} + 4a) \leq \frac{16a^2 + 9}{2a + 3} + 10a + 3 = \frac{18(2a^2 + 2a + 1)}{2a + 3}.$$

Thus, it suffices to show that

$$\sum \frac{2a + 3}{2a^2 + 2a + 1} \geq 3.$$

If the inequality

$$\frac{2a+3}{2a^2+2a+1} \ge \frac{3}{a^{8/5}+a^{4/5}+1}$$

holds for all $a > 0$, then it suffices to show that

$$\sum \frac{1}{a^{8/5}+a^{4/5}+1} \ge 1,$$

which follows immediately from the inequality in P 1.45. Therefore, using the substitution $x = a^{1/5}$, $x > 0$, we need to show that

$$\frac{2x^5+3}{2x^{10}+2x^5+1} \ge \frac{3}{x^8+x^4+1},$$

which is equivalent to

$$2x^4(x^5-3x^2+x+1)+x^4-4x+3 \ge 0.$$

This is true since, by the AM-GM inequality, we have

$$x^5+x+1 \ge 3\sqrt[3]{x^5\cdot x\cdot 1} = 3x^2$$

and

$$x^4+3 = x^4+1+1+1 \ge 4\sqrt[4]{x^4\cdot 1\cdot 1\cdot 1} = 4x.$$

The equality holds for $a = b = c = 1$.

Second Solution. Making the substitution

$$x = \sqrt{16a^2+9}-4a, \quad y = \sqrt{16b^2+9}-4b, \quad z = \sqrt{16c^2+9}-4c, \quad x,y,z > 0,$$

which involves

$$a = \frac{9-x^2}{8x}, \quad b = \frac{9-y^2}{8y}, \quad c = \frac{9-z^2}{8z},$$

we need to show that

$$(9-x^2)(9-y^2)(9-z^2) = 512xyz$$

yields

$$x+y+z \ge 3.$$

Use the contradiction method. Assume that

$$x+y+z < 3,$$

and show that

$$(9-x^2)(9-y^2)(9-z^2) > 512xyz.$$

According to the AM-GM inequality, we get

$$3+x = 1+1+1+x \ge 4\sqrt[4]{x}, \quad 3+y \ge 4\sqrt[4]{y}, \quad 3+z \ge 4\sqrt[4]{z},$$

hence

$$(3+x)(3+y)(3+z) \geq 64\sqrt[4]{xyz}.$$

Therefore, it suffices to prove that

$$(3-x)(3-y)(3-z) > 8\sqrt[4]{x^3y^3z^3}.$$

Since

$$1 > \left(\frac{x+y+z}{3}\right)^3 \geq xyz,$$

we have

$$\begin{aligned}(3-x)(3-y)(3-z) &= 9(3-x-y-z) + 3(xy+yz+zx) - xyz \\ &> 3(xy+yz+zx) - xyz \geq 9(xyz)^{2/3} - xyz \\ &> 8(xyz)^{2/3} > 8(xyz)^{3/4}.\end{aligned}$$

□

P 2.75. *Let a, b, c be positive real numbers such that $abc = 1$. Prove that*

$$\sqrt{25a^2+144} + \sqrt{25b^2+144} + \sqrt{25c^2+144} \leq 5(a+b+c) + 24.$$

(Vasile Cîrtoaje, 2012)

First Solution. Since

$$\sqrt{25a^2+144} - 5a = \frac{144}{\sqrt{25a^2+144}+5a},$$

the inequality is equivalent to

$$\sum \frac{1}{\sqrt{25a^2+144}+5a} \leq \frac{1}{6}.$$

If the inequality

$$\frac{1}{\sqrt{25a^2+144}+5a} \leq \frac{1}{6\sqrt{5a^{18/13}+4}}$$

holds for all $a > 0$, then it suffices to show that

$$\sum \frac{1}{\sqrt{5a^{18/13}+4}} \leq 1,$$

which follows immediately from P 2.33. Using the substitution $x = a^{1/13}$, $x > 0$, we only need to show that

$$\sqrt{25x^{26}+144} + 5x^{13} \geq 6\sqrt{5x^{18}+4}.$$

By squaring, the inequality becomes

$$10x^{13}(\sqrt{25x^{26}+144}+5x^{13}-18x^5) \ge 0.$$

This is true if

$$25x^{26}+144 \ge (18x^5-5x^{13})^2,$$

which is equivalent to

$$5x^{18}+4 \ge 9x^{10}.$$

By the AM-GM inequality, we have

$$\begin{aligned} 5x^{18}+4 &= x^{18}+x^{18}+x^{18}+x^{18}+x^{18}+1+1+1+1 \\ &\ge 9\sqrt[9]{x^{18}\cdot x^{18}\cdot x^{18}\cdot x^{18}\cdot x^{18}\cdot 1\cdot 1\cdot 1\cdot 1} = 9x^{10}. \end{aligned}$$

The equality holds for $a=b=c=1$.

Second Solution. Making the substitution

$$8x=\sqrt{25a^2+144}-5a, \quad 8y=\sqrt{25b^2+144}-5b, \quad 8z=\sqrt{25c^2+144}-5c,$$

which involves

$$a=\frac{9-4x^2}{5x}, \quad b=\frac{9-4y^2}{5y}, \quad c=\frac{9-4z^2}{5z}, \quad x,y,z \in \left(0,\frac{3}{2}\right),$$

we need to show that

$$(9-4x^2)(9-4y^2)(9-4z^2)=125xyz$$

involves

$$x+y+z \le 3.$$

Use the contradiction method. Assume that

$$x+y+z>3,$$

and show that

$$(9-4x^2)(9-4y^2)(9-4z^2)<125xyz.$$

Since

$$9-4x^2<3(x+y+z)-\frac{12x^2}{x+y+z}=\frac{3(y+z-x)(y+z+3x)}{x+y+z},$$

it suffices to prove the homogeneous inequality

$$27AB \le 125xyz(x+y+z)^3,$$

where

$$A=(y+z-x)(z+x-y)(x+y-z),$$

$$B=(y+z+3x)(z+x+3y)(x+y+3z).$$

Consider the nontrivial case $A \ge 0$. By the AM-GM inequality, we have

$$B \le \frac{125}{27}(x+y+z)^3.$$

Therefore, it suffices to show that

$$A \le xyz,$$

which is a well known inequality (equivalent to Schur's inequality of degree three).

□

P 2.76. *If a, b are positive real numbers such that $ab + bc + ca = 3$, then*

(a) $$\sqrt{a^2+3}+\sqrt{b^2+3}+\sqrt{b^2+3} \ge a+b+c+3;$$

(b) $$\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a} \ge \sqrt{4(a+b+c)+6}.$$

(Lee Sang Hoon, 2007)

Solution. (a) ***First Solution*** (by *Pham Thanh Hung*). By squaring, the inequality becomes

$$\sum \sqrt{(b^2+3)(c^2+3)} \ge 3(1+a+b+c).$$

Since

$$\begin{aligned}(b^2+3)(c^2+3) &= (b+c)(b+a)(c+a)(c+b) = (b+c)^2(a^2+3)\\ &\ge \frac{1}{4}(b+c)^2(a+3)^2,\end{aligned}$$

we have

$$\begin{aligned}\sum \sqrt{(b^2+3)(c^2+3)} &\ge \frac{1}{2}\sum(b+c)(a+3) = \frac{1}{2}\left(6\sum a + 2\sum bc\right)\\ &= 3(a+b+c+1).\end{aligned}$$

The equality holds for $a = b = c = 1$.

Second Solution. Use the SOS method. Write the inequality as follows:

$$\sqrt{(a+b)(a+c)}+\sqrt{(b+c)(b+a)}+\sqrt{(c+a)(c+b)} \ge a+b+c+3,$$

$$2\left[a+b+c-\sqrt{3(ab+bc+ca)}\right] \ge \sum\left(\sqrt{a+b}-\sqrt{a+c}\right)^2,$$

$$\frac{1}{a+b+c+\sqrt{3(ab+bc+ca)}}\sum(b-c)^2 \ge \sum\frac{(b-c)^2}{\left(\sqrt{a+b}+\sqrt{a+c}\right)^2},$$

$$\sum\frac{S_a(b-c)^2}{\left(\sqrt{a+b}+\sqrt{a+c}\right)^2} \ge 0,$$

where

$$S_a = \left(\sqrt{a+b}+\sqrt{a+c}\right)^2 - a - b - c - \sqrt{3(ab+bc+ca)}.$$

The inequality is true since

$$S_a = 3(a+b+c) + 2\sqrt{(a+b)(a+c)} - \sqrt{3(ab+bc+ca)}$$

$$> 2\sqrt{a^2+(ab+bc+ca)} - \sqrt{3(ab+bc+ca)} > 0.$$

Third Solution. Use the substitution

$$x = \sqrt{a^2+3} - a, \quad y = \sqrt{b^2+3} - b, \quad z = \sqrt{c^2+3} - c, \quad x, y, z > 0.$$

We need to show that

$$x + y + z \ge 3.$$

We have

$$\sum yz = \sum \left[\sqrt{(b+a)(b+c)} - b\right]\left[\sqrt{(c+a)(c+b)} - c\right]$$

$$= \sum (b+c)\sqrt{(a+b)(a+c)} - \sum b\sqrt{(c+a)(c+b)} - \sum c\sqrt{(b+a)(b+c)} + \sum bc$$

$$= \sum (b+c)\sqrt{(a+b)(a+c)} - \sum c\sqrt{(a+b)(a+c)} - \sum b\sqrt{(a+c)(a+b)} + \sum bc$$

$$= \sum bc = 3.$$

Thus, we get

$$x + y + z \ge \sqrt{3(xy+yz+zx)} = 3.$$

(b) By squaring, we get the inequality in (a).

□

P 2.77. *If a, b, c are nonnegative real numbers such that $a+b+c=3$, then*

$$\sqrt{(5a^2+3)(5b^2+3)} + \sqrt{(5b^2+3)(5c^2+3)} + \sqrt{(5c^2+3)(5a^2+3)} \ge 24.$$

(Nguyen Van Quy, 2012)

Solution. Assume that

$$a \ge b \ge c, \qquad 1 \le a \le 3, \quad b + c \le 2.$$

Using the notation

$$A = 5a^2+3, \qquad B = 5b^2+3, \qquad C = 5c^2+3,$$

we can write the inequality as follows:

$$\sqrt{A}\left(\sqrt{B}+\sqrt{C}\right) + \sqrt{BC} \ge 24,$$

$$\sqrt{A\left(B+C+2\sqrt{BC}\right)} \geq 24-\sqrt{BC}.$$

Consider the nontrivial case $\sqrt{BC} < 24$. The inequality is true if

$$A\left(B+C+2\sqrt{BC}\right) \geq \left(24-\sqrt{BC}\right)^2,$$

which is equivalent to

$$A(A+B+C+48) \geq \left(A+24-\sqrt{BC}\right)^2.$$

Applying Lemma below for $k=5/3$ and $m=4/15$ yields

$$5\sqrt{BC} \geq 25bc+15+4(b-c)^2.$$

Therefore, it suffices to show that

$$25A(A+B+C+48) \geq [5A+120-25bc-15-4(b-c)^2]^2,$$

which is equivalent to

$$25(5a^2+3)[5(a^2+b^2+c^2)+57] \geq [25a^2+120-25bc-4(b-c)^2]^2.$$

Since

$$5(a^2+b^2+c^2)+57 = 5a^2+5(b+c)^2-10bc+57 = 2(5a^2-15a+51-5bc)$$

and

$$\begin{aligned}25a^2+120-25bc-4(b-c)^2 &= 25a^2+120-4(b+c)^2-9bc\\ &= 3(7a^2+8a+28-3bc),\end{aligned}$$

we need to show that

$$50(5a^2+3)(5a^2-15a+51-5bc) \geq 9(7a^2+8a+28-3bc)^2.$$

From $bc \leq (b+c)^2/4$ and $(a-b)(a-c) \geq 0$, we get

$$bc \leq \frac{(3-a)^2}{4}, \qquad bc \geq a(b+c)-a^2 = 3a-2a^2.$$

Consider a fixed, $a \geq 1$, and denote $x=bc$. So, we only need to prove that $f(x) \geq 0$ for

$$3a-2a^2 \leq x \leq \frac{a^2-6a+9}{4},$$

where

$$f(x) = 50(5a^2+3)(5a^2-15a+51-5x)-9(7a^2+8a+28-3x)^2.$$

Since f is concave, it suffices to show that $f(3a-2a^2) \geq 0$ and $f\left(\dfrac{a^2-6a+9}{4}\right) \geq 0$. Indeed, we have

$$\begin{aligned} f(3a-2a^2) &= 3(743a^4 - 2422a^3 + 2813a^2 - 1332a + 198) \\ &= 3(a-1)^2[(a-1)(743a-193)+5] \geq 0, \end{aligned}$$

$$\begin{aligned} f\left(\frac{a^2-6a+9}{4}\right) &= \frac{375}{16}(25a^4 - 140a^3 + 286a^2 - 252a + 81) \\ &= \frac{375}{16}(a-1)^2(5a-9)^2 \geq 0. \end{aligned}$$

Thus, the proof is completed. The equality holds for $a = b = c = 1$, and also for $a = 9/5$ and $b = c = 3/5$ (or any cyclic permutation).

Lemma. *Let $b, c \geq 0$ such that $b + c \leq 2$. If $k > 0$ and $0 \leq m \leq \dfrac{k}{2k+2}$, then*

$$\sqrt{(kb^2+1)(kc^2+1)} \geq kbc + 1 + m(b-c)^2.$$

Proof. By squaring, the inequality becomes

$$(b-c)^2[k - 2m - 2kmbc - m^2(b-c)^2] \geq 0.$$

This is true since

$$k - 2m - 2kmbc - m^2(b-c)^2 = k - 2m - 2m(k-2m)bc - m^2(b+c)^2$$

$$\geq k - 2m - \frac{m(k-2m)}{2}(b+c)^2 - m^2(b+c)^2$$

$$= k - 2m - \frac{km}{2}(b+c)^2 \geq k - 2m - 2km \geq 0.$$

□

P 2.78. *If a, b, c are nonnegative real numbers such that $a + b + c = 3$, then*

$$\sqrt{a^2+1} + \sqrt{b^2+1} + \sqrt{c^2+1} \geq \sqrt{\frac{4(a^2+b^2+c^2)+42}{3}}.$$

(Vasile Cîrtoaje, 2014)

Solution. Assume that

$$a \geq b \geq c, \qquad a \geq 1, \quad b + c \leq 2.$$

By squaring, the inequality becomes

$$\sqrt{A}\left(\sqrt{B}+\sqrt{C}\right) + \sqrt{BC} \geq \frac{a^2+b^2+c^2+33}{6},$$

$$\sqrt{A\left(B+C+2\sqrt{BC}\right)}+\sqrt{BC} \ge \frac{a^2+b^2+c^2+33}{6},$$

where

$$A = a^2+1, \qquad B = b^2+1, \qquad C = c^2+1.$$

Applying Lemma from the preceding problem P 2.77 for $k = 1$ and $m = 1/4$ gives

$$\sqrt{BC} \ge bc+1+\frac{1}{4}(b-c)^2.$$

Therefore, it suffices to show that

$$\sqrt{A\left[B+C+2bc+2+\frac{1}{2}(b-c)^2\right]}+bc+1+\frac{1}{4}(b-c)^2 \ge \frac{a^2+b^2+c^2+33}{6},$$

which is equivalent to

$$6\sqrt{2(a^2+1)[3(b+c)^2+8-4bc]} \ge 2a^2-(b+c)^2+54-4bc,$$

$$6\sqrt{2(a^2+1)(3a^2-18a+35-4bc)} \ge a^2+6a+45-4bc.$$

From $bc \le (b+c)^2/4$ and $(a-b)(a-c) \ge 0$, we get

$$bc \le \frac{(3-a)^2}{4}, \qquad bc \ge a(b+c)-a^2 = 3a-2a^2.$$

Consider a fixed, $a \ge 1$, and denote $x = bc$. So, we only need to prove that $f(x) \ge 0$ for

$$3a-2a^2 \le x \le \frac{a^2-6a+9}{4},$$

where

$$f(x) = 72(a^2+1)(3a^2-18a+35-4x)-(a^2+6a+45-4x)^2.$$

Since f is concave, it suffices to show that $f(3a-2a^2) \ge 0$ and $f\left(\frac{a^2-6a+9}{4}\right) \ge 0$. Indeed,

$$\begin{aligned} f(3a-2a^2) &= 9(79a^4-228a^3+274a^2-180a+55) \\ &= 9(a-1)^2(79a^2-70a+55 \ge 0, \end{aligned}$$

$$\begin{aligned} f\left(\frac{a^2-6a+9}{4}\right) &= 144(a^4-6a^3+13a^2-12a+4) \\ &= 144(a-1)^2(a-2)^2 \ge 0. \end{aligned}$$

The equality holds for $a = b = c = 1$, and also for $a = 2$ and $b = c = 1/2$ (or any cyclic permutation).

□

P 2.79. *If a,b,c are nonnegative real numbers such that $a+b+c=3$, then*

(a) $$\sqrt{a^2+3}+\sqrt{b^2+3}+\sqrt{c^2+3}\ge\sqrt{2(a^2+b^2+c^2)+30};$$

(b) $$\sqrt{3a^2+1}+\sqrt{3b^2+1}+\sqrt{3c^2+1}\ge\sqrt{2(a^2+b^2+c^2)+30}.$$

(Vasile Cîrtoaje, 2014)

Solution. Assume that
$$a\ge b\ge c,\qquad a\ge 1,\quad b+c\le 2.$$

(a) By squaring, the inequality becomes
$$\sqrt{A}\left(\sqrt{B}+\sqrt{C}\right)+\sqrt{BC}\ge\frac{a^2+b^2+c^2+21}{2},$$
$$\sqrt{A\left(B+C+2\sqrt{BC}\right)}+\sqrt{BC}\ge\frac{a^2+b^2+c^2+21}{2},$$
where
$$A=a^2+3,\quad B=b^2+3,\quad C=c^2+3.$$
Applying Lemma from problem P 2.77 for $k=1/3$ and $m=1/9$ gives
$$\sqrt{BC}\ge bc+3+\frac{1}{3}(b-c)^2.$$
Therefore, it suffices to show that
$$\sqrt{A\left[B+C+2bc+6+\frac{2}{3}(b-c)^2\right]}+bc+3+\frac{1}{3}(b-c)^2\ge\frac{a^2+b^2+c^2+21}{2},$$
which is equivalent to
$$2\sqrt{3(a^2+3)[5(b+c)^2+36-8bc]}\ge 3a^2+(b+c)^2+45-4bc,$$
$$\sqrt{3(a^2+3)(5a^2-30a+81-8bc)}\ge 2a^2-3a+27-2bc.$$
From $bc\le(b+c)^2/4$ and $(a-b)(a-c)\ge 0$, we get
$$bc\le\frac{(3-a)^2}{4},\qquad bc\ge a(b+c)-a^2=3a-2a^2.$$
Consider a fixed, $a\ge 1$, and denote $x=bc$. So, we only need to prove that $f(x)\ge 0$ for
$$3a-2a^2\le x\le\frac{a^2-6a+9}{4},$$
where
$$f(x)=3(a^2+3)(5a^2-30a+81-8x)-(2a^2-3a+27-2x)^2.$$

Since f is concave, it suffices to show that $f(3a-2a^2) \ge 0$ and $f\left(\frac{a^2-6a+9}{4}\right) \ge 0$. Indeed,

$$f(3a-2a^2) = 27a^2(a-1)^2 \ge 0,$$

$$f\left(\frac{a^2-6a+9}{4}\right) = \frac{27}{4}(a^4-8a^3+22a^2-24a+9)$$
$$= \frac{27}{4}(a-1)^2(a-3)^2 \ge 0.$$

The equality holds for $a=b=c=1$, and also for $a=3$ and $b=c=0$ (or any cyclic permutation).

(b) By squaring, the inequality becomes

$$\sqrt{A}\left(\sqrt{B}+\sqrt{C}\right) + \sqrt{BC} \ge \frac{27-a^2-b^2-c^2}{2},$$

$$\sqrt{A\left(B+C+2\sqrt{BC}\right)} + \sqrt{BC} \ge \frac{27-a^2-b^2-c^2}{2},$$

where

$$A = 3a^2+1, \qquad B = 3b^2+1, \qquad C = 3c^2+1.$$

Applying Lemma from problem P 2.77 for $k=3$ and $m=1/3$ gives

$$\sqrt{BC} \ge 3bc+1+\frac{1}{3}(b-c)^2.$$

Therefore, it suffices to show that

$$\sqrt{A\left[B+C+6bc+2+\frac{2}{3}(b-c)^2\right]} + 3bc+1+\frac{1}{3}(b-c)^2 \ge \frac{27-a^2-b^2-c^2}{2},$$

which is equivalent to

$$2\sqrt{3(3a^2+1)[11(b+c)^2+12-8bc]} \ge 75-3a^2-5(b+c)^2-4bc,$$

$$\sqrt{3(3a^2+1)(11a^2-66a+111-8bc)} \ge 15+15a-4a^2-2bc.$$

From $bc \le (b+c)^2/4$ and $(a-b)(a-c) \ge 0$, we get

$$bc \le \frac{(3-a)^2}{4}, \qquad bc \ge a(b+c)-a^2 = 3a-2a^2.$$

Consider a fixed, $a \ge 1$, and denote $x = bc$. So, we only need to prove that $f(x) \ge 0$ for

$$3a-2a^2 \le x \le \frac{a^2-6a+9}{4},$$

where

$$f(x) = 3(3a^2+1)(11a^2-66a+111-8x)-(15+15a-4a^2-2x)^2.$$

Since f is concave, it suffices to show that $f(3a-2a^2) \geq 0$ and $f\left(\dfrac{a^2-6a+9}{4}\right) \geq 0$. Indeed,

$$f(3a-2a^2) = 27(a-1)^2(3a-2)^2 \geq 0,$$

$$f\left(\frac{a^2-6a+9}{4}\right) = \frac{27}{4}(9a^4-48a^3+94a^2-80a+25)$$
$$= \frac{27}{4}(a-1)^2(3a-5)^2 \geq 0.$$

The equality holds for $a=b=c=1$, and also for $a=5/3$ and $b=c=2/3$ (or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- *Let a,b,c be nonnegative real numbers such that $a+b+c=3$. If $k>0$, then*

$$\sqrt{ka^2+1}+\sqrt{kb^2+1}+\sqrt{kc^2+1} \geq \sqrt{\frac{8k(a^2+b^2+c^2)+3(9k^2+10k+9)}{3(k+1)}},$$

with equality for $a=b=c=1$, and also for $a=\dfrac{3k+1}{2k}$ and $b=c=\dfrac{3k-1}{4k}$ (or any cyclic permutation).

□

P 2.80. *If a,b,c are nonnegative real numbers such that $a+b+c=3$, then*

$$\sqrt{(32a^2+3)(32b^2+3)}+\sqrt{(32b^2+3)(32c^2+3)}+\sqrt{(32c^2+3)(32a^2+3)} \leq 105.$$

(Vasile Cîrtoaje, 2014)

Solution. Assume that

$$a \leq b \leq c, \qquad a \leq 1, \quad b+c \geq 2.$$

Denote

$$A = 32a^2+3, \qquad B = 32b^2+3, \qquad C = 32c^2+3,$$

and write the inequality as follows:

$$\sqrt{A}\left(\sqrt{B}+\sqrt{C}\right)+\sqrt{BC} \leq 105,$$

$$\sqrt{A}\cdot\sqrt{B+C+2\sqrt{BC}} \leq 105-\sqrt{BC}.$$

By Lemma below, we have

$$\sqrt{BC} \le 5(b+c)^2 + 12bc + 3 \le 8(b+c)^2 + 3 \le 8(a+b+c)^2 + 3 = 75 < 105.$$

Therefore, we can write the desired inequality as

$$A\left(B + C + 2\sqrt{BC}\right) \le \left(105 - \sqrt{BC}\right)^2,$$

which is equivalent to

$$A(A + B + C + 210) \le (A + 105 - \sqrt{BC})^2.$$

According to Lemma below, it suffices to show that

$$A(A + B + C + 210) \le [A + 105 - 5(b^2 + c^2) - 22bc - 3]^2,$$

which is equivalent to

$$[32a^2 + 105 - 5(b^2 + c^2) - 22bc]^2 \ge (32a^2 + 3)[32(a^2 + b^2 + c^2) + 219].$$

Since

$$32(a^2 + b^2 + c^2) + 219 = 32a^2 + 32(b+c)^2 - 64bc + 219 = 64a^2 - 192a + 507 - 64bc$$

and

$$32a^2 + 105 - 5(b^2 + c^2) - 22bc = 32a^2 + 105 - 5(b+c)^2 - 12bc = 3(9a^2 + 10a + 20 - 4bc),$$

we need to show that

$$9(9a^2 + 10a + 20 - 4bc)^2 \ge (32a^2 + 3)(64a^2 - 192a + 507 - 64bc).$$

From $bc \le (b+c)^2/4$, we get

$$bc \le \frac{(3-a)^2}{4}.$$

Consider a fixed, $0 \le a \le 1$, and denote $x = bc$. So, we only need to prove that $f(x) \ge 0$ for

$$0 \le x \le \frac{a^2 - 6a + 9}{4},$$

where

$$f(x) = 9(9a^2 + 10a + 20 - 4x)^2 - (32a^2 + 3)(64a^2 - 192a + 507 - 64x).$$

Since

$$\begin{aligned} f'(x) &= 72(4x - 9a^2 - 10a - 20) + 64(32a^2 + 3) \\ &\le 72[(a^2 - 6a + 9) - 9a^2 - 10a - 20) + 64(32a^2 + 3) \\ &= 8[184a(a-1) + (44a - 75)] < 0, \end{aligned}$$

f is decreasing, hence $f(x) \geq f\left(\frac{a^2-6a+9}{4}\right)$. Therefore, it suffices to show that $f\left(\frac{a^2-6a+9}{4}\right) \geq 0$. We have

$$\begin{aligned} f\left(\frac{a^2-6a+9}{4}\right) =& 9[9a^2+10a+20-(a^2-6a+9)]^2 \\ & -(32a^2+3)[64a^2-192a+507-16(a^2-6a+9)] \\ =& 9(8a^2+16a+11)^2-(32a^2+3)(48a^2-96a+363) \\ =& 192a(a-1)^2(18-5a) \geq 0. \end{aligned}$$

Thus, the proof is completed. The equality holds for $a=b=c=1$, and also for $a=0$ and $b=c=3/2$ (or any cyclic permutation).

Lemma. *If $b,c \geq 0$ such that $b+c \geq 2$, then*

$$\sqrt{(32b^2+3)(32c^2+3)} \leq 5(b^2+c^2)+22bc+3.$$

Proof. By squaring, the inequality becomes

$$(5b^2+5c^2+22bc)^2-32^2b^2c^2 \geq 96(b^2+c^2)-6(5b^2+5c^2+22bc),$$

$$5(b-c)^2(5b^2+5c^2+54bc) \geq 66(b-c)^2.$$

It suffices to show that

$$5(5b^2+5c^2+10bc) \geq 100,$$

which is equivalent to the obvious inequality $(b+c)^2 \geq 4$.

□

P 2.81. *If a,b,c are positive real numbers, then*

$$\left|\frac{b+c}{a}-3\right|+\left|\frac{c+a}{b}-3\right|+\left|\frac{a+b}{c}-3\right| \geq 2.$$

(Vasile Cîrtoaje, 2012)

Solution. Without loss of generality, assume that $a \geq b \geq c$.

Case 1: $a > b+c$. We have

$$\left|\frac{b+c}{a}-3\right|+\left|\frac{a+b}{c}-3\right|+\left|\frac{c+a}{b}-3\right| \geq \left|\frac{b+c}{a}-3\right| = 3-\frac{b+c}{a} > 2.$$

Case 2: $a \leq b+c$. We have

$$\left|\frac{b+c}{a}-3\right|+\left|\frac{a+b}{c}-3\right|+\left|\frac{c+a}{b}-3\right| \geq \left|\frac{b+c}{a}-3\right|+\left|\frac{c+a}{b}-3\right|$$

$$=\left(3-\frac{b+c}{a}\right)+\left(3-\frac{c+a}{b}\right) \geq 6-\frac{b+b}{a}-\frac{b+a}{b} = 2+\frac{(a-b)(2b-a)}{ab} \geq 2.$$

Thus, the proof is completed. The equality holds for $\frac{a}{2}=b=c$ (or any cyclic permutation).

□

P 2.82. *If a, b, c are real numbers such that $abc \neq 0$, then*

$$\left|\frac{b+c}{a}\right| + \left|\frac{c+a}{b}\right| + \left|\frac{a+b}{c}\right| \geq 2.$$

First Solution. Let

$$|a| = \max\{|a|, |b|, |c|\}.$$

We have

$$\left|\frac{b+c}{a}\right| + \left|\frac{c+a}{b}\right| + \left|\frac{a+b}{c}\right| \geq \left|\frac{b+c}{a}\right| + \left|\frac{c+a}{a}\right| + \left|\frac{a+b}{a}\right|$$

$$\geq \frac{|(-b-c)+(c+a)+(a+b)|}{|a|} = 2.$$

The equality holds for $a = 1$, $b = -1$ and $|c| \leq 1$ (or any permutation).

Second Solution. Since the inequality remains unchanged by replacing a, b, c with $-a, -b, -c$, it suffices to consider two cases: $a, b, c > 0$, and $a < 0$, $b, c > 0$.

Case 1: $a, b, c > 0$. We have

$$\left|\frac{b+c}{a}\right| + \left|\frac{c+a}{b}\right| + \left|\frac{a+b}{c}\right| = \left(\frac{a}{b} + \frac{b}{a}\right) + \left(\frac{b}{c} + \frac{c}{b}\right) + \left(\frac{c}{a} + \frac{a}{c}\right) \geq 6.$$

Case 2: $a < 0$ and $b, c > 0$. Replacing a by $-a$, we need to show that

$$\frac{b+c}{a} + \frac{|a-c|}{b} + \frac{|a-b|}{c} \geq 2$$

for all $a, b, c > 0$. Without loss of generality, assume that $b \geq c$. There are three case to consider: $b \geq c \geq a$, $b \geq a \geq c$ and $a \geq b \geq c$.
For $b \geq c \geq a$, we have

$$\frac{b+c}{a} + \frac{|a-c|}{b} + \frac{|a-b|}{c} \geq \frac{b+c}{a} \geq 2.$$

For $b \geq a \geq c$, we have

$$\frac{b+c}{a} + \frac{|a-c|}{b} + \frac{|a-b|}{c} - 2 \geq \frac{b+c}{a} + \frac{a-c}{b} - 2 = \frac{(a-b)^2 + c(b-a)}{ab} \geq 0.$$

For $a \geq b \geq c$, we have

$$\frac{b+c}{a} + \frac{|a-c|}{b} + \frac{|a-b|}{c} - 2 = \frac{b+c}{a} + \frac{a-c}{b} + \frac{a-b}{c} - 2$$

$$= \left(\frac{a}{b} + \frac{b}{a} - 2\right) + \frac{a-b}{c} + c\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{(a-b)^2}{ab} + \frac{(a-b)(ab-c^2)}{abc} \geq 0.$$

Third Solution. Using the substitution

$$x = \frac{b+c}{a}, \quad y = \frac{c+a}{b}, \quad z = \frac{a+b}{c},$$

we need to show that

$$x + y + z + 2 = xyz, \quad x, y, z \in \mathbb{R},$$

involves

$$|x| + |y| + |z| \geq 2.$$

If $xyz \leq 0$, then

$$-x - y - z = 2 - xyz \geq 2,$$

hence

$$|x| + |y| + |z| \geq |x + y + z| = |-x - y - z| \geq -x - y - z \geq 2.$$

If $xyz > 0$, then either $x, y, z > 0$ or only one of x, y, z is positive (for instance, $x > 0$ and $y, z < 0$).

Case 1: $x, y, z > 0$. We need to show that $x + y + z \geq 2$. We have

$$xyz = x + y + z + 2 > 2$$

and, by the AM-GM inequality, we get

$$x + y + z \geq 3\sqrt[3]{xyz} > 3\sqrt[3]{2} > 2,$$

Case 2: $x > 0$ and $y, z < 0$. Replacing y, z by $-y, -z$, we need to prove that

$$x - y - z + 2 = xyz$$

involves

$$x + y + z \geq 2$$

for all $x, y, z > 0$. Since

$$x + y + z - 2 = x + y + z - (xyz - x + y + z) = x(2 - yz),$$

we need to show that $yz \leq 2$. Indeed, we have

$$x + 2 = y + z + xyz \geq 2\sqrt{yz} + xyz,$$

$$x(1 - yz) + 2(1 - \sqrt{yz}\,) \geq 0,$$

$$(1 - \sqrt{yz}\,)\,[x(1 + \sqrt{yz}\,) + 2] \geq 0,$$

hence

$$yz \leq 1 < 2.$$

□

P 2.83. *Let a, b, c be nonnegative real numbers, no two of which are zero, and let*

$$x = \frac{2a}{b+c}, \quad y = \frac{2b}{c+a}, \quad z = \frac{2c}{a+b}.$$

Prove that

(a) $$\sqrt{xy} + \sqrt{yz} + \sqrt{zx} \ge xyz + 2;$$

(b) $$x + y + z + \sqrt{xy} + \sqrt{yz} + \sqrt{zx} \ge 6;$$

(c) $$\sqrt{x} + \sqrt{y} + \sqrt{z} \ge \sqrt{8 + xyz};$$

(d) $$\frac{\sqrt{yz}}{x+2} + \frac{\sqrt{zx}}{y+2} + \frac{\sqrt{xy}}{z+2} \ge 1.$$

Solution. (a) Since

$$\begin{aligned}\sqrt{yz} &= \frac{2\sqrt{bc(a+b)(c+a)}}{(a+b)(c+a)} \ge \frac{2\sqrt{bc}\,(a+\sqrt{bc})}{(a+b)(c+a)} \\ &= \frac{2a(b+c)\sqrt{bc} + 2bc(b+c)}{(a+b)(b+c)(c+a)} \ge \frac{4abc + 2bc(b+c)}{(a+b)(b+c)(c+a)},\end{aligned}$$

we have

$$\begin{aligned}\sum \sqrt{yz} &\ge \frac{12abc + 2\sum bc(b+c)}{(a+b)(b+c)(c+a)} \\ &= \frac{8abc}{(a+b)(b+c)(c+a)} + 2 = xyz + 2.\end{aligned}$$

The equality holds for $a = b = c$, and also for $a = 0$ or $b = 0$ or $c = 0$.

(b) **First Solution.** Taking into account the inequality (a), it suffices to show that

$$x + y + z + xyz \ge 4,$$

which is equivalent to Schur's inequality of degree three

$$a^3 + b^3 + c^3 + 3abc \ge \sum ab(a+b).$$

The equality holds for $a = b = c$, and also for $a = 0$ and $b = c$ (or any cyclic permutation).

Second Solution. We use the SOS technique. Write the inequality as

$$4\sum(x-1) \ge \sum \left(\sqrt{y} - \sqrt{z}\right)^2.$$

Since

$$\sum(x-1)=\sum\frac{(a-b)+(a-c)}{b+c}=\sum\frac{a-b}{b+c}+\sum\frac{b-a}{c+a}$$
$$=\sum\frac{(a-b)^2}{(b+c)(c+a)}=\sum\frac{(b-c)^2}{(a+b)(a+c)}$$

and

$$\left(\sqrt{y}-\sqrt{z}\right)^2=\frac{(y-z)^2}{\left(\sqrt{y}+\sqrt{z}\right)^2}=\frac{2(b-c)^2(a+b+c)^2}{(a+b)(a+c)\left(\sqrt{b^2+ab}+\sqrt{c^2+ac}\right)^2},$$

we can write the inequality as

$$\sum(b-c)^2S_a\ge 0,$$

where

$$S_a=(b+c)\left[2-\frac{(a+b+c)^2}{\left(\sqrt{b^2+ab}+\sqrt{c^2+ac}\right)^2}\right].$$

By Minkowski's inequality, we have

$$\left(\sqrt{b^2+ab}+\sqrt{c^2+ac}\right)^2\ge(b+c)^2+a\left(\sqrt{b}+\sqrt{c}\right)^2$$
$$\ge(b+c)^2+a(b+c)=(b+c)(a+b+c),$$

hence

$$S_a\ge(b+c)\left(2-\frac{a+b+c}{b+c}\right)=b+c-a.$$

Thus, it suffices to show that

$$\sum(b-c)^2(b+c-a)\ge 0,$$

which is just Schur's inequality of third degree.

Third Solution. Using the Cauchy-Schwarz inequality yields

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\ge\frac{(a+b+c)^2}{a(b+c)+b(c+a)+c(a+b)}=\frac{(a+b+c)^2}{2(ab+bc+ca)}.$$

Also, using Hölder's inequality, we have

$$\left(\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}\right)^2\ge\frac{(a+b+c)^3}{a^2(b+c)+b^2(c+a)+c^2(a+b)}.$$

Thus, it suffices to prove that

$$\frac{(a+b+c)^2}{ab+bc+ca}+\frac{2(a+b+c)^3}{a^2(b+c)+b^2(c+a)+c^2(a+b)}\ge 12.$$

Due to homogeneity, we may assume that $a+b+c=1$. Substituting

$$q = ab+bc+ca, \qquad 3q \le 1,$$

the inequality becomes

$$\frac{1}{q}+\frac{2}{q-3abc} \ge 12.$$

The fourth degree Schur's inequality

$$6abcp \ge (p^2-q)(4q-p^2), \quad p=a+b+c,$$

gives

$$6abc \ge (1-q)(4q-1).$$

Therefore,

$$\frac{1}{q}+\frac{2}{q-3abc}-12 \ge \frac{1}{q}+\frac{4}{2q-(1-q)(4q-1)}-12 = \frac{(1-3q)(1-4q)^2}{q(4q^2-3q+1)} \ge 0.$$

(c) By squaring, the inequality becomes

$$x+y+z+2\sqrt{xy}+2\sqrt{yz}+2\sqrt{zx} \ge 8+xyz.$$

Based on the inequality in (a), it suffices to show that

$$x+y+z+2(xyz+2) \ge 8+xyz,$$

which is equivalent to

$$x+y+z+xyz \ge 4,$$
$$a^3+b^3+c^3+3abc \ge \sum ab(a+b).$$

The last form is just Schur's inequality of third degree. The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

(d) Write the inequality as

$$\sum (b+c)\sqrt{yz} \ge 2(a+b+c).$$

First Solution. Since

$$\sqrt{yz} = \frac{2\sqrt{bc(a+b)(c+a)}}{(a+b)(c+a)} \ge \frac{2\sqrt{bc}\,(a+\sqrt{bc})}{(a+b)(c+a)}$$
$$= \frac{2a(b+c)\sqrt{bc}+2bc(b+c)}{(a+b)(b+c)(c+a)} \ge \frac{4abc+2bc(b+c)}{(a+b)(b+c)(c+a)},$$

it suffices to show that

$$\sum (b+c)[2abc+bc(b+c)] \ge (a+b+c)(a+b)(b+c)(c+a),$$

which is an identity. The equality holds for $a = b = c$, and also for $a = 0$ or $b - 0$ or $c = 0$.

Second Solution. Let

$$q = ab + bc + ca.$$

Since

$$\sqrt{yz} = \sqrt{\frac{2b}{a+b} \cdot \frac{2c}{c+a}} \geq \frac{2 \cdot \frac{2b}{a+b} \cdot \frac{2c}{c+a}}{\frac{2b}{a+b} + \frac{2c}{c+a}} = \frac{4bc}{bc+q},$$

it suffices to show that

$$\sum \frac{2bc(b+c)}{bc+q} \geq a+b+c,$$

which is equivalent to

$$\sum \left[\frac{2bc(b+c)}{bc+q} - a\right] \geq 0,$$

$$\sum \frac{bc(b-a) + bc(c-a) + b(c^2-a^2) + c(b^2-a^2)}{bc+q} \geq 0,$$

$$\sum \frac{c(b-a)(2b+a) + b(c-a)(2c+a)}{bc+q} \geq 0,$$

$$\sum \frac{c(b-a)(2b+a)}{bc+q} + \sum \frac{c(a-b)(2a+b)}{ca+q} \geq 0,$$

$$\sum c(a-b)\left[\frac{2a+b}{ca+q} - \frac{2b+a}{bc+q}\right] \geq 0,$$

$$\sum \frac{c(a-b)[q(a-b) - c(a^2-b^2)]}{(ca+q)(bc+q)} \geq 0,$$

$$abc \sum \frac{(a-b)^2}{(ca+q)(bc+q)} \geq 0.$$

□

P 2.84. *Let a, b, c be nonnegative real numbers, no two of which are zero, and let*

$$x = \frac{2a}{b+c}, \quad y = \frac{2b}{c+a}, \quad z = \frac{2c}{a+b}.$$

Prove that

$$\sqrt{1+24x} + \sqrt{1+24y} + \sqrt{1+24z} \geq 15.$$

(Vasile Cîrtoaje, 2005)

Solution *(by Vo Quoc Ba Can).* Assume that $c = \min\{a,b,c\}$, hence $z \le 1$. By Hölder's inequality

$$\left(\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}\right)^2\left[a^2(b+c)+b^2(c+a)\right] \ge (a+b)^3,$$

we get

$$\left(\sqrt{x}+\sqrt{y}\right)^2 \ge \frac{2(a+b)^3}{c(a^2+b^2)+ab(a+b)} = \frac{2(a+b)^3}{c(a+b)^2+ab(a+b-2c)}$$
$$\ge \frac{2(a+b)^3}{c(a+b)^2+\frac{1}{4}(a+b)^2(a+b-2c)} = \frac{8(a+b)}{a+b+2c} = \frac{8}{1+z}.$$

Using this result and Minkowski's inequality, we have

$$\sqrt{1+24x}+\sqrt{1+24y} \ge \sqrt{(1+1)^2+24(\sqrt{x}+\sqrt{y})^2} \ge 2\sqrt{1+\frac{48}{1+z}}.$$

Therefore, it suffices to show that

$$2\sqrt{1+\frac{48}{1+z}}+\sqrt{1+24z} \ge 15.$$

Using the substitution

$$\sqrt{1+24z} = 5t, \quad \frac{1}{5} \le t \le 1,$$

the inequality turns into

$$2\sqrt{\frac{t^2+47}{25t^2+23}} \ge 3-t.$$

By squaring, this inequality becomes

$$25t^4-150t^3+244t^2-138t+19 \le 0,$$

which is equivalent to the obvious inequality

$$(t-1)^2(5t-1)(5t-19) \le 0.$$

The equality holds for $a=b=c$, and also for $a=b$ and $c=0$ (or any cyclic permutation).

□

P 2.85. *If a,b,c are positive real numbers, then*

$$\sqrt{\frac{7a}{a+3b+3c}}+\sqrt{\frac{7b}{b+3c+3a}}+\sqrt{\frac{7c}{c+3a+3b}} \le 3.$$

(Vasile Cîrtoaje, 2005)

First Solution. Using the substitution

$$x=\sqrt{\frac{7a}{a+3b+3c}},\quad y=\sqrt{\frac{7b}{b+3c+3a}},\quad z=\sqrt{\frac{7c}{c+3a+3b}},$$

we have

$$\begin{cases} (x^2-7)a+3x^2b+3x^2c=0 \\ 3y^2a+(y^2-7)b+3y^2c=0 \\ 3z^2a+3z^2b+(z^2-7)c=0 \end{cases},$$

which involves

$$\begin{vmatrix} x^2-7 & 3x^2 & 3x^2 \\ 3y^2 & y^2-7 & 3y^2 \\ 3z^2 & 3z^2 & z^2-7 \end{vmatrix}=0\ ;$$

that is,

$$F(x,y,z)=0,$$

where

$$F(x,y,z)=4x^2y^2z^2+8\sum x^2y^2+7\sum x^2-49.$$

We need to show that $F(x,y,z)=0$ involves $x+y+z\le 3$, where $x,y,z>0$. To do this, we use the contradiction method. Assume that $x+y+z>3$ and show that $F(x,y,z)>0$. Since $F(x,y,z)$ is strictly increasing in each of its arguments, it is enough to prove that $x+y+z=3$ involves $F(x,y,z)\ge 0$. We will use the mixing variables technique. Assume that $x=\max\{x,y,z\}$ and denote

$$t=\frac{y+z}{2},\quad 0<t\le 1\le x.$$

We will show that

$$F(x,y,z)\ge F(x,t,t)\ge 0.$$

We have

$$\begin{aligned} F(x,y,z)-F(x,t,t) &= (8x^2+7)(y^2+z^2-2t^2)-4(x^2+2)(t^4-y^2z^2) \\ &= \frac{1}{2}(8x^2+7)(y-z)^2-(x^2+2)(t^2+yz)(y-z)^2 \\ &\ge \frac{1}{2}(8x^2+7)(y-z)^2-2(x^2+2)t^2(y-z)^2 \\ &= \frac{1}{2}(4x^2-1)(y-z)^2\ge 0 \end{aligned}$$

and

$$F(x,t,t)=F\left(x,\frac{3-x}{2},\frac{3-x}{2}\right)=\frac{1}{4}(x-1)^2(x-2)^2(x^2-6x+23)\ge 0.$$

The equality holds for $a=b=c$, and also for $\frac{a}{8}=b=c$ (or any cyclic permutation).

Second Solution. Due to homogeneity, we may assume that $a+b+c=3$, when the inequality becomes

$$\sum \sqrt{\frac{7a}{9-2a}} \le 3.$$

Using the substitution

$$x=\sqrt{\frac{7a}{9-2a}}, \quad y=\sqrt{\frac{7b}{9-2b}}, \quad z=\sqrt{\frac{7c}{9-2c}},$$

we need to show that if x,y,z are positive real numbers such that

$$\sum \frac{1}{2x^2+7}=\frac{1}{3},$$

then

$$x+y+z\le 3.$$

For the sake of contradiction, assume that $x+y+z>3$ and show that $F(x,y,z)<0$, where

$$F(x,y,z)=\sum \frac{1}{2x^2+7}-\frac{1}{3}.$$

Since $F(x,y,z)$ is strictly decreasing in each of its arguments, it is enough to prove that $x+y+z=3$ involves $F(x,y,z)\le 0$. This is just the inequality in P 1.33.

□

P 2.86. *If a,b,c are positive real numbers such that $a+b+c=3$, then*

$$\sqrt[3]{a^2(b^2+c^2)}+\sqrt[3]{b^2(c^2+a^2)}+\sqrt[3]{c^2(a^2+b^2)}\le 3\sqrt[3]{2}.$$

(Michael Rozenberg, 2013)

Solution. By Hölder's inequality, we have

$$\left[\sum \sqrt[3]{a^2(b^2+c^2)}\right]^3 \le \left[\sum a(b+c)\right]^2 \cdot \sum \frac{b^2+c^2}{(b+c)^2}.$$

Therefore, it suffices to show that

$$\sum \frac{b^2+c^2}{(b+c)^2} \le \frac{27}{2(ab+bc+ca)^2},$$

which is equivalent to the homogeneous inequalities

$$\sum \left[\frac{b^2+c^2}{(b+c)^2}-1\right] \le \frac{p^4}{6q^2}-3,$$

$$\sum \frac{2bc}{(b+c)^2}+\frac{p^4}{6q^2}\ge 3,$$

where

$$p = a + b + c, \qquad q = ab + bc + ca.$$

According to P 1.62, the following inequality holds

$$\sum \frac{2bc}{(b+c)^2} + \frac{p^2}{q} \ge \frac{9}{2}.$$

Thus, it is enough to show that

$$\frac{9}{2} - \frac{p^2}{q} + \frac{p^4}{6q^2} \ge 3,$$

which is equivalent to

$$\left(\frac{p^2}{q} - 3\right)^2 \ge 0.$$

The equality holds for $a = b = c = 1$.

□

P 2.87. *If a, b, c are nonnegative real numbers, no two of which are zero, then*

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \ge \frac{1}{a+b+c} + \frac{2}{\sqrt{ab+bc+ca}}.$$

(Vasile Cîrtoaje, 2005)

Solution. Using the notation

$$p = a + b + c, \qquad q = ab + bc + ca, \qquad r = abc,$$

we can write the inequality as

$$\frac{p^2 + q}{pq - r} \ge \frac{1}{p} + \frac{2}{\sqrt{q}}.$$

According to P 3.57-(a) in Volume 1, for fixed p and q, the product $r = abc$ is minimum when two of a, b, c are equal or one of a, b, c is zero. Therefore, it suffices to prove the inequality for $b = c = 1$ and for $a = 0$. For $a = 0$, the inequality reduces to

$$\frac{1}{b} + \frac{1}{c} \ge \frac{2}{\sqrt{bc}},$$

which is obvious. For $b = c = 1$, the inequality becomes as follows:

$$\frac{1}{2} + \frac{2}{a+1} \ge \frac{1}{a+2} + \frac{2}{\sqrt{2a+1}},$$

$$\frac{1}{2} - \frac{1}{a+2} \ge \frac{2}{\sqrt{2a+1}} - \frac{2}{a+1},$$

$$\frac{a}{2(a+2)} \ge \frac{2\left(a+1-\sqrt{2a+1}\right)}{(a+1)\sqrt{2a+1}},$$

$$\frac{a}{2(a+2)} \ge \frac{2a^2}{(a+1)\sqrt{2a+1}\left(a+1+\sqrt{2a+1}\right)}.$$

So, we need to show that

$$\frac{1}{2(a+2)} \ge \frac{2a}{(a+1)\sqrt{2a+1}\left(a+1+\sqrt{2a+1}\right)}.$$

Consider two cases: $0 \le a \le 1$ and $a > 1$.

Case 1: $0 \le a \le 1$. Since

$$\sqrt{2a+1}(a+1+\sqrt{2a+1}) \ge \sqrt{2a+1}\left(\sqrt{2a+1}+\sqrt{2a+1}\right) = 2(2a+1),$$

it suffices to prove that

$$\frac{1}{2(a+2)} \ge \frac{a}{(a+1)(2a+1)},$$

which is equivalent to $1 - a \ge 0$.

Case 2: $a > 1$. Write the desired inequality as

$$\frac{1}{2(a+2)} \ge \frac{2a}{(a+1)\left[(a+1)\sqrt{2a+1}+2a+1\right]}.$$

First, we will show that

$$(a+1)\sqrt{2a+1} > 3a.$$

Indeed, by squaring, we get the obvious inequality

$$a^3 + a(a-2)^2 + 1 > 0.$$

Therefore, it suffices to show that

$$\frac{1}{2(a+2)} \ge \frac{2a}{(a+1)(3a+2a+1)},$$

which is equivalent to $(a-1)^2 \ge 0$.

The equality holds for $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 2.88. *If $a, b \ge 1$, then*

$$\frac{1}{\sqrt{3ab+1}} + \frac{1}{2} \ge \frac{1}{\sqrt{3a+1}} + \frac{1}{\sqrt{3b+1}}.$$

Solution. Using the substitution

$$x = \frac{2}{\sqrt{3a+1}}, \quad y = \frac{2}{\sqrt{3b+1}}, \qquad x, y \in (0, 1],$$

the desired inequality can be written as

$$xy\sqrt{\frac{3}{x^2y^2 - x^2 - y^2 + 4}} \geq x + y - 1.$$

Consider the nontrivial case $x + y - 1 \geq 0$, and denote

$$t = x + y - 1, \quad p = xy.$$

We have

$$1 \geq p \geq t \geq 0.$$

Since

$$x^2 + y^2 = (x+y)^2 - 2xy = (t+1)^2 - 2p,$$

we need to prove that

$$p\sqrt{\frac{3}{p^2 + 2p - t^2 - 2t + 3}} \geq t.$$

By squaring, we get the inequality

$$(p - t)[(3 - t^2)p + t(1 - t)(3 + t)] \geq 0,$$

which is clearly true. The equality holds for $a = b = 1$.

□

P 2.89. *Let a, b, c be positive real numbers such that $a + b + c = 3$. If $k \geq \frac{1}{\sqrt{2}}$, then*

$$(abc)^k(a^2 + b^2 + c^2) \leq 3.$$

(Vasile Cîrtoaje, 2006)

Solution. Since

$$abc \leq \left(\frac{a+b+c}{3}\right)^3 = 1,$$

it suffices to prove the desired inequality for $k = 1/\sqrt{2}$. Write the inequality in the homogeneous form

$$(abc)^k(a^2 + b^2 + c^2) \leq 3\left(\frac{a+b+c}{3}\right)^{3k+2}.$$

According to P 3.57-(a) in Volume 1, for fixed $a+b+c$ and $ab+bc+ca$, the product abc is maximum when two of a,b,c are equal. Therefore, it suffices to prove the homogeneous inequality for $b=c=1$; that is, $f(a)\ge 0$, where

$$f(a)=(3k+2)\ln(a+2)-(3k+1)\ln 3-k\ln a-\ln(a^2+2).$$

From

$$\begin{aligned}f'(a)=&\frac{3k+2}{a+2}-\frac{k}{a}-\frac{2a}{a^2+2}=\frac{2(a-1)(ka^2-2a+2k)}{a(a+2)(a^2+2)}\\=&\frac{\sqrt{2}(a-1)(a-\sqrt{2})^2}{a(a+2)(a^2+2)},\end{aligned}$$

it follows that f is decreasing on $(0,1]$ and increasing on $[1,\infty)$; therefore, $f(a)\ge f(1)=0$. This completes the proof. The equality holds for $a=b=c=1$.

□

P 2.90. *If $a,b,c\in[0,4]$ and $ab+bc+ca=4$, then*

$$\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}\le 3+\sqrt{5}.$$

(Vasile Cîrtoaje, 2019)

Solution. Assume that $a\ge b\ge c$, $1\le a\le 4$, and write the inequality as follows

$$\sqrt{b+c}+\sqrt{(a+b)+(a+c)+2\sqrt{(a+b)(a+c)}}\le 3+\sqrt{5},$$

$$\sqrt{b+c}+\sqrt{2a+b+c+2\sqrt{a^2+4}}\le 3+\sqrt{5},$$

From $4-a(b+c)=bc\ge 0$, we get

$$b+c\le\frac{4}{a}.$$

Thus, it suffices to show that

$$\frac{2}{\sqrt{a}}+\sqrt{2a+\frac{4}{a}+2\sqrt{a^2+4}}\le 3+\sqrt{5},$$

which is equivalent to

$$\frac{2}{\sqrt{a}}+\frac{a+\sqrt{a^2+4}}{\sqrt{a}}\le 3+\sqrt{5},$$

$$a-3\sqrt{a}+2\le\sqrt{5a}-\sqrt{a^2+4},$$

$$(\sqrt{a}-1)(\sqrt{a}-2)\le\frac{(a-1)(4-a)}{\sqrt{5a}+\sqrt{a^2+4}}.$$

This is true if

$$1 \le \frac{(\sqrt{a}+1)(\sqrt{a}+2)}{\sqrt{5a}+\sqrt{a^2+4}},$$

that can be written in the obvious form

$$(a+2-\sqrt{a^2+4}) + (3-\sqrt{5})\sqrt{a} \ge 0.$$

The equality occurs for $a = 4$, $b = 1$ and $c = 0$ (or any permutation).

□

P 2.91. *Let a, b, c be positive real numbers such that $a = \min\{a, b, c\}$ and $a^4bc \ge 1$, and let*

$$F(a,b,c) = \sqrt{\frac{a^2+b^2+c^2}{3}} - \frac{a+b+c}{3}.$$

Then,

$$F(a,b,c) \ge F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

(Vasile Cîrtoaje and Vasile Mircea Popa, 2020)

Solution. Write the inequality as $E(a,b,c) \ge 0$, where

$$E(a,b,c) = \sqrt{3(a^2+b^2+c^2)} - (a+b+c) - \sqrt{3\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)} + \frac{1}{a}+\frac{1}{b}+\frac{1}{c},$$

and show that

$$E(a,b,c) \ge E(a,x,x) \ge 0,$$

where

$$x = \sqrt{bc} \ge a, \quad a^2x \ge 1, \quad x \ge 1.$$

Write the inequality $E(a,b,c) \ge E(a,x,x)$ it in the form

$$A - C \ge B - D,$$

where

$$A = \sqrt{3(a^2+b^2+c^2)} - \sqrt{3(a^2+2x^2)} = \frac{3(b-c)^2}{\sqrt{3(a^2+b^2+c^2)}+\sqrt{3(a^2+2x^2)}}$$

$$\ge \frac{3(b-c)^2}{\sqrt{3(x^2+b^2+c^2)}+3x},$$

$$B = (a+b+c) - (a+2x) = \left(\sqrt{b}-\sqrt{c}\right)^2,$$

$$C = \sqrt{3\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)} - \sqrt{3\left(\frac{1}{a^2}+\frac{2}{x^2}\right)}$$

$$= \frac{3}{x^4}\cdot\frac{(b-c)^2}{\sqrt{3\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}+\sqrt{3\left(\frac{1}{a^2}+\frac{2}{x^2}\right)}} \le \frac{3}{x^4}\cdot\frac{(b-c)^2}{\sqrt{3\left(\frac{1}{x^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}+\frac{3}{x}}$$

$$= \frac{3}{x^2}\cdot\frac{(b-c)^2}{\sqrt{3\left(x^2+\frac{x^4}{b^2}+\frac{x^4}{c^2}\right)}+3x} = \frac{3}{x^2}\cdot\frac{(b-c)^2}{\sqrt{3(x^2+c^2+b^2)}+3x}\ ,$$

$$D = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{1}{a}-\frac{2}{x} = \frac{\left(\sqrt{b}-\sqrt{c}\right)^2}{x^2}\ .$$

Thus, we need to show that

$$3\left(\sqrt{b}+\sqrt{c}\right)^2\left[\frac{1}{\sqrt{3(x^2+b^2+c^2)}+3x} - \frac{1}{x^2}\cdot\frac{1}{\sqrt{3(x^2+c^2+b^2)}+3x}\right] \ge \frac{x^2-1}{x^2}.$$

This inequality is true if

$$\frac{3\left(\sqrt{b}+\sqrt{c}\right)^2}{\sqrt{3(x^2+b^2+c^2)}+3x} \ge 1,$$

that is equivalent to

$$\sqrt{3}\ \left(b+c+\sqrt{bc}\right) \ge \sqrt{bc+b^2+c^2},$$

which is true.

Write now the inequality $E(a,x,x) \ge 0$ in the form

$$\sqrt{3(a^2+2x^2)}-a-2x \ge \sqrt{3\left(\frac{1}{a^2}+\frac{2}{x^2}\right)}-\frac{1}{a}-\frac{2}{x}.$$

Since both sides of the inequality are nonnegative and $a^2x \ge 1$, it suffices to prove the homogeneous inequality

$$\sqrt{3(a^2+2x^2)}-a-2x \ge (a^2x)^{2/3}\left[\sqrt{3\left(\frac{1}{a^2}+\frac{2}{x^2}\right)}-\frac{1}{a}-\frac{2}{x}\right].$$

Due to homogeneity, we may set $x=1$. Thus, we need to show that $a \le x = 1$ yields

$$\sqrt{3(a^2+2)}-a-2 \ge a^{1/3}\left[\sqrt{3(1+2a^2)}-1-2a\right],$$

which is equivalent to

$$\frac{2(a-1)^2}{\sqrt{3(a^2+2)}+a+2} \ge a^{1/3}\frac{2(a-1)^2}{\sqrt{3(1+2a^2)}+1+2a}.$$

It is true if

$$\sqrt{3(1+2a^2)}+1+2a \ge a^{1/3}\left[\sqrt{3(a^2+2)}+a+2\right].$$

For $t=a^{1/3}$, $t\in(0,1]$, the inequality becomes

$$\sqrt{3(1+2t^6)}+1+2t^3 \ge \sqrt{3(t^8+2t^2)}+t^4+2t,$$

which is true because

$$1+2t^6-(t^8+2t^2)=(1-t^4)(1-t^2)^2\ge 0,$$

$$1+2t^3-(t^4+2t)=(1-t^2)(1-t)^2\ge 0.$$

The equality occurs for $a=b=c\ge 1$.

Remark. The inequality is true in the particular case $a,b,c\ge 1$ (which involves $a^4bc\ge 1$).

□

P 2.92. *Let a,b,c be positive real numbers such that $a=\min\{a,b,c\}$ and $a^2(b+c)\ge 2$, and let*

$$F(a,b,c)=\sqrt{\frac{a^2+b^2+c^2}{3}}-\frac{a+b+c}{3}.$$

Then,

$$F(a,b,c)\ge F\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right).$$

(Vasile Cîrtoaje, 2020)

Solution. The proof follows the same way as the proof of the preceding P 2.91. Assume that $a\le b\le c$, write the inequality as $E(a,b,c)\ge 0$, where

$$E(a,b,c)=\sqrt{a^2+b^2+c^2}-\frac{a+b+c}{\sqrt{3}}-\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}+\frac{1}{\sqrt{3}}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right),$$

and show that

$$E(a,b,c)\ge E(a,x,x)\ge 0,$$

where

$$x=\frac{b+c}{2}\ge b,\qquad a^2x\ge 1,\qquad x\ge 1.$$

Write the inequality $E(a,b,c)\ge E(a,x,x)$ it in the form

$$A+B\ge C,$$

where

$$A=\sqrt{a^2+b^2+c^2}-\sqrt{a^2+2x^2}$$

$$= \frac{(b-c)^2}{2} \cdot \frac{1}{\sqrt{a^2+b^2+c^2}+\sqrt{a^2+2x^2}} \geq \frac{(b-c)^2}{2} \cdot \frac{1}{\sqrt{2b^2+c^2}+\sqrt{b^2+2x^2}} \ ,$$

$$B = \frac{1}{\sqrt{3}}\left(\frac{1}{b}+\frac{1}{c}-\frac{2}{x}\right) = \frac{(b-c)^2}{\sqrt{3}bc(b+c)} \ ,$$

$$C = \sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}} - \sqrt{\frac{1}{a^2}+\frac{2}{x^2}}$$

$$= \frac{(b-c)^2(b^2+4bc+c^2)}{b^2c^2(b+c)^2} \cdot \frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}+\sqrt{\frac{1}{a^2}+\frac{2}{x^2}}}$$

$$\leq \frac{(b-c)^2(b^2+4bc+c^2)}{b^2c^2(b+c)^2} \cdot \frac{1}{\sqrt{\frac{2}{b^2}+\frac{1}{c^2}}+\sqrt{\frac{1}{b^2}+\frac{2}{x^2}}} \ .$$

Thus, we need to show that

$$\frac{1}{2} \cdot \frac{1}{\sqrt{2b^2+c^2}+\sqrt{b^2+2x^2}} + \frac{1}{\sqrt{3}bc(b+c)} \geq \frac{b^2+4bc+c^2}{b^2c^2(b+c)^2} \cdot \frac{1}{\sqrt{\frac{2}{b^2}+\frac{1}{c^2}}+\sqrt{\frac{1}{b^2}+\frac{2}{x^2}}}.$$

Since

$$b^2+4bc+c^2 = 4bc+(b^2+c^2),$$

it suffices to show that

$$\frac{1}{\sqrt{3}bc(b+c)} \geq \frac{4bc}{b^2c^2(b+c)^2} \cdot \frac{1}{\sqrt{\frac{2}{b^2}+\frac{1}{c^2}}+\sqrt{\frac{1}{b^2}+\frac{2}{x^2}}}$$

and

$$\frac{1}{2} \cdot \frac{1}{\sqrt{2b^2+c^2}+\sqrt{b^2+2x^2}} \geq \frac{b^2+c^2}{b^2c^2(b+c)^2} \cdot \frac{1}{\sqrt{\frac{2}{b^2}+\frac{1}{c^2}}+\sqrt{\frac{1}{b^2}+\frac{2}{x^2}}}.$$

Write the first inequality as

$$\sqrt{\frac{2}{b^2}+\frac{1}{c^2}}+\sqrt{\frac{1}{b^2}+\frac{2}{x^2}} \geq \frac{4\sqrt{3}}{b+c}.$$

Since

$$\sqrt{\frac{2}{b^2}+\frac{1}{c^2}}+\sqrt{\frac{1}{b^2}+\frac{2}{x^2}} \geq \frac{1}{\sqrt{3}}\left(\frac{2}{b}+\frac{1}{c}\right)+\frac{1}{\sqrt{3}}\left(\frac{1}{b}+\frac{2}{x}\right)$$

$$\geq \frac{1}{\sqrt{3}}\left(\frac{2}{b}+\frac{1}{c}\right)+\frac{1}{\sqrt{3}}\left(\frac{1}{c}+\frac{2}{x}\right) = \frac{2}{\sqrt{3}}\left(\frac{1}{b}+\frac{1}{c}+\frac{2}{b+c}\right)$$

$$\geq \frac{2}{\sqrt{3}}\left(\frac{4}{b+c}+\frac{2}{b+c}\right) = \frac{4\sqrt{3}}{b+c},$$

the inequality is proved.

The second inequality can be obtained by multiplying the inequalities

$$\frac{1}{2}\cdot\frac{1}{\sqrt{2b^2+c^2}+\sqrt{b^2+2x^2}} \geq \frac{1}{2bc}\cdot\frac{1}{\sqrt{\frac{2}{b^2}+\frac{1}{c^2}}+\sqrt{\frac{1}{b^2}+\frac{2}{x^2}}}$$

and

$$bc(b+c)^2 \geq 2(b^2+c^2).$$

Write the first inequality as

$$\sqrt{b^2+2c^2}+\sqrt{c^2+\frac{2b^2c^2}{x^2}} \geq \sqrt{2b^2+c^2}+\sqrt{b^2+2x^2}.$$

Since $\sqrt{b^2+2c^2} \geq \sqrt{2b^2+c^2}$, it is sufficient to prove that

$$c^2+\frac{2b^2c^2}{x^2} \geq b^2+2x^2,$$

that is

$$c^2-b^2 \geq \frac{2(x^2-bc)(x^2+bc)}{x^2}.$$

Since $x^2+bc \leq 2x^2$, it is true if

$$c^2-b^2 \geq 4(x^2-bc).$$

Indeed,

$$c^2-b^2-4(x^2-bc) = c^2-b^2-(c-b)^2 = 2b(c-b) \geq 0.$$

Since

$$\frac{b^2(b+c)}{2} \geq \frac{a^2(b+c)}{2} \geq 1$$

to prove the second inequality it suffices to show that

$$bc(b+c)^2 \geq 2(b^2+c^2)\left[\frac{b^2(b+c)}{2}\right]^{2/3}.$$

Due to homogeneity, we may set $b=1$, hence $c \geq 1$. The inequality becomes

$$c(c+1)^2 \geq 2(c^2+1)\left(\frac{c+1}{2}\right)^{2/3}.$$

It is true if

$$c^3(c+1)^4 \geq 2(c^2+1)^3,$$

that is

$$c^7+2c^6+6c^5-2c^4+c^3-6c^2-2 \geq 0,$$

$$(c^7+c^3-2)+2c^4(c^2-1)+6c^2(c^3-1) \geq 0.$$

To complete the proof, we need to show that $E(a,x,x) \ge 0$ for $a^2x \ge 1$, $x \ge a$. This inequality was proved at the preceding P 2.91.

The equality occurs for $a = b = c \ge 1$.

Remark. Since $a^4bc \ge 1$ yields $a^2(b+c) \ge 2$, the inequality in P 2.91 follows from the inequality in P 2.92.

□

P 2.93. *Let a,b,c be positive real numbers such that $a = \min\{a,b,c\}$ and $a^4(b^2+c^2) \ge 2$, and let*

$$F(a,b,c) = \sqrt{\frac{a^2+b^2+c^2}{3}} - \frac{a+b+c}{3}.$$

Then,

$$F(a,b,c) \ge F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

(Vasile Cîrtoaje, 2020)

Solution. The proof follows the same way as the proof of P 2.92. Assume that $a \le b \le c$, write the inequality as $E(a,b,c) \ge 0$, where

$$E(a,b,c) = \sqrt{a^2+b^2+c^2} - \frac{a+b+c}{\sqrt{3}} - \sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}} + \frac{1}{\sqrt{3}}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right),$$

and show that

$$E(a,b,c) \ge E(a,x,x) \ge 0,$$

where

$$x = \sqrt{\frac{b^2+c^2}{2}} \ge b, \qquad a^2x \ge 1, \qquad x \ge 1.$$

Write the inequality $E(a,b,c) \ge E(a,x,x)$ it in the form

$$A + B \ge C,$$

where

$$A = \frac{2x-b-c}{\sqrt{3}} = \frac{(b-c)^2}{\sqrt{3}\,(2x+b+c)}\ ,$$

$$B = \frac{1}{\sqrt{3}}\left(\frac{1}{b}+\frac{1}{c}-\frac{2}{x}\right) = \frac{(b-c)^2(b^2+c^2+4bc)}{2\sqrt{3}\,b^2c^2x^2\left(\frac{1}{b}+\frac{1}{c}+\frac{2}{x}\right)}\ ,$$

$$C = \sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}} - \sqrt{\frac{1}{a^2}+\frac{2}{x^2}}$$

$$= \frac{(b^2-c^2)^2}{2b^2c^2x^2} \cdot \frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}} + \sqrt{\frac{1}{a^2}+\frac{2}{x^2}}} \le \frac{\sqrt{3}(b^2-c^2)^2}{2b^2c^2x^2} \cdot \frac{1}{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+\left(\frac{1}{a}+\frac{2}{x}\right)}$$

$$\leq \frac{\sqrt{3}(b^2-c^2)^2}{2b^2c^2x^2} \cdot \frac{1}{\frac{3}{b}+\frac{1}{c}+\frac{2}{x}} \, .$$

Thus, we need to show that

$$\frac{1}{2x+b+c} + \frac{b^2+c^2+4bc}{2b^2c^2x^2\left(\frac{1}{b}+\frac{1}{c}+\frac{2}{x}\right)} \geq \frac{3(b+c)^2}{2b^2c^2x^2} \cdot \frac{1}{\frac{3}{b}+\frac{1}{c}+\frac{2}{x}} \, .$$

Since

$$b^2x \geq a^2x \geq 1,$$

it suffices to prove the homogeneous inequality

$$\frac{1}{(b^2x)^{2/3}(2x+b+c)} + \frac{b^2+c^2+4bc}{2b^2c^2x^2\left(\frac{1}{b}+\frac{1}{c}+\frac{2}{x}\right)} \geq \frac{3(b+c)^2}{2b^2c^2x^2} \cdot \frac{1}{\frac{3}{b}+\frac{1}{c}+\frac{2}{x}} \, .$$

Since

$$2\left(\frac{3}{b}+\frac{1}{c}+\frac{2}{x}\right) - 3\left(\frac{1}{b}+\frac{1}{c}+\frac{2}{x}\right) = \frac{3}{b} - \frac{2}{c} - \frac{2}{x} \geq 0,$$

it is enough to show that

$$\frac{1}{(b^2x)^{2/3}(2x+b+c)} + \frac{b^2+c^2+4bc}{2b^2c^2x^2\left(\frac{1}{b}+\frac{1}{c}+\frac{2}{x}\right)} \geq \frac{2(b+c)^2}{2b^2c^2x^2} \cdot \frac{1}{\frac{1}{b}+\frac{1}{c}+\frac{2}{x}} \, ,$$

that is

$$\frac{1}{(b^2x)^{2/3}(2x+b+c)} \geq \frac{1}{b^2c^2} \cdot \frac{1}{\frac{1}{b}+\frac{1}{c}+\frac{2}{x}} \, ,$$

$$c\left(b+c+\frac{2bc}{x}\right) \geq b^{1/3}\left[2x^{5/3}+(b+c)x^{2/3}\right].$$

Since $x \leq c$, it suffices to show that

$$c\left(b+c+\frac{2bc}{c}\right) \geq b^{1/3}\left[2cx^{2/3}+(b+c)x^{2/3}\right],$$

that is

$$c(3b+c) \geq (b+3c)(bx^2)^{1/3}.$$

Due to homogeneity, we may set $c = 1$, when $0 < b \leq 1$ and

$$x = \sqrt{\frac{b^2+1}{2}}.$$

Thus, we need to show that

$$3b+1 \geq (b+3)\sqrt[3]{\frac{b^3+b}{2}},$$

which is true if

$$2(3b+1)^3 \geq b(b^2+1)(b+3)^3.$$

Since

$$(b+3)^3 = b^3 + 39b^2 + 27b + 27 \le 37b + 27 \le 32(b+1),$$

it suffices to sow that

$$(3b+1)^3 \ge 16(b^2+1)(b+1),$$

which is equivalent to

$$1 - 7b + 11b^2 + 11b^3 - 16b^4 \ge 0,$$

$$(1-b)(1-6b+5b^2+16b^3) \ge 0.$$

This is true because

$$1 - 6b + 5b^2 + 16b^3 = (1-4b)^2 + b(2 - 11b + 16b^2) > 0.$$

To complete the proof, we need to show that $E(a,x,x) \ge 0$ for $a^2x \ge 1$, $x \ge a$. This inequality was proved at P 2.91.

The equality occurs for $a = b = c \ge 1$.

Remark. Since $a^2(b+c) \ge 2$ yields $a^4(b^2+c^2) \ge 2$, the inequality in P 2.92 follows from the inequality in P 2.93.

□

P 2.94. *Let a, b, c be positive real numbers such that $a = \max\{a,b,c\}$ and $a^4b^7c^7 \ge 1$, and let*

$$F(a,b,c) = \sqrt[3]{abc} - \frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}.$$

Then,

$$F(a,b,c) \ge F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

(Vasile Cîrtoaje and Vasile Mircea Popa, 2019)

Solution. By the AM-GM inequality, both sides of the inequality are nonnegative. Denote

$$x = \sqrt{bc}.$$

We have

$$a \ge 1, \quad x \le a, \quad a^2x^7 \ge 1.$$

From

$$x \ge \frac{1}{a^{2/7}} \ge \frac{1}{a^{1/2}},$$

it follows that

$$a \ge \frac{1}{x^2}.$$

Write the inequality as $E(a,b,c) \geq 0$, where

$$E(a,b,c) = \sqrt[3]{abc} - \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} - \frac{1}{\sqrt[3]{abc}} + \frac{3}{a+b+c},$$

and prove that

$$E(a,b,c) \geq E(a,x,x) \geq 0.$$

We will show that the left inequality is true for $a \geq 1$ and $a \geq \frac{1}{x^2}$. Write the inequality as follows

$$\frac{1}{a+b+c} - \frac{1}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} \geq \frac{1}{a+2\sqrt{bc}} - \frac{1}{\frac{1}{a}+\frac{2}{\sqrt{bc}}} \ ,$$

$$\frac{1}{\frac{1}{a}+\frac{2}{\sqrt{bc}}} - \frac{1}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} \geq \frac{1}{a+2\sqrt{bc}} - \frac{1}{a+b+c} \ ,$$

$$\frac{\left(\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{c}}\right)^2}{\left(\frac{1}{a}+\frac{2}{\sqrt{bc}}\right)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} \geq \frac{(\sqrt{b}-\sqrt{c})^2}{(a+2\sqrt{bc})(a+b+c)} \ .$$

After dividing by $(\sqrt{b}-\sqrt{c})^2$, we need to show that

$$(a+2x)(a+b+c) \geq x^2\left(\frac{1}{a}+\frac{2}{x}\right)\left(\frac{1}{a}+\frac{b+c}{x^2}\right). \tag{*}$$

Write this inequality as

$$A(b+c)+B \geq 0,$$

where

$$A = a+2x-\frac{1}{a}-\frac{2}{x}\ , \qquad B = a^2+2ax-\frac{x^2}{a^2}-\frac{2x}{a}\ .$$

Clearly, $A \geq 0$ for $x \geq 1$. Also, $A \geq 0$ for $x \leq 1$, because

$$A \geq \frac{1}{x^2}+2x-x^2-\frac{2}{x} = \frac{(1-x)^3(1+x)}{x^2} \geq 0\ .$$

Since $A \geq 0$ and $b+c \geq 2\sqrt{bc}$, it suffices to replace $b+c$ in (*) with $2x$. So, we need to show that

$$(a+2x)(a+2x) \geq x^2\left(\frac{1}{a}+\frac{2}{x}\right)\left(\frac{1}{a}+\frac{2}{x}\right)\ ,$$

which is equivalent to

$$a+2x \geq x\left(\frac{1}{a}+\frac{2}{x}\right),$$

$$a+2x \geq \frac{x}{a}+2.$$

For $x \geq 1$, we have

$$a+2x-\frac{x}{a}-2 = a-2+\left(2-\frac{1}{a}\right)x \geq a-2+\left(2-\frac{1}{a}\right) = a-\frac{1}{a} \geq 0\ ,$$

and for $x \le 1$, we have

$$a+2x-\frac{x}{a}-2 \ge \frac{1}{x^2}+2x-x^3-2=\frac{(1-x)(1+x-x^2+x^3+x^4)}{x^2} \ge 0 \ .$$

Write the right inequality $E(a,x,x) \ge 0$, as follows

$$\sqrt[3]{ax^2}-\frac{3ax}{2a+x} \ge \frac{1}{\sqrt[3]{ax^2}}-\frac{3}{a+2x} \ .$$

Since $a^{4/7}x^2 \ge 1$, it suffices to prove the homogeneous inequality

$$\sqrt[3]{ax^2}-\frac{3ax}{2a+x} \ge \left(a^{4/7}x^2\right)^{7/9}\left(\frac{1}{\sqrt[3]{ax^2}}-\frac{3}{a+2x}\right) \ .$$

Setting $x=1$ and substituting

$$a=d^9, \qquad d \ge 1,$$

the inequality becomes

$$d^3-\frac{3d^9}{2d^9+1} \ge d^4\left(\frac{1}{d^3}-\frac{3}{d^9+2}\right) \ ,$$

$$\frac{d^2(d^3-1)^2(2d^3+1)}{2d^9+1} \ge \frac{(d^3-1)^2(d^3+2)}{d^9+2} \ .$$

Thus, we need to show that

$$d^2(2d^3+1)(d^9+2) \ge (d^3+2)(2d^9+1) \ ,$$

that is

$$2(d^{12}+1)(d^2-1)+d^3(d^8-1)-4d^5(d^4-1) \ge 0 \ ,$$

$$(d^2-1)A \ge 0,$$

where

$$\begin{aligned} A &= 2(d^{12}+1)+d^3(d^6+d^4+d^2+1)-4d^5(d^2+1) \\ &= 2d^7(d^5-1)+d^7(d^2-1)-3d^3(d^2-1)-2(d^3-1) \\ &\ge 2d(d^5-1)+(d^2-1)-3d^3(d^2-1)-2(d^3-1)=(d-1)B \ , \end{aligned}$$

where

$$\begin{aligned} B &= 2d(d^4+d^3+d^2+d+1)+(d+1)-3d^3(d+1)-2(d^2+d+1) \\ &= 2d^5-d^4-d^3+d-1=(d-1)(2d^4+d^3+1) \ge 0 \ . \end{aligned}$$

The equality holds for $a=b=c \ge 1$.

Remark. The inequality is true in the particular case $a,b,c \ge 1$ (which involves $a^4b^7c^7 \ge 1$). In addition, if

$$F(a_1,a_2,\ldots,a_n) = \sqrt[n]{a_1a_2\cdots a_n}-\frac{n}{\dfrac{1}{a_1}+\dfrac{1}{a_2}+\cdots+\dfrac{1}{a_n}}$$

and all numbers $a_1, a_2, \ldots, a_n$ are larger than or equal to 1, then the inequality

$$F(a_1, a_2, \ldots, a_n) \ge F\left(\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}\right)$$

holds for $n \le 5$ and does not hold for $n \ge 6$ (problem P 12196 from AMM, 7, 2020, solved in AMM, 4, 2022).

□

P 2.95. *Let a, b, c be positive real numbers such that $a = \min\{a, b, c\}$ and $a^4bc \ge 1$, and let*

$$F(a, b, c) = \sqrt[3]{abc} - \frac{3}{\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}}.$$

Prove that

$$F(a, b, c) \ge F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

(Vasile Cîrtoaje, RMM, 39, 2025)

Solution. Since $F(a, b, c) \ge 0$ and $F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right) \ge 0$ (by the AM-GM inequality), it suffices to prove the homogeneous inequality

$$F(a, b, c) \ge (a^4bc)^{1/3} \cdot F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$$

for $a = \min\{a, b, c\}$. Due to homogeneity, we may set $a = 1$, hence $b, c \ge 1$. Thus, we need to show that

$$(bc)^{1/3} - \frac{3bc}{b + c + bc} \ge (bc)^{1/3} \cdot \left[\frac{1}{(bc)^{1/3}} - \frac{3}{1 + b + c}\right].$$

Denote

$$s = \frac{b + c}{2}, \qquad p = \sqrt{bc},$$

with $s \ge p \ge 1$. The desired inequality is equivalent to

$$p^{2/3} - \frac{3p^2}{2s + p^2} \ge 1 - \frac{3p^{2/3}}{2s + 1},$$

$$p^{2/3} - 1 \ge 3p^{2/3}\left(\frac{p^{4/3}}{2s + p^2} - \frac{1}{2s + 1}\right).$$

$$\frac{p^{2/3} - 1}{3p^{2/3}} \ge \frac{2s(p^{4/3} - 1) - p^{4/3}(p^{2/3} - 1)}{(2s + p^2)(2s + 1)}.$$

It is true if

$$\frac{1}{3p^{2/3}} \ge \frac{2s(p^{2/3}+1)-p^{4/3}}{(2s+p^2)(2s+1)},$$

i.e.

$$4s^2 - 2As + 4p^2 \ge 0, \qquad A = 3p^{4/3} + 3p^{2/3} - p^2 - 1.$$

For the nontrivial case $A \ge 0$, since

$$4(4s^2 - 2As + 4p^2) = (4s - A)^2 + 16p^2 - A^2 \ge 16p^2 - A^2 = (4p - A)(4p + A),$$

it suffices to show that $4p - A \ge 0$, which is equivalent to

$$p^2 - 3p^{4/3} + 4p - 3p^{2/3} + 1 \ge 0.$$

Denoting $p = x^3$, we need to show that

$$x^6 - 3x^4 + 4x^3 - 3x^2 + 1 \ge 0,$$

that is

$$(x-1)^2(x^4 + 2x^3 + 2x + 1) \ge 0.$$

The proof is completed. The equality occurs for $a = b = c \ge 1$.

Remark. The inequality $F(a,b,c) \le F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$ is true in the particular case $a, b, c \ge 1$ (which involves $a^4bc \ge 1$).

□

P 2.96. *Let a, b, c be positive real numbers such that $a = \min\{a, b, c\}$ and $a^4bc \ge 1$, and let*

$$F(a,b,c) = \frac{a+b+c}{3} - \sqrt{\frac{ab+bc+ca}{3}}.$$

Then,

$$F(a,b,c) \ge F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

(Vasile Mircea Popa and Vasile Cîrtoaje, Math. Reflections, 5, 2023)

Solution. Since $F(a,b,c) \ge 0$ and $F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right) \ge 0$, it suffices to prove the homogeneous inequality

$$F(a,b,c) \ge (a^4bc)^{1/3} \cdot F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$$

for $a = \min\{a, b, c\}$. Due to homogeneity, we may set $a = 1$, hence $b, c \ge 1$. Thus, we need to show the inequality

$$F(1, b, c) \ge (bc)^{1/3} \cdot F\left(1, \frac{1}{b}, \frac{1}{c}\right),$$

which is equivalent to

$$\frac{1+b+c}{3} - \sqrt{\frac{b+c+bc}{3}} \ge (bc)^{1/3}\left(\frac{b+c+bc}{3bc} - \sqrt{\frac{1+b+c}{3bc}}\right).$$

Denote

$$s = \frac{b+c}{2}, \qquad p = \sqrt{bc},$$

with $s \ge p \ge 1$. For fixed p, the desired inequality is equivalent to $F(s) \ge 0$, where

$$F(s) = 2s + 1 - \sqrt{3(2s+p^2)} - p^{-4/3}\left[2s + p^2 - p\sqrt{3(2s+1)}\right].$$

We have

$$F'(s) = A - p^{-4/3}B,$$

where

$$A = 2 - \sqrt{\frac{3}{2s+p^2}}, \qquad B = 2 - p\sqrt{\frac{3}{2s+1}}.$$

We will show that $F'(s) \ge 0$. Since $A \ge 2 - 1 > 0$, it suffices to consider the case $B > 0$, when

$$F'(s) \ge A - B = p\sqrt{\frac{3}{2s+1}} - \sqrt{\frac{3}{2s+p^2}} \ge p\sqrt{\frac{3}{2s+1}} - \sqrt{\frac{3}{2s+1}} \ge 0.$$

From $F'(s) \ge 0$, it follows that $F(s)$ is increasing, hence $F(s) \ge F(p)$. So, we need to show that $F(p) \ge 0$, i.e.

$$2p + 1 - \sqrt{3(p^2+2p)} - p^{-1/3}\left[2 + p - \sqrt{3(2p+1)}\right] \ge 0,$$

$$\frac{(p-1)^2}{2p+1+\sqrt{3(p^2+2p)}} - \frac{p^{-1/3}(p-1)^2}{2+p+\sqrt{3(2p+1)}} \ge 0.$$

It is true if

$$\frac{p^{1/3}}{2p+1+\sqrt{3(p^2+2p)}} - \frac{1}{2+p+\sqrt{3(2p+1)}} \ge 0.$$

Substituting $p = x^3$, where $x \ge 1$, we need to prove that

$$\frac{x}{2x^3+1+\sqrt{3(x^6+2x^3)}} - \frac{1}{2+x^3+\sqrt{3(2x^3+1)}} \ge 0,$$

i.e.

$$x^4 - 2x^3 + 2x - 1 \ge \sqrt{3}x\left(\sqrt{x^4+2x} - \sqrt{2x^3+1}\right),$$

$$(x-1)^3(x+1) \ge \frac{\sqrt{3}x(x-1)^3(x+1)}{\sqrt{x^4+2x}+\sqrt{2x^3+1}}.$$

Thus, we need to show that

$$\sqrt{x^4+2x}+\sqrt{2x^3+1} \ge \sqrt{3}x.$$

Indeed,

$$\sqrt{x^4+2x}+\sqrt{2x^3+1}-\sqrt{3}x > \sqrt{2x^3+1}-\sqrt{3}x = \frac{(x-1)^2(2x+1)}{\sqrt{2x^3+1}+\sqrt{3}x} \ge 0.$$

The equality occurs for $a=b=c \ge 1$.

Remark. The inequality is true in the particular case $a,b,c \ge 1$ (which involves $a^4bc \ge 1$).

□

P 2.97. *Let a,b,c be positive real numbers such that $a = \max\{a,b,c\}$ and $a^2b^5c^5 \ge 1$, and let*

$$F(a,b,c) = \sqrt{\frac{ab+bc+ca}{3}} - \sqrt[3]{abc}.$$

Prove that

$$F(a,b,c) \ge F\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right).$$

(Vasile Cîrtoaje and Vasile Mircea Popa, RMM, 37, 2025)

Solution. Since $F(a,b,c) \ge 0$ and $F\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right) \ge 0$, it suffices to prove the homogeneous inequality

$$F(a,b,c) \ge (a^2b^5c^5)^{1/6} \cdot F\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right)$$

for $a = \max\{a,b,c\}$. Due to homogeneity, we may set $a=1$, hence $b,c \le 1$. Thus, we need to show that

$$F(1,b,c) \ge (bc)^{5/6} \cdot F\left(1,\frac{1}{b},\frac{1}{c}\right),$$

which is equivalent to

$$\sqrt{\frac{a+b+ab}{3}} - \sqrt[3]{bc} \ge (bc)^{5/6}\left[\sqrt{\frac{b+c+1}{bc}} - \frac{1}{\sqrt[3]{bc}}\right].$$

Denote

$$s = \frac{b+c}{2}, \qquad p = \sqrt{bc},$$

with $p \le s \le 1$. For fixed p, the desired inequality is equivalent to $F(s) \ge 0$, where

$$F(s) = \sqrt{2s+p^2} - p^{2/3}\sqrt{3} - p^{2/3}\left(\sqrt{2s+1} - p^{1/3}\sqrt{3}\right).$$

Since

$$F'(s) = \frac{1}{\sqrt{2s+p^2}} - \frac{p^{2/3}}{\sqrt{2s+1}} = \frac{\sqrt{2s+1} - p^{2/3}\sqrt{2s+p^2}}{\sqrt{(2s+p^2)(2s+1)}} \ge 0,$$

$F(s)$ is increasing, hence $F(s) \ge F(p)$. So, we need to prove that $F(p) \ge 0$, i.e.

$$\sqrt{2p+p^2} - p^{2/3}\sqrt{3} \ge p^{2/3}\left(\sqrt{2p+1} - p^{1/3}\sqrt{3}\right).$$

Putting $p = x^3$, where $x \le 1$,the inequality becomes

$$\sqrt{x^3+2} - \sqrt{3x} \ge \sqrt{x}\left(\sqrt{2x^3+1} - x\sqrt{3}\right).$$

$$\frac{(x+2)(x-1)^2}{\sqrt{x^3+2}+\sqrt{3x}} \ge \frac{\sqrt{x}(2x+1)(x-1)^2}{\sqrt{2x^3+1}+x\sqrt{3}}.$$

It is true if

$$\frac{x+2}{\sqrt{x^3+2}+\sqrt{3x}} \ge \frac{\sqrt{x}(2x+1)}{\sqrt{2x^3+1}+x\sqrt{3}},$$

which is equivalent to

$$x\sqrt{3}(1-x) + (x+2)\sqrt{2x^3+1} - (2x+1)\sqrt{x^4+2x} \ge 0.$$

It suffices to show that

$$(x+2)\sqrt{2x^3+1} \ge (2x+1)\sqrt{x^4+2x}.$$

It is true since

$$x+2-(2x+1) = 1-x \ge 0$$

and

$$2x^3+1-(x^4+2x) = (1-x)^3(1+x) \ge 0.$$

The proof is completed. The equality occurs for $a = b = c \ge 1$.

Remark. The inequality is true in the particular case $a, b, c \ge 1$ (which involves $a^2b^5c^5 \ge 1$).

□

P 2.98. *Let a, b, c be positive real numbers such that $\min\{ab, bc, ca\} \ge 1$, and let*

$$F(a,b,c) = \sqrt{\frac{3}{ab+bc+ca}} - \frac{3}{a+b+c}.$$

Prove that

$$F(a,b,c) \le F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

(Vasile Cîrtoaje and Vasile Mircea Popa, Crux Mathematicorum, 8, 2024)

Solution. Assume $a = \max\{a, b, c\}$. Since $bc \ge 1$ and $F(a, b, c) \ge 0$, it suffices to prove the homogeneous inequality

$$bc \cdot F(a, b, c) \le F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$$

for $a = \max\{a, b, c\}$. Due to homogeneity, we may set $a = 1$, hence $b, c \le 1$. Thus, we need to show that

$$bc \cdot F(1, b, c) \le F\left(1, \frac{1}{b}, \frac{1}{c}\right),$$

which is equivalent to

$$bc\left(\sqrt{\frac{3}{b+c+bc}} - \frac{3}{1+b+c}\right) \le \sqrt{\frac{3bc}{1+b+c}} - \frac{3bc}{b+c+bc}.$$

Denote

$$s = \frac{b+c}{2}, \qquad p = \sqrt{bc},$$

with $1 \ge s \ge p$. The desired inequality is equivalent to

$$p\left(\sqrt{\frac{3}{2s+p^2}} - \frac{3}{2s+1}\right) \le \sqrt{\frac{3}{2s+1}} - \frac{3p}{2s+p^2},$$

$$3p\left(\frac{1}{2s+p^2} - \frac{1}{2s+1}\right) \le \sqrt{3}\left(\frac{1}{\sqrt{2s+1}} - \frac{p}{\sqrt{2s+p^2}}\right),$$

$$\frac{\sqrt{3}p(1-p^2)}{(2s+1)(2s+p^2)} \le \frac{2s(1-p^2)}{\sqrt{(2s+1)(2s+p^2)}\left(\sqrt{2s+p^2} + p\sqrt{2s+1}\right)}.$$

It is true if

$$\sqrt{3}p\left(\sqrt{2s+p^2} + p\sqrt{2s+1}\right) \le 2s\sqrt{(2s+1)(2s+p^2)},$$

that is equivalent to

$$p\left(\sqrt{\frac{3}{2s+1}} + p\sqrt{\frac{3}{2s+p^2}}\right) \le 2s.$$

Since $s \ge p$, it suffices to prove the inequality for $s = p$, i.e.

$$\sqrt{\frac{3}{2p+1}} + \sqrt{\frac{3p}{p+2}} \le 2.$$

Putting

$$t = \sqrt{\frac{3}{2p+1}} < \sqrt{3}, \qquad p = \frac{3-t^2}{2t^2},$$

the inequality becomes

$$t + \sqrt{\frac{3-t^2}{1+t^2}} \le 2.$$

It is true if

$$\frac{3-t^2}{1+t^2} \le (2-t)^2,$$

which is equivalent to

$$(t-1)^4 \ge 0.$$

The proof is completed. The equality occurs for $a = b = c \ge 1$.

Remark. The inequality is true in the particular case $a, b, c \ge 1$ (which involves $\min\{ab, bc, ca\} \ge 1$).

□

P 2.99. *Let a, b, c, d be positive real numbers such that $ab \ge 1$ and $cd \ge 1$, and let*

$$F(a,b,c,d) = \sqrt[4]{abcd} - \frac{4}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}}.$$

Then,

$$F(a,b,c,d) \ge F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}\right).$$

(Vasile Cîrtoaje, 2019)

Solution. Write the inequality as $E(a,b,c,d) \ge 0$, where

$$E(a,b,c,d) = \sqrt[4]{abcd} - \frac{4}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}} - \frac{1}{\sqrt[4]{abcd}} + \frac{4}{a+b+c+d},$$

assume that

$$ab \ge cd \ge 1,$$

and show that

$$E(a,b,c,d) \ge E(a,b,\sqrt{cd},\sqrt{cd}) \ge E(\sqrt{ab},\sqrt{ab},\sqrt{cd},\sqrt{cd}) \ge 0.$$

Since

$$1 - \frac{\sqrt{cd}}{ab} \ge 1 - \frac{cd}{ab} \ge 0$$

and

$$\sqrt{cd} - 1 \ge 0,$$

the left inequality $E(a,b,c,d) \ge E(a,b,\sqrt{cd},\sqrt{cd})$ follows from Lemma below, point (a). The inequality $E(a,b,\sqrt{cd},\sqrt{cd}) \ge E(\sqrt{ab},\sqrt{ab},\sqrt{cd},\sqrt{cd})$ follows also from Lemma below by replacing c and d with $\sqrt{cd}$. We only need to show that

$$(\sqrt{cd}+\sqrt{cd})\left(1 - \frac{\sqrt{ab}}{cd}\right) + 2(\sqrt{ab}-1) \ge 0,$$

which is equivalent to the obvious inequality

$$(\sqrt{cd}-1)\left(\sqrt{\frac{ab}{cd}}+1\right)\geq 0.$$

The inequality $E(\sqrt{ab},\sqrt{ab},\sqrt{cd},\sqrt{cd})\geq 0$, is true if the inequality $E(a,b,c,d)\geq 0$ holds for $a=b=x^2$ and $c=d=y^2$, where $x\geq 1$, $y\geq 1$. We need to show that

$$xy-\frac{4}{\frac{2}{x^2}+\frac{2}{y^2}}\geq\frac{1}{xy}-\frac{4}{2x^2+2y^2},$$

that is

$$(x^2y^2-1)(x-y)^2\geq 0.$$

This completes the proof. The equality holds for $a=b=c=d\geq 1$, and for $ab=cd=1$.

Lemma. *Let*

$$E(a,b,c,d)=\sqrt[4]{abcd}-\frac{4}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}}-\frac{1}{\sqrt[4]{abcd}}+\frac{4}{a+b+c+d},$$

where a,b,c,d are positive real numbers such that $ab\geq 1$ and $cd\geq 1$.

(a) *If*

$$(a+b)\left(1-\frac{\sqrt{cd}}{ab}\right)+2(\sqrt{cd}-1)\geq 0,$$

then

$$E(a,b,c,d)\geq E(a,b,\sqrt{cd},\sqrt{cd}).$$

(b) *If*

$$(c+d)\left(1-\frac{\sqrt{ab}}{cd}\right)+2(\sqrt{ab}-1)\geq 0,$$

then

$$E(a,b,c,d)\geq E(\sqrt{ab},\sqrt{ab},c,d).$$

Proof. (a) Write the inequality $E(a,b,c,d)\geq E(a,b,\sqrt{cd},\sqrt{cd})$ as follows:

$$\frac{1}{a+b+c+d}-\frac{1}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}}\geq\frac{1}{a+b+2\sqrt{cd}}-\frac{1}{\frac{1}{a}+\frac{1}{b}+\frac{2}{\sqrt{cd}}},$$

$$\frac{1}{\frac{1}{a}+\frac{1}{b}+\frac{2}{\sqrt{cd}}}-\frac{1}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}}\geq\frac{1}{a+b+2\sqrt{cd}}-\frac{1}{a+b+c+d},$$

$$\frac{(\sqrt{c}-\sqrt{d})^2}{cd\left(\frac{1}{a}+\frac{1}{b}+\frac{2}{\sqrt{cd}}\right)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)}\geq\frac{(\sqrt{c}-\sqrt{d})^2}{(a+b+2\sqrt{cd})(a+b+c+d)},$$

After dividing by $(\sqrt{c}-\sqrt{d})^2$, we need to show that

$$(a+b+2\sqrt{cd})(a+b+c+d) \ge \left(\frac{a+b}{ab}cd+2\sqrt{cd}\right)\left(\frac{a+b}{ab}+\frac{c+d}{cd}\right), \tag{*}$$

that is

$$A(c+d)+B \ge 0,$$

where

$$A = a+b+\sqrt{cd}-\frac{a+b}{ab}-\frac{2}{\sqrt{cd}} = (a+b)\left(1-\frac{1}{ab}\right)+2\left(\sqrt{cd}-\frac{1}{\sqrt{cd}}\right) \ge 0,$$

$$B = (a+b)\left[\frac{a+b}{a^2b^2}cd+\frac{2\sqrt{cd}}{ab}-a-b-2\sqrt{cd}\right].$$

Since

$$A(c+d)+B \ge 2A\sqrt{cd}+b,$$

we need to show that $2A\sqrt{cd}+b \ge 0$. This is equivalent to (*) if the sum $c+d$ is replaced by $2\sqrt{cd}$:

$$(a+b+2\sqrt{cd})(a+b+2\sqrt{cd}) \ge \left(\frac{a+b}{ab}cd+2\sqrt{cd}\right)\left(\frac{a+b}{ab}+\frac{2\sqrt{cd}}{cd}\right),$$

that is

$$(a+b+2\sqrt{cd})^2 \ge \left(\frac{a+b}{ab}cd+2\sqrt{cd}\right)^2,$$

$$a+b+2\sqrt{cd} \ge \frac{a+b}{ab}cd+2\sqrt{cd},$$

$$(a+b)\left(1-\frac{\sqrt{cd}}{ab}\right)+2(\sqrt{cd}-1) \ge 0.$$

The last inequality is true by hypothesis.

(b) Due to symmetry, this follows from (a).

Remark. The inequality is true in the particular case $a,b,c,d \ge 1$ (which involves $ab \ge 1$ and $cd \ge 1$).

□

P 2.100. *Let a, b, c, d be positive real numbers such that $a^2+b^2+c^2+d^2=1$. Prove that*

$$\sqrt{1-a}+\sqrt{1-b}+\sqrt{1-c}+\sqrt{1-d} \ge \sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}.$$

(Vasile Cîrtoaje, 2007)

First Solution. We can obtain the desired inequality by summing the inequalities

$$\sqrt{1-a}+\sqrt{1-b} \ge \sqrt{c}+\sqrt{d},$$

$$\sqrt{1-c}+\sqrt{1-d} \ge \sqrt{a}+\sqrt{b}.$$

Since

$$\sqrt{1-a}+\sqrt{1-b} \ge 2\sqrt[4]{(1-a)(1-b)}$$

and

$$\sqrt{c}+\sqrt{d} \le 2\sqrt{\frac{c+d}{2}} \le 2\sqrt[4]{\frac{c^2+d^2}{2}},$$

the former inequality holds if

$$(1-a)(1-b) \ge \frac{c^2+d^2}{2}.$$

Indeed,

$$2(1-a)(1-b)-c^2-d^2 = 2(1-a)(1-b)+a^2+b^2-1 = (a+b-1)^2 \ge 0.$$

Similarly, we can prove the second inequality. The equality holds for

$$a=b=c=d=\frac{1}{2}.$$

Second Solution. We can obtain the desired inequality by summing the inequalities

$$\sqrt{1-a}-\sqrt{a} \ge \frac{1}{2\sqrt{2}}(1-4a^2), \qquad \sqrt{1-b}-\sqrt{b} \ge \frac{1}{2\sqrt{2}}(1-4b^2),$$

$$\sqrt{1-c}-\sqrt{c} \ge \frac{1}{2\sqrt{2}}(1-4c^2), \qquad \sqrt{1-d}-\sqrt{d} \ge \frac{1}{2\sqrt{2}}(1-4d^2).$$

To prove the first inequality, we write it as

$$\frac{1-2a}{\sqrt{1-a}+\sqrt{a}} \ge \frac{1}{2\sqrt{2}}(1-2a)(1+2a).$$

Case 1: $0 < a \le \frac{1}{2}$. We need to show that

$$2\sqrt{2} \ge (1+2a)(\sqrt{1-a}+\sqrt{a}).$$

Since $\sqrt{1-a}+\sqrt{a} \le \sqrt{2[(1-a)+a]} = \sqrt{2}$, we have

$$2\sqrt{2}-(1+2a)(\sqrt{1-a}+\sqrt{a}) \ge \sqrt{2}(1-2a) \ge 0.$$

Case 2: $\frac{1}{2} \le a < 1$. We need to show that

$$2\sqrt{2} \le (1+2a)(\sqrt{1-a}+\sqrt{a}).$$

Since $1+2a \ge 2\sqrt{2a}$, it suffices to prove that

$$1 \le \sqrt{a(1-a)} + a.$$

Indeed,

$$1-a-\sqrt{a(1-a)} = \sqrt{1-a}\,(\sqrt{1-a}-\sqrt{a}) = \frac{\sqrt{1-a}\,(1-2a)}{\sqrt{1-a}+\sqrt{a}} \le 0.$$

□

P 2.101. *Let a, b, c, d be positive real numbers, and let*

$$A = (a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right), \qquad B = (a^2+b^2+c^2+d^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}\right).$$

Prove that

(a) $$\sqrt{B-12} \le A-14;$$

(b) $$2\sqrt{B} \ge A-8.$$

(Vasile Cîrtoaje, 2004)

Solution. Denote

$$A_1 = A-16, \qquad B_1 = B-16,$$

and

$$f(x,y,z) = \frac{x}{y}+\frac{y}{z}+\frac{z}{x}-3, \qquad F(x,y,z) = \frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2}-3,$$

where $x,y,z>0$. By the AM-GM inequality, we get $f(x,y,z) \ge 0$. Since

$$\begin{aligned} F(x,y,z) &= [f(x,y,z)+3]^2 - 2[f(x,z,y)+3] - 3 \\ &= f^2(x,y,z) + 6f(x,y,z) - 2f(x,z,y) \end{aligned}$$

and

$$\begin{aligned} A_1 &= f(a,b,c)+f(b,d,c)+f(c,d,a)+f(d,b,a) \\ &= f(a,c,b)+f(b,c,d)+f(c,a,d)+f(d,a,b), \end{aligned}$$

we have

$$\begin{aligned} B_1 &= F(a,b,c)+F(b,d,c)+F(c,d,a)+F(d,b,a) \\ &= f^2(a,b,c)+f^2(b,d,c)+f^2(c,d,a)+f^2(d,b,a)+4A_1, \end{aligned}$$

therefore

$$B_1 - 4A_1 = f^2(a,b,c)+f^2(b,d,c)+f^2(c,d,a)+f^2(d,b,a).$$

(a) Write the inequality as follows:

$$\sqrt{B_1+4} \le A_1+2,$$

$$A_1^2 \ge B_1 - 4A_1,$$

$$[f(a,b,c)+f(b,d,c)+f(c,d,a)+f(d,b,a)]^2 \ge f^2(a,b,c)+f^2(b,d,c)+f^2(c,d,a)+f^2(d,b,a).$$

The last inequality is clearly true. The equality occurs for $a=b=c=d$.

(b) ***First Solution.*** Write the inequality as follows:

$$2\sqrt{B_1+16} \ge A_1+8,$$

$$4(B_1-4A_1) \ge A_1^2,$$

$$4[f^2(a,b,c)+f^2(b,d,c)+f^2(c,d,a)+f^2(d,b,a)] \ge [f(a,b,c)+f(b,d,c)+f(c,d,a)+f(d,b,a)]^2.$$

Since the last inequality is well-known, the proof is finished. For $a \ge b \ge c \ge d$, the equality occurs when $a=b$ and $c=d$.

Second Solution (by *Nquyen Van Huyen*). Since

$$4(a^2+b^2+c^2+d^2) = \sum(-a+b+c+d)^2,$$

by the Cauchy-Schwarz inequality we get

$$4B = \sum(-a+b+c+d)^2\left(\sum\frac{1}{a^2}\right) \ge \left(\sum\frac{-a+b+c+d}{a}\right)^2 = (A-8)^2.$$

Remark 1. Using the second solution of the inequality (b), we can prove that if n is an even integer, $a_1,a_2,\ldots,a_n$ are positive real numbers and

$$A = \left(\sum_{i=1}^n a_i\right)\left(\sum_{i=1}^n \frac{1}{a_i}\right), \qquad B = \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n \frac{1}{a_i^2}\right),$$

then

$$n\sqrt{B} \ge 2A - n^2,$$

with equality when half of the numbers a_i are equal to each other and the others are also equal. Let $k=\dfrac{n-2}{2}$. Since

$$(n-1-k)^2\left(\sum_{i=1}^n a_i^2\right) = \sum(-ka_1+a_2+\cdots+a_n)^2,$$

by the Cauchy-Schwarz inequality we get

$$(n-1-k)^2B = \sum(-ka_1+a_2+\cdots+a_n)^2\left(\sum\frac{1}{a_1^2}\right) \ge \left(\sum\frac{-ka_1+a_2+\cdots+a_n}{a_1}\right)^2 = [A-(k+1)n]^2,$$

hence $n\sqrt{B} \ge 2A - n^2$.

Remark 2. We claim that

$$\sqrt{3B-12} \le A - 10$$

for $A \le 10 + 4\sqrt{6}$, with equality when $a = b = c$ (or any cyclic permutation), and

$$\sqrt[4]{B} \le \sqrt{A} - 2$$

for $A \ge 10 + 4\sqrt{6}$, with equality when $a = b = \sqrt{cd}$ (or any permutation).

□

P 2.102. *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = 1$. Prove that*

$$\sqrt{3a_1+1} + \sqrt{3a_2+1} + \cdots + \sqrt{3a_n+1} \ge n+1.$$

First Solution. Without loss of generality, assume that $a_1 = \max\{a_1, a_2, \ldots, a_n\}$. Write the inequality as follows:

$$(\sqrt{3a_1+1} - 2) + (\sqrt{3a_2+1} - 1) + \cdots + (\sqrt{3a_n+1} - 1) \ge 0,$$

$$\frac{a_1 - 1}{\sqrt{3a_1+1}+2} + \frac{a_2}{\sqrt{3a_2+1}+1} + \cdots + \frac{a_n}{\sqrt{3a_n+1}+1} \ge 0,$$

$$\frac{a_2}{\sqrt{3a_2+1}+1} + \cdots + \frac{a_n}{\sqrt{3a_n+1}+1} \ge \frac{a_2 + \cdots + a_n}{\sqrt{3a_1+1}+2},$$

$$a_2\left(\frac{1}{\sqrt{3a_2+1}+1} - \frac{1}{\sqrt{3a_1+1}+2}\right) + \cdots + a_n\left(\frac{1}{\sqrt{3a_n+1}+1} - \frac{1}{\sqrt{3a_1+1}+2}\right) \ge 0.$$

The last inequality is clearly true. The equality holds for $a_1 = 1$ and $a_2 = \cdots = a_n = 0$ (or any cyclic permutation).

Second Solution. We use the induction method. For $n = 1$, the inequality is an equality. We claim that

$$\sqrt{3a_1+1} + \sqrt{3a_n+1} \ge \sqrt{3(a_1+a_n)+1} + 1.$$

By squaring, this inequality becomes

$$\sqrt{(3a_1+1)(a_n+1)} \ge \sqrt{3(a_1+a_n)+1},$$

which is equivalent to $a_1a_n \ge 0$. Thus, to prove the original inequality, it suffices to show that

$$\sqrt{3(a_1+a_n)+1} + \sqrt{3a_2+1} + \cdots + \sqrt{3a_{n-1}+1} \ge n.$$

Using the substitution $b_1 = a_1 + a_n$ and $b_2 = a_2, \ldots, b_{n-1} = a_{n-1}$, this inequality turns into

$$\sqrt{3b_1+1} + \sqrt{3b_2+1} + \cdots + \sqrt{3b_{n-1}+1} \ge n$$

for $b_1 + b_2 + \cdots + b_{n-1} = 1$. Clearly, this is true by the induction hypothesis.

□

P 2.103. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1a_2\cdots a_n = 1$. Prove that*

$$\frac{1}{\sqrt{1+(n^2-1)a_1}} + \frac{1}{\sqrt{1+(n^2-1)a_2}} + \cdots + \frac{1}{\sqrt{1+(n^2-1)a_n}} \geq 1.$$

First Solution. For the sake of contradiction, assume that

$$\frac{1}{\sqrt{1+(n^2-1)a_1}} + \frac{1}{\sqrt{1+(n^2-1)a_2}} + \cdots + \frac{1}{\sqrt{1+(n^2-1)a_n}} < 1.$$

It suffices to show that $a_1a_2\cdots a_n > 1$. Let

$$x_i = \frac{1}{\sqrt{1+(n^2-1)a_i}}, \quad 0 < x_i < 1, \qquad i = 1, 2, \cdots, n.$$

Since $a_i = \dfrac{1-x_i^2}{(n^2-1)x_i^2}$ for all i, we need to show that

$$x_1 + x_2 + \cdots + x_n < 1$$

implies

$$(1-x_1^2)(1-x_2^2)\cdots(1-x_n^2) > (n^2-1)^n x_1^2x_2^2\cdots x_n^2.$$

Using the AM-GM inequality gives

$$\prod(1-x_1^2) > \prod\left[\left(\sum x_1\right)^2 - x_1^2\right] = \prod(x_2+\cdots+x_n)(2x_1+x_2+\cdots+x_n)$$

$$\geq (n^2-1)^n \prod\left(\sqrt[n-1]{x_2\cdots x_n}\cdot \sqrt[n+1]{x_1^2x_2\cdots x_n}\right) = (n^2-1)^n x_1^2x_2^2\cdots x_n^2.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Second Solution. We will show that

$$\frac{1}{\sqrt{1+(n^2-1)x}} \geq \frac{1}{1+(n-1)x^k}$$

for $x > 0$ and $k = \dfrac{n+1}{2n}$. By squaring, the inequality becomes

$$(n-1)x^{2k-1} + 2x^{k-1} \geq n+1.$$

Applying the AM-GM inequality, we get

$$(n-1)x^{2k-1} + 2x^{k-1} \geq (n+1)\sqrt[n+1]{x^{(n-1)(2k-1)}\cdot x^{2(k-1)}} = n+1.$$

Using this result, it suffices to show that

$$\frac{1}{1+(n-1)a_1^k} + \frac{1}{1+(n-1)a_2^k} + \cdots + \frac{1}{1+(n-1)a_n^k} \geq 1.$$

Since $a_1^ka_2^k\cdots a_n^k = 1$, this inequality follows immediately from P 1.203-(a).

□

P 2.104. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1a_2\cdots a_n = 1$. Prove that*

$$\sum_{i=1}^{n} \frac{1}{1+\sqrt{1+4n(n-1)a_i}} \ge \frac{1}{2}.$$

First Solution. Write the inequality as follows:

$$\sum_{i=1}^{n} \frac{\sqrt{1+4n(n-1)a_i}-1}{a_i} \ge 2n(n-1),$$

$$\sum_{i=1}^{n} \sqrt{\frac{1}{a_i^2}+\frac{4n(n-1)}{a_i}} \ge 2n(n-1)+\sum \frac{1}{a_i}.$$

By squaring, the inequality becomes

$$\sum_{1\le i<j\le n} \sqrt{\left[\frac{1}{a_i^2}+\frac{4n(n-1)}{a_i}\right]\left[\frac{1}{a_j^2}+\frac{4n(n-1)}{a_j}\right]} \ge 2n^2(n-1)^2+\sum_{1\le i<j\le n} \frac{1}{a_ia_j}.$$

The Cauchy-Schwarz inequality gives

$$\sqrt{\left[\frac{1}{a_i^2}+\frac{4n(n-1)}{a_i}\right]\left[\frac{1}{a_j^2}+\frac{4n(n-1)}{a_j}\right]} \ge \frac{1}{a_ia_j}+\frac{4n(n-1)}{\sqrt{a_ia_j}}.$$

Thus, it suffices to show that

$$\sum_{1\le i<j\le n} \frac{1}{\sqrt{a_ia_j}} \ge \frac{n(n-1)}{2},$$

which follows immediately from the AM-GM inequality. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Second Solution. For the sake of contradiction, assume that

$$\sum_{i=1}^{n} \frac{1}{1+\sqrt{1+4n(n-1)a_i}} < \frac{1}{2}.$$

It suffices to show that $a_1a_2\cdots a_n > 1$. Using the substitution

$$\frac{x_i}{2n} = \frac{1}{1+\sqrt{1+4n(n-1)a_i}}, \quad i = 1, 2, \cdots, n,$$

which yields

$$a_i = \frac{n-x_i}{(n-1)x_i^2}, \quad 0 < x_i < n, \quad i = 1, 2, \cdots, n,$$

we need to show that

$$x_1 + x_2 + \cdots + x_n < n$$

implies

$$(n - x_1)(n - x_2)\cdots(n - x_n) > (n-1)^n x_1^2 x_2^2 \cdots x_n^2.$$

By the AM-GM inequality, we have

$$x_1 x_2 \cdots x_n \le \left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^n < 1$$

and

$$n - x_i > (x_1 + x_2 + \cdots + x_n) - x_i \ge (n-1)\sqrt[n-1]{\frac{x_1 x_2 \cdots x_n}{x_i}}, \qquad i = 1, 2, \cdots, n.$$

Therefore, we get

$$(n - x_1)(n - x_2)\cdots(n - x_n) > (n-1)^n x_1 x_2 \cdots x_n > (n-1)^n x_1^2 x_2^2 \cdots x_n^2.$$

□

P 2.105. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 a_2 \cdots a_n = 1$, then*

$$a_1 + a_2 + \cdots + a_n \ge n - 1 + \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}}.$$

Solution. Let us denote

$$a = \frac{a_1 + a_2 + \cdots + a_n}{n}, \qquad b = \sqrt{\frac{2\sum_{1\le i<j\le n} a_i a_j}{n(n-1)}},$$

where $a \ge 1$ and $b \ge 1$ (by the AM-GM inequality). We need to show that

$$na - n + 1 \ge \sqrt{\frac{n^2a^2 - n(n-1)b^2}{n}}.$$

By squaring, this inequality becomes

$$(n-1)[n(a-1)^2 + b^2 - 1] \ge 0,$$

which is clearly true. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 2.106. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1a_2\cdots a_n = 1$, then*

$$\sqrt{(n-1)(a_1^2+a_2^2+\cdots+a_n^2)}+n-\sqrt{n(n-1)} \ge a_1+a_2+\cdots+a_n.$$

(Vasile Cîrtoaje, 2006)

Solution. We use the induction method. For $n=2$, the inequality is equivalent to the obvious inequality

$$a_1+\frac{1}{a_1}\ge 2.$$

Assume now that the inequality holds for $n-1$ numbers, $n\ge 3$, and prove that it holds also for n numbers. Let $a_1=\min\{a_1,a_2,\ldots,a_n\}$, and denote

$$x=\frac{a_2+a_3+\cdots+a_n}{n-1},\quad y=\sqrt[n-1]{a_2a_3\cdots a_n}\ ,$$

$$f(a_1,a_2,\ldots,a_n)=\sqrt{(n-1)(a_1^2+a_2^2+\cdots+a_n^2)}+n-\sqrt{n(n-1)}-(a_1+a_2+\cdots+a_n).$$

By the AM-GM inequality, we have $x\ge y$. We will show that

$$f(a_1,a_2,\ldots,a_n)\ge f(a_1,y,\cdots,y)\ge 0. \tag{*}$$

Write the left inequality as

$$\sqrt{a_1^2+a_2^2+\cdots+a_n^2}-\sqrt{a_1^2+(n-1)y^2}\ge\sqrt{n-1}\ (x-y).$$

To prove this inequality, we use the induction hypothesis, written in the homogeneous form

$$\sqrt{(n-2)(a_2^2+a_3^2+\cdots+a_n^2)}+\left[n-1-\sqrt{(n-1)(n-2)}\right]y\ge(n-1)x,$$

which is equivalent to

$$a_2^2+\cdots+a_n^2\ge(n-1)A^2,$$

where

$$A=kx-(k-1)y,\quad k=\sqrt{\frac{n-1}{n-2}}.$$

So, we need to prove that

$$\sqrt{a_1^2+(n-1)A^2}-\sqrt{a_1^2+(n-1)y^2}\ge\sqrt{n-1}\ (x-y).$$

Write this inequality as

$$\frac{A^2-y^2}{\sqrt{a_1^2+(n-1)A^2}+\sqrt{a_1^2+(n-1)y^2}}\ge\frac{x-y}{\sqrt{n-1}}.$$

Since $x\ge y$ and

$$A^2-y^2=k(x-y)[kx-(k-2)y]=k(x-y)(A+y),$$

we need to show that

$$\frac{k(A+y)}{\sqrt{a_1^2+(n-1)A^2}+\sqrt{a_1^2+(n-1)y^2}} \ge \frac{1}{\sqrt{n-1}}.$$

In addition, since $a_1 \le y$, it suffices to show that

$$\frac{k(A+y)}{\sqrt{y^2+(n-1)A^2}+\sqrt{n}y} \ge \frac{1}{\sqrt{n-1}}.$$

From

$$kA-y = k^2x-(k^2-k+1)y \ge k^2y-(k^2-k+1)y = (k-1)y > 0,$$

it follows that

$$y^2+(n-1)A^2 < k^2A^2+(n-1)^2 = (n-1)k^2A^2.$$

Therefore, it is enough to prove that

$$\frac{k(A+y)}{\sqrt{n-1}\;kA+\sqrt{n}\;y} \ge \frac{1}{\sqrt{n-1}},$$

which is equivalent to

$$\left(k\sqrt{n-1}-\sqrt{n}\right)y \ge 0.$$

This is true since

$$k\sqrt{n-1}-\sqrt{n} = \frac{n-1}{\sqrt{n-2}}-\sqrt{n} = \frac{1}{n-1+\sqrt{n(n-2)}} > 0.$$

The proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 2.107. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1a_2\cdots a_n \ge 1$. If $k > 1$, then*

$$\sum \frac{a_1^k}{a_1^k+a_2+\cdots+a_n} \ge 1.$$

(Vasile Cîrtoaje, 2006)

First Solution. Let us denote $r = \sqrt[n]{a_1a_2\cdots a_n}$ and $b_i = a_i/r$ for $i = 1, 2, \cdots, n$. Note that $r \ge 1$ and $b_1b_2\cdots b_n = 1$. The desired inequality becomes

$$\sum \frac{b_1^k}{b_1^k+(b_2+\cdots+b_n)/r^{k-1}} \ge 1,$$

and we see that it suffices to prove it for $r = 1$; that is, for $a_1a_2\cdots a_n = 1$. On this hypothesis, we will show that there exists a positive number p, $1 < p < k$, such that

$$\frac{a_1^k}{a_1^k+a_2+\cdots+a_n} \ge \frac{a_1^p}{a_1^p+a_2^p+\cdots+a_n^p}.$$

Clearly, by adding this inequality and the analogous inequalities for $a_2, \ldots, a_n$, we get the desired inequality. Write the claimed inequality as

$$a_2^p + \cdots + a_n^p \ge (a_2 \cdots a_n)^{k-p}(a_2 + \cdots + a_n).$$

Based on the AM-GM inequality

$$a_2 \cdots a_n \le \left(\frac{a_2 + \cdots + a_n}{n-1}\right)^{n-1},$$

it suffices to show that

$$a_2^p + \cdots + a_n^p \ge (n-1)\left(\frac{a_2 + \cdots + a_n}{n-1}\right)^{(n-1)(k-p)+1}.$$

Choosing

$$p = \frac{(n-1)k+1}{n}, \quad 1 < p < k,$$

the inequality becomes

$$a_2^p + \cdots + a_n^p \ge (n-1)\left(\frac{a_2 + \cdots + a_n}{n-1}\right)^p,$$

which is just Jensen's inequality applied to the convex function $f(x) = x^p$. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Second Solution. By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a_1^k}{a_1^k + a_2 + \cdots + a_n} \ge \frac{\left(\sum a_1^{\frac{k+1}{2}}\right)^2}{\sum a_1(a_1^k + a_2 + \cdots + a_n)} = \frac{\sum a_1^{k+1} + 2\sum_{1\le i<j\le n}(a_ia_j)^{\frac{k+1}{2}}}{\sum a_1^{k+1} + 2\sum_{1\le i<j\le n} a_ia_j}.$$

Thus, it suffices to show that

$$\sum_{1\le i<j\le n}(a_ia_j)^{\frac{k+1}{2}} \ge \sum_{1\le i<j\le n} a_ia_j.$$

Jensen's inequality applied to the convex function $f(x) = x^{\frac{k+1}{2}}$ yields

$$\sum_{1\le i<j\le n}(a_ia_j)^{\frac{k+1}{2}} \ge \frac{n(n-1)}{2}\left(\frac{2\sum_{1\le i<j\le n} a_ia_j}{n(n-1)}\right)^{\frac{k+1}{2}}.$$

On the other hand, by the AM-GM inequality, we get

$$\frac{2}{n(n-1)}\sum_{1\le i<j\le n} a_ia_j \ge (a_1a_2\cdots a_n)^{\frac{2}{n}} \ge 1.$$

Therefore,

$$\left(\frac{2\sum_{1\le i<j\le n} a_ia_j}{n(n-1)}\right)^{\frac{k+1}{2}} = \left(\frac{2\sum_{1\le i<j\le n} a_ia_j}{n(n-1)}\right)^{\frac{k-1}{2}} \cdot \frac{2\sum_{1\le i<j\le n} a_ia_j}{n(n-1)} \ge \frac{2\sum_{1\le i<j\le n} a_ia_j}{n(n-1)}.$$

hence

$$\sum_{1\le i<j\le n} (a_ia_j)^{\frac{k+1}{2}} \ge \frac{n(n-1)}{2} \cdot \frac{2\sum_{1\le i<j\le n} a_ia_j}{n(n-1)} = \sum_{1\le i<j\le n} a_ia_j.$$

□

P 2.108. *Let $a_1, a_2, \dots, a_n$ be positive real numbers such that $a_1a_2\cdots a_n \ge 1$. If*

$$\frac{-2}{n-2} \le k < 1,$$

then

$$\sum \frac{a_1^k}{a_1^k + a_2 + \cdots + a_n} \le 1.$$

(Vasile Cîrtoaje, 2006)

Solution. Let us denote $r = \sqrt[n]{a_1a_2\cdots a_n}$ and $b_i = a_i/r$ for $i = 1, 2, \cdots, n$. Clearly, $r \ge 1$ and $b_1b_2\cdots b_n = 1$. The desired inequality becomes

$$\sum \frac{b_1^k}{b_1^k + (b_2 + \cdots + b_n)r^{1-k}} \le 1,$$

and we see that it suffices to prove it for $r = 1$; that is, for $a_1a_2\cdots a_n = 1$. On this hypothesis, we will show that there exists a real number p such that

$$\frac{a_1^k}{a_1^k + a_2 + \cdots + a_n} \le \frac{a_1^p}{a_1^p + a_2^p + \cdots + a_n^p}.$$

By adding this inequality and the analogous inequalities for $a_2, \dots, a_n$, we get the desired inequality. Write the claimed inequality as

$$a_2 + \cdots + a_n \ge (a_2^p + \cdots + a_n^p)a_1^{k-p},$$

$$a_2 + \cdots + a_n \ge (a_2^p + \cdots + a_n^p)(a_2\cdots a_n)^{p-k}.$$

This inequality is homogeneous when $1 = p + (n-1)(p-k)$; that is, for

$$p = \frac{(n-1)k+1}{n}, \quad \frac{-1}{n-2} \le p < 1.$$

Rewrite the homogeneous inequality as

$$a_2 + \cdots + a_n \ge (a_2^p + \cdots + a_n^p)(a_2\cdots a_n)^{\frac{1-p}{n-1}}. \tag{*}$$

To prove it, we use the weighted AM-GM inequality

$$ma_2 + a_3 + \cdots + a_n \ge (m+n-2)a_2^{\frac{m}{m+n-2}}(a_3 \cdots a_n)^{\frac{1}{m+n-2}}, \qquad m \ge 0,$$

which can be rewritten as

$$ma_2 + a_3 + \cdots + a_n \ge (m+n-2)a_2^{\frac{m-1}{m+n-2}}(a_2 \cdots a_n)^{\frac{1}{m+n-2}}.$$

Choosing m such that $\dfrac{m-1}{m+n-2} = p$, i.e.

$$m = \frac{1+(n-2)p}{1-p} \ge 0,$$

we get

$$\frac{1+(n-2)p}{1-p}a_2 + a_3 + \cdots + a_n \ge \frac{n-1}{1-p}a_2^p(a_2a_3\cdots a_n)^{\frac{1-p}{n-1}}.$$

Adding this inequality and the analogous inequalities for $a_3, \cdots, a_n$ yields the inequality (*). Thus, the proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 2.109. *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n \ge n$. If $1 < k \le n+1$, then*

$$\sum \frac{a_1}{a_1^k + a_2 + \cdots + a_n} \le 1.$$

(Vasile Cîrtoaje, 2006)

Solution. Using the substitutions

$$s = \frac{a_1 + a_2 + \cdots + a_n}{n},$$

and

$$x_1 = \frac{a_1}{s}, \quad x_2 = \frac{a_2}{s}, \quad \cdots, \quad x_n = \frac{a_n}{s},$$

the desired inequality becomes

$$\frac{x_1}{s^{k-1}x_1^k + x_2 + \cdots + x_n} + \cdots + \frac{x_n}{x_1 + x_2 + \cdots + s^{k-1}x_n^k} \le 1,$$

where $s \ge 1$ and $x_1+x_2+\cdots+x_n = n$. Clearly, if this inequality holds for $s = 1$, then it holds for any $s \ge 1$. Therefore, we only need to consider the case $s = 1$, when $a_1+a_2+\cdots+a_n = n$, and the desired inequality is equivalent to

$$\frac{a_1}{a_1^k - a_1 + n} + \frac{a_2}{a_2^k - a_2 + n} + \cdots + \frac{a_n}{a_n^k - a_n + n} \le 1.$$

By Bernoulli's inequality, we have

$$a_1^k - a_1 + n \geq 1 + k(a_1 - 1) - a_1 + n = n - k + 1 + (k-1)a_1 \geq 0.$$

Consequently, it suffices to prove that

$$\sum_{i=1}^{n} \frac{a_i}{n-k+1+(k-1)a_i} \leq 1.$$

For $k = n+1$, this inequality is an equality. Otherwise, for $1 < k < n+1$, we rewrite the inequality as

$$\sum_{i=1}^{n} \frac{1}{n-k+1+(k-1)a_i} \geq 1,$$

which follows from the AM-HM inequality as follows:

$$\sum_{i=1}^{n} \frac{1}{n-k+1+(k-1)a_i} \geq \frac{n^2}{\sum_{i=1}^{n}[n-k+1+(k-1)a_i]} = 1.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 2.110. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1a_2\cdots a_n \geq 1$. If $k > 1$, then*

$$\sum \frac{a_1}{a_1^k + a_2 + \cdots + a_n} \leq 1.$$

(Vasile Cîrtoaje, 2006)

Solution. Consider two cases: $1 < k \leq n+1$ and $k \geq n - \dfrac{1}{n-1}$.

Case 1: $1 < k \leq n+1$. By the AM-GM inequality, we have

$$a_1 + a_2 + \cdots + a_n \geq n\sqrt[n]{a_1a_2\cdots a_n} \geq n.$$

Thus, the desired inequality follows from the preceding P 2.109.

Case 2: $k \geq n - \dfrac{1}{n-1}$. Let $r = \sqrt[n]{a_1a_2\cdots a_n}$ and $b_i = a_i/r$ for $i = 1, 2, \cdots, n$. Note that $r \geq 1$ and $b_1b_2\cdots b_n = 1$. The desired inequality can be rewritten as

$$\sum \frac{b_1}{r^{k-1}b_1^k + b_2 + \cdots + b_n} \leq 1.$$

Obviously, it suffices to prove this inequality for $r = 1$; that is, for

$$a_1a_2\cdots a_n = 1.$$

On this hypothesis, it suffices to show that there exists a real p such that

$$\frac{(n-1)a_1}{a_1^k+a_2+\cdots+a_n}+\frac{a_1^p}{a_1^p+a_2^p+\cdots+a_n^p}\le 1.$$

Then, adding this inequality and the analogous inequalities for $a_2,\cdots,a_n$ yields the desired inequality. Let us denote $t=\sqrt[n-1]{a_2\cdots a_n}$. By the AM-GM inequality, we have

$$a_2+\cdots+a_n\ge(n-1)t,\quad a_2^p+\cdots+a_n^p\ge(n-1)t^p.$$

Thus, it suffices to show that

$$\frac{(n-1)a_1}{a_1^k+(n-1)t}+\frac{a_1^p}{a_1^p+(n-1)t^p}\le 1.$$

Since $a_1=1/t^{n-1}$, this inequality is equivalent to

$$(n-1)t^q(t^n-1)-(t^{q-np}-1)\ge 0,$$

where

$$q=(n-1)(k-1).$$

Choose p such that $(n-1)n=q-np$, i.e.

$$p=\frac{(n-1)(k-n-1)}{n}.$$

The inequality becomes as follows:

$$(n-1)t^q(t^n-1)-\left[t^{n(n-1)}-1\right]\ge 0,$$

$$(n-1)t^q(t^n-1)-(t^n-1)(t^{n^2-2n}+t^{n^2-3n}+\cdots+1)\ge 0,$$

$$(t^n-1)[(t^q-t^{n^2-2n})+(t^q-t^{n^2-3n})+\cdots+(t^q-1)]\ge 0.$$

The last inequality is clearly true for $q\ge n^2-2n$; that is, for $k\ge n-\dfrac{1}{n-1}$.

The equality holds for $a_1=a_2=\cdots=a_n=1$.

□

P 2.111. *Let $a_1,a_2,\ldots,a_n$ be positive real numbers such that $a_1a_2\cdots a_n\ge 1$. If*

$$-1-\frac{2}{n-2}\le k<1,$$

then

$$\sum\frac{a_1}{a_1^k+a_2+\cdots+a_n}\ge 1.$$

(Vasile Cîrtoaje, 2006)

Solution. Let us denote $r = \sqrt[n]{a_1a_2\cdots a_n}$ and $b_i = a_i/r$ for $i = 1, 2, \cdots, n$. Note that $r \ge 1$ and $b_1b_2\cdots b_n = 1$. The desired inequality becomes

$$\sum \frac{b_1}{b_1^k/r^{1-k} + b_2 + \cdots + b_n} \ge 1,$$

and we see that it suffices to prove it for $r = 1$; that is, for $a_1a_2\cdots a_n = 1$. On this hypothesis, by the Cauchy-Schwarz inequality, we have

$$\sum \frac{a_1}{a_1^k + a_2 + \cdots + a_n} \ge \frac{\left(\sum a_1\right)^2}{\sum a_1(a_1^k + a_2 + \cdots + a_n)} = \frac{\left(\sum a_1\right)^2}{\sum a_1^{1+k} + \left(\sum a_1\right)^2 - \sum a_1^2}.$$

Thus, we still have to show that

$$\sum a_1^2 \ge \sum a_1^{1+k}.$$

Case 1: $-1 \le k < 1$. Using Chebyshev's inequality and the AM-GM inequality yields

$$\sum a_1^2 \ge \frac{1}{n}\left(\sum a_1^{1-k}\right)\left(\sum a_1^{1+k}\right) \ge (a_1a_2\cdots a_n)^{\frac{1-k}{n}} \sum a_1^{1+k} = \sum a_1^{1+k}.$$

Case 2: $-1 - \dfrac{2}{n-1} \le k < -1$. It is convenient to replace $a_1, a_2, \cdots, a_n$ by

$$a_1^{(n-1)/2}, a_2^{(n-1)/2}, \cdots, a_n^{(n-1)/2},$$

respectively. Thus, we need to show that $a_1a_2\cdots a_n = 1$ involves

$$\sum a_1^{n-1} \ge \sum a_1^q,$$

where

$$q = \frac{(n-1)(1+k)}{2}, \qquad -1 \le q < 0.$$

By the AM-GM inequality, we get

$$\sum a_1^{n-1} = \sum \frac{a_2^{n-1} + \cdots + a_n^{n-1}}{n-1} \ge \sum a_2\cdots a_n = \sum \frac{1}{a_1}.$$

Thus, it suffice to show that

$$\sum \frac{1}{a_1} \ge \sum a_1^q.$$

By Chebyshev's inequality and the AM-GM inequality, we have

$$\sum \frac{1}{a_1} \ge \frac{1}{n}\left(\sum a_1^{-1-q}\right)\left(\sum a_1^q\right) \ge (a_1a_2\cdots a_n)^{-(1+q)/n}\left(\sum a_1^q\right) = \sum a_1^q.$$

Thus, the proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 2.112. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1a_2\cdots a_n = 1$. If $k \ge 0$, then*

$$\sum \frac{1}{a_1^k + a_2 + \cdots + a_n} \le 1.$$

(Vasile Cîrtoaje, 2006)

Solution. Consider two cases: $0 \le k \le 1$ and $k \ge 1$.

Case 1: $0 \le k \le 1$. By the Cauchy-Schwarz inequality and the AM-GM inequality, we have

$$\frac{1}{a_1^k + a_2 + \cdots + a_n} \le \frac{a_1^{1-k} + 1 + \cdots + 1}{\left(\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n}\right)^2}$$

$$= \frac{a_1^{1-k} + n - 1}{\sum a_1 + 2\sum_{1\le i<j\le n}\sqrt{a_ia_j}} \le \frac{a_1^{1-k} + n - 1}{\sum a_1 + n(n-1)},$$

hence

$$\sum \frac{1}{a_1^k + a_2 + \cdots + a_n} \le \frac{\sum a_1^{1-k} + n(n-1)}{\sum a_1 + n(n-1)}.$$

Therefore, it suffices to show that

$$\sum a_1^{1-k} \le \sum a_1.$$

Indeed, by Chebyshev's inequality and the AM-GM inequality, we have

$$\sum a_1 = \sum a_1^k \cdot a_1^{1-k} \ge \frac{1}{n}\left(\sum a_1^k\right)\left(\sum a_1^{1-k}\right) \ge (a_1a_2\cdots a_n)^{k/n}\left(\sum a_1^{1-k}\right) = \sum a_1^{1-k}.$$

Case 2: $k > 1$. Write the inequality as

$$\sum \left(\frac{n-1}{a_1^k + a_2 + \cdots + a_n} + \frac{a_1^p}{a_1^p + a_2^p + \cdots + a_n^p} - 1\right) \le 0,$$

where $p > 0$. It suffices to show that there exists a positive number p such that

$$\frac{n-1}{a_1^k + a_2 + \cdots + a_n} + \frac{a_1^p}{a_1^p + a_2^p + \cdots + a_n^p} \le 1.$$

Let

$$x = \sqrt[n-1]{a_1}, \quad x > 0.$$

By the AM-GM inequality, we have

$$a_2 + \cdots + a_n \ge (n-1)\sqrt[n-1]{a_2\cdots a_n} = \frac{n-1}{\sqrt[n-1]{a_1}} = \frac{n-1}{x}$$

and

$$a_2^p + \cdots + a_n^p \ge \sqrt[n-1]{(a_2\cdots a_n)^p} = \frac{n-1}{\sqrt[n-1]{a_1^p}} = \frac{n-1}{x^p}.$$

Thus, it is enough to show that

$$\frac{n-1}{x^{(n-1)k}+\frac{n-1}{x}}+\frac{x^{(n-1)p}}{x^{(n-1)p}+\frac{n-1}{x^p}}\le 1,$$

which is equivalent to

$$\frac{x}{x^{(n-1)k+1}+n-1}\le\frac{1}{x^{np}+n-1},$$

$$x^{(n-1)k+1}-x^{np+1}-(n-1)(x-1)\ge 0,$$

$$x^{np+1}\left[(x^{(n-1)k-np}-1\right]-(n-1)(x-1)\ge 0.$$

Choose p such that $(n-1)k-np=n-1$, i.e.

$$p=\frac{(k-1)(n-1)}{n}>0.$$

The inequality becomes as follows:

$$x^{np+1}\left[(x^{n-1}-1\right]-(n-1)(x-1)\ge 0,$$

$$(x-1)\left[(x^{np+n-1}-1)+(x^{np+n-2}-1)+\cdots+(x^{np+1}-1)\right]\ge 0.$$

Since the last inequality is obvious true, the proof is completed. The equality holds for $a_1=a_2=\cdots=a_n=1$.

□

P 2.113. *Let $a_1,a_2,\ldots,a_n$ be nonnegative real numbers such that $a_1+a_2+\cdots+a_n\le n$. If $0\le k<1$, then*

$$\frac{1}{a_1^k+a_2+\cdots+a_n}+\frac{1}{a_1+a_2^k+\cdots+a_n}+\cdots+\frac{1}{a_1+a_2+\cdots+a_n^k}\ge 1.$$

Solution. By the AM-HM inequality

$$\sum\frac{1}{a_1^k+a_2+\cdots+a_n}\ge\frac{n^2}{\sum(a_1^k+a_2+\cdots+a_n)}=\frac{n^2}{\sum a_1^k+(n-1)\sum a_1}$$

and Jensen's inequality

$$\sum a_1^k\le n\left(\frac{1}{n}\sum a_1\right)^k,$$

we get

$$\sum\frac{1}{a_1^k+a_2+\cdots+a_n}\ge\frac{n^2}{n\left(\frac{1}{n}\sum a_1\right)^k+(n-1)\sum a_1}\ge 1.$$

The equality holds for $a_1=a_2=\cdots=a_n=1$.

□

P 2.114. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers. If $k > 1$, then*

$$\sum \frac{a_2^k + a_3^k + \cdots + a_n^k}{a_2 + a_3 + \cdots + a_n} \le \frac{n(a_1^k + a_2^k + \cdots + a_n^k)}{a_1 + a_2 + \cdots + a_n}.$$

(Wolfgang Berndt and Vasile Cîrtoaje, 2006)

Solution. Due to homogeneity, we may assume that $a_1 + a_2 + ... + a_n = 1$. Write the inequality as follows:

$$\sum \left(1 + \frac{a_1}{a_2 + a_3 + \cdots + a_n}\right)(a_2^k + a_3^k + \cdots + a_n^k) \le n(a_1^k + a_2^k + \cdots + a_n^k);$$

$$\sum \frac{a_1(a_2^k + a_3^k + \cdots + a_n^k)}{a_2 + a_3 + \cdots + a_n} \le a_1^k + a_2^k + \cdots + a_n^k;$$

$$\sum a_1 \left(a_1^{k-1} - \frac{a_2^k + a_3^k + \cdots + a_n^k}{a_2 + a_3 + \cdots + a_n}\right) \ge 0;$$

$$\sum \frac{a_1 a_2(a_1^{k-1} - a_2^{k-1}) + a_1 a_3(a_1^{k-1} - a_3^{k-1}) + \cdots + a_1 a_n(a_1^{k-1} - a_n^{k-1})}{a_2 + a_3 + \cdots + a_n} \ge 0;$$

$$\sum_{1 \le i < j \le n} a_i a_j \left(\frac{a_i^{k-1} - a_j^{k-1}}{1 - a_i} + \frac{a_j^{k-1} - a_i^{k-1}}{1 - a_j}\right) \ge 0;$$

$$\sum_{1 \le i < j \le n} \frac{a_i a_j (a_i^{k-1} - a_j^{k-1})(a_i - a_j)}{(1 - a_i)(1 - a_j)} \ge 0.$$

Since the last inequality is true for $k > 1$, the proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n$.

□

P 2.115. *Let f be a convex function on the closed interval $[a, b]$, and let $a_1, a_2, \ldots, a_n \in [a, b]$ such that*

$$a_1 + a_2 + \cdots + a_n = pa + qb,$$

where $p, q \ge 0$ such that $p + q = n$. Prove that

$$f(a_1) + f(a_2) + \cdots + f(a_n) \le pf(a) + qf(b).$$

(Vasile Cîrtoaje, 2009)

Solution. Consider the nontrivial case $a < b$. Since $a_1, a_2, \ldots, a_n \in [a, b]$, there exist $\lambda_1, \lambda_2, \ldots, \lambda_n \in [0, 1]$ such that

$$a_i = \lambda_i a + (1 - \lambda_i)b, \quad i = 1, 2, \ldots, n.$$

From

$$\lambda_i = \frac{a_i - b}{a - b}, \quad i = 1, 2, \ldots, n,$$

we have

$$\sum_{i=1}^{n} \lambda_i = \frac{1}{a-b}\left(\sum_{i=1}^{n} a_i - nb\right) = \frac{(pa+qb)-(p+q)b}{a-b} = p.$$

Since f is convex on $[a, b]$, we get

$$\begin{aligned}
\sum_{i=1}^{n} f(a_i) &\le \sum_{i=1}^{n} [\lambda_i f(a) + (1-\lambda_i) f(b)] \\
&= \left(\sum_{i=1}^{n} \lambda_i\right) [f(a) - f(b)] + nf(b) \\
&= p\,[f(a) - f(b)] + (p+q) f(b) \\
&= pf(a) + qf(b).
\end{aligned}$$

□

P 2.116. *If a, b, c are positive real numbers such that*

$$a \ge 1 \ge b \ge c, \qquad abc = 1,$$

then

$$\frac{1}{\sqrt{3a+1}} + \frac{1}{\sqrt{3b+1}} + \frac{1}{\sqrt{3c+1}} \ge \frac{3}{2}.$$

(Vasile Cîrtoaje, 2007)

Solution. Let

$$b_1 = 1/b, \qquad b_1 \ge 1.$$

We claim that

$$\frac{1}{\sqrt{3b+1}} + \frac{1}{\sqrt{3b_1+1}} \ge \frac{1}{2}.$$

This inequality is equivalent to

$$\frac{1}{\sqrt{3b+1}} + \sqrt{\frac{b}{b+3}} \ge \frac{1}{2}.$$

Making the substitution

$$\frac{1}{\sqrt{3b+1}} = t, \quad \frac{1}{2} \le t < 1,$$

the inequality becomes

$$\sqrt{\frac{1-t^2}{1+8t^2}} \ge 1 - t.$$

By squaring, we get

$$t(1-t)(1-2t)^2 \ge 0,$$

which is clearly true. Similarly, we have

$$\frac{1}{\sqrt{3c+1}} + \frac{1}{\sqrt{3c_1+1}} \ge \frac{1}{2},$$

where

$$c_1 = 1/c, \qquad c_1 \ge 1.$$

Using these inequalities, it suffices to show that

$$\frac{1}{\sqrt{3a+1}} + \frac{1}{2} \ge \frac{1}{\sqrt{3b_1+1}} + \frac{1}{\sqrt{3c_1+1}},$$

which is equivalent to

$$\frac{1}{\sqrt{3b_1c_1+1}} + \frac{1}{2} \ge \frac{1}{\sqrt{3b_1+1}} + \frac{1}{\sqrt{3c_1+1}}.$$

According to P 2.88, the conclusion follows. The equality holds for $a = b = c = 1$.

□

P 2.117. *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that*

$$a_1 \le 1 \le a_2 \le \cdots \le a_n, \qquad a_1 + a_2 + \cdots + a_n = n,$$

then

$$\sqrt{\frac{3a_1}{4-a_1}} + \sqrt{\frac{3a_2}{4-a_2}} + \cdots + \sqrt{\frac{3a_n}{4-a_n}} \le n.$$

(Vasile Cîrtoaje, 2009)

Solution. Write the inequality as follows:

$$\left(\sqrt{\frac{3a_1}{4-a_1}} - 1\right) + \left(\sqrt{\frac{3a_2}{4-a_2}} - 1\right) + \cdots + \left(\sqrt{\frac{3a_n}{4-a_n}} - 1\right) \le 0.$$

$$\frac{a_1-1}{4-a_1+\sqrt{3a_1(4-a_1)}} + \frac{a_2-1}{4-a_2+\sqrt{3a_2(4-a_2)}} + \cdots + \frac{a_n-1}{4-a_n+\sqrt{3a_n(4-a_n)}} \le 0,$$

$$\frac{a_2-1}{4-a_2+\sqrt{3a_2(4-a_2)}} + \cdots + \frac{a_n-1}{4-a_n+\sqrt{3a_n(4-a_n)}} \le \frac{(a_2-1)+\cdots+(a_n-1)}{4-a_1+\sqrt{3a_1(4-a_1)}},$$

$$(a_2-1)E_2 + \cdots + (a_n-1)E_n \ge 0,$$

where

$$E_j = \frac{1}{4-a_1+\sqrt{3a_1(4-a_1)}} - \frac{1}{4-a_j+\sqrt{3a_j(4-a_j)}}, \qquad j = 2, \ldots, n.$$

It suffices to show that all $E_j \ge 0$. The inequality $E_j \ge 0$ is equivalent to

$$\sqrt{3a_j(4-a_j)} - \sqrt{3a_1(4-a_1)} \ge a_j - a_1,$$

$$\frac{3(a_j-a_1)(4-a_1-a_j)}{\sqrt{3a_j(4-a_j)} + \sqrt{3a_1(4-a_1)}} \ge a_j - a_1.$$

This is true if

$$\sqrt{3a_1(4-a_1)} + \sqrt{3a_j(4-a_j)} \le 3(4-a_1-a_j).$$

We have

$$a_1 + a_j - 2 \le a_1 + a_n - 2 = (1-a_2) + \cdots + (1-a_{n-1}) \le 0.$$

Denote

$$x = a_1 + a_j, \qquad x \le 2.$$

Since

$$\sqrt{3a_1(4-a_1)} + \sqrt{3a_j(4-a_j)} \le \sqrt{2[3a_1(4-a_1) + 3a_j(4-a_j)]} \le \sqrt{24x - 3x^2},$$

it suffices to show that

$$\sqrt{24x - 3x^2} \le 3(4-x),$$

which is equivalent to the obvious inequality

$$(2-x)(6-x) \ge 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 2.118. *If a, b, c are positive real numbers and*

$$x = \frac{2a}{b+c}, \qquad y = \frac{2b}{c+a}, \qquad z = \frac{2c}{a+b},$$

then

$$\frac{1}{\sqrt{5x+4}} + \frac{1}{\sqrt{5y+4}} + \frac{1}{\sqrt{5z+4}} \ge 1.$$

(Vasile Cîrtoaje, 2021)

Solution. Write the inequality as follows:

$$\sum \left(\sqrt{\frac{3(b+c)}{10a+4b+4c}} - 1 \right) \ge 0,$$

$$\sum \frac{5(b-a) + 5(c-a)}{A} \ge 0,$$

$$\sum(a-b)\left(\frac{1}{B}-\frac{1}{A}\right)\geq 0,$$

$$\sum(a-b)(A-B)\geq 0,$$

where

$$A=10a+4b+4c+3\sqrt{(b+c)(10a+4b+4c)},$$

$$B=10b+4c+4a+3\sqrt{(c+a)(10b+4c+4a)}.$$

Since

$$\frac{A-B}{3}=2(a-b)+\sqrt{(b+c)(10a+4b+4c)}-\sqrt{(c+a)(10b+4c+4a)}$$

$$=2(a-b)+\frac{2(a-b)(c-2a-2b)}{\sqrt{(b+c)(10a+4b+4c)}+\sqrt{(c+a)(10b+4c+4a)}}\ ,$$

we only need to show that

$$\sum(a-b)^2C_1\geq 0,$$

where

$$C_1=1+\frac{c-2a-2b}{\sqrt{(b+c)(10a+4b+4c)}+\sqrt{(c+a)(10b+4c+4a)}}\ .$$

Since

$$C_1\geq 1-\frac{2(a+b)}{\sqrt{b(10a+4b)}+\sqrt{a(10b+4a)}}$$

$$\geq 1-\frac{2(a+b)}{\sqrt{b(10a+4b)+a(10b+4a)}}\geq 1-\frac{2(a+b)}{\sqrt{4(a^2+b^2+2ab)}}=0,$$

the conclusion follows.
The equality holds for $a=b=c$.

□

P 2.119. *If a,b,c are nonnegative real numbers such that $ab+bc+ca=3$, then*

$$\sqrt{(a+3b)(a+3c)}+\sqrt{(b+3c)(b+3a)}+\sqrt{(c+3a)(c+3b)}\geq 12.$$

(Vasile Cîrtoaje, Crux Mathematicorum, 5, 2023)

Solution. Since

$$2\sqrt{(a+3b)(a+3c)}-2a-3(b+c)=\frac{-9(b-c)^2}{2\sqrt{(a+3b)(a+3c)}+2a+3(b+c)},$$

we may write the inequality as follows:

$$8(a+b+c-3)\geq 9\sum\frac{(b-c)^2}{2\sqrt{(a+3b)(a+3c)}+2a+3(b+c)},$$

$$8\left[a+b+c-\sqrt{3(ab+bc+ca)}\right] \geq 9\sum \frac{(b-c)^2}{2\sqrt{(a+3b)(a+3c)}+2a+3(b+c)},$$

$$4\sum \frac{(b-c)^2}{a+b+c+\sqrt{3(ab+bc+ca)}} \geq 9\sum \frac{(b-c)^2}{2\sqrt{(a+3b)(a+3c)}+2a+3(b+c)},$$

$$A(b-c)^2+B(c-a)^2+C(a-b)^2 \geq 0,$$

where

$$A = \frac{8\sqrt{(a+3b)(a+3c)}-a+3(b+c)-9\sqrt{3(ab+bc+ca)}}{2\sqrt{(a+3b)(a+3c)}+2a+3(b+c)}.$$

If $A \geq 0$, $B \geq 0$, $C \geq 0$, then the conclusion follows. The inequality $A \geq 0$ is equivalent to

$$8\sqrt{(a+3b)(a+3c)}-a+3(b+c)-27 \geq 0,$$

$$8\sqrt{a^2+9+6bc}-4a+3(a+b+c)-27 \geq 0.$$

Since $6bc \geq 0$ and

$$a+b+c \geq \sqrt{3(ab+bc+ca)} \geq 3,$$

it suffices to show that

$$8\sqrt{a^2+9}-4a-18 \geq 0,$$

that is

$$4\sqrt{a^2+9} \geq 2a+9.$$

By squaring, the inequality becomes

$$4a^2-12a+21 \geq 0,$$

$$(2a-3)^2+12 \geq 0.$$

The equality holds for $a=b=c=1$.

Remark. The following generalization is valid (Vasile Cîrtoaje):

- *The largest positive value of the constant k such that*

$$\sqrt{(a+kb)(a+kc)}+\sqrt{(b+kc)(b+ka)}+\sqrt{(c+ka)(c+kb)} \geq 3(1+k)$$

for all nonnegative real numbers a,b,c satisfying $ab+bc+ca=3$ is

$$k_0 = \frac{(1+\sqrt{2})(1+\sqrt{3})}{2} \approx 3.2979.$$

Proof. For $a=0$ and $b=c=\sqrt{3}$, the inequality becomes

$$k+2\sqrt{1+k} \geq (1+k)\sqrt{3}, \qquad 2\sqrt{1+k} \geq (\sqrt{3}-1)k+\sqrt{3}, \qquad 2(\sqrt{3}+1)\sqrt{1+k} \geq 2k+3+\sqrt{3}.$$

By squaring, we get

$$2k^2-2(1+\sqrt{3})k-2-\sqrt{3} \leq 0,$$

which is equivalent to $k \le k_0$. Thus, we only need to show that the inequality holds for $k = k_0$. Since

$$(a+kb)(a+kc) = a^2 + ka(b+c) + k^2bc = a^2 + (k^2-k)bc + 3k,$$

we write the required inequality as

$$\sqrt{a^2+mbc+3k} + \sqrt{b^2+mca+3k} + \sqrt{c^2+mab+3k} \ge 3(1+k),$$

where

$$m = k^2 - k \approx 7.5781\ .$$

Assume that $a \ge b \ge c$, $c \le 1$. If $c = 1$, then $a = b = 1$ and the inequality is true. Consider next c fixed, $c < 1$, b as function of a,

$$b = \frac{3-ca}{a+c},$$

and the desired inequality as

$$F(a) \ge 3(1+k).$$

From $a \ge b$, we get

$$a \ge m_1, \qquad m_1 = -c + \sqrt{c^2+3}.$$

Note that a has the minimum value m_1 for $a = b$. We will show that $F(a) \ge F(m_1) \ge 3(1+k)$. To prove that $F(a) \ge F(m_1)$, we show that $F'(a) \ge 0$. From the hypothesis equation $ab + bc + ca = 3$, we get

$$b' = -\frac{b+c}{a+c}$$

and

$$2F'(a) = \frac{2a+mb'c}{\sqrt{a^2+mbc+3k}} + \frac{2bb'+mc}{\sqrt{b^2+mca+3k}} + \frac{m(b+ab')}{\sqrt{c^2+mab+3k}},$$

$$2(a+c)F'(a) = \frac{2a(a+c)-mc(b+c)}{\sqrt{A}} + \frac{mc(a+c)-2b(b+c)}{\sqrt{B}} - \frac{mc(a-b)}{\sqrt{C}}\ ,$$

where

$$A = a^2+mbc+3k, \qquad B = b^2+mca+3k, \qquad C = c^2+mab+3k.$$

Since $a - b \ge 0$ and $C - B = (b-c)(ma-b-c) \ge 0$, we have

$$2(a+c)F'(a) \ge \frac{2a(a+c)-mc(b+c)}{\sqrt{A}} + \frac{mc(a+c)-2b(b+c)}{\sqrt{B}} - \frac{mc(a-b)}{\sqrt{B}}$$

$$= \frac{2a(a+c)-mc(b+c)}{\sqrt{A}} + \frac{mc(b+c)-2b(b+c)}{\sqrt{B}}$$

$$\ge \frac{2a(b+c)-mc(b+c)}{\sqrt{A}} + \frac{mc(b+c)-2b(b+c)}{\sqrt{B}}\ ,$$

hence

$$\frac{2(a+c)}{b+c}F'(a) \ge \frac{2a-mc}{\sqrt{A}} + \frac{mc-2b}{\sqrt{B}} = \frac{2(a\sqrt{B}-b\sqrt{A})+mc(\sqrt{A}-\sqrt{B})}{\sqrt{AB}}.$$

Clearly, we have $F'(a) \ge 0$ if

$$\frac{2(a^2B-b^2A)}{a\sqrt{B}+b\sqrt{A}} + \frac{mc(A-B)}{\sqrt{A}+\sqrt{B}} \ge 0.$$

Since

$$a^2B - b^2A = mc(a^3-b^3) + 3k(a^2-b^2) \ge mc(a-b)(a^2+ab+b^2)$$

and $A-B = a^2-b^2+mbc-mca = (a-b)(a+b-mc)$, we need to show that

$$\frac{2(a^2+ab+b^2)}{a\sqrt{B}+b\sqrt{A}} + \frac{a+b-mc}{\sqrt{A}+\sqrt{B}} \ge 0,$$

that is

$$C\sqrt{A} + D\sqrt{B} \ge 0,$$

where

$$C = 2(a^2+ab+b^2) + b(a+b-mc), \qquad D = 2(a^2+ab+b^2) + a(a+b-mc).$$

Since

$$C) > 2(a^2+ab+b^2) + b(a+b-8b) = 2a^2+3ab-5b^2 = (a-b)(2a+5b) \ge 0\ ,$$

$$D > 2(a^2+ab+b^2) + a(a+b-8b) = 3a^2-5ab+2b^2 = (a-b)(3a-2b) \ge 0\ ,$$

the inequality is true. Finally, we need to show that $F(m_1) \ge 3(1+k)$. Since $a = m_1$ when $a = b$, it suffices to prove the homogeneous inequality

$$\sqrt{(a+kb)(a+kc)} + \sqrt{(b+kc)(b+ka)} + \sqrt{(c+ka)(c+kb)} \ge (1+k)\sqrt{3(ab+bc+ca)}$$

for $a = b = 1$ and $0 \le c < 1$. So, we need to show that

$$2\sqrt{(1+k)(1+kc)} + c + k \ge (1+k)\sqrt{3(2c+1)}\ .$$

By squaring, the inequality becomes

$$c^2 - 2(k^2+3k+3)c - 2k^2 - 2k + 1 + 4(c+k)\sqrt{(k+1)(kc+1)} \ge 0.$$

Using the substitution $kc+1 = (k+1)x^2$, hence

$$c = \frac{(k+1)x^2-1}{k}, \qquad x \ge \frac{1}{\sqrt{k+1}},$$

we need to show that

$$(k+1)^2[x^4+4kx^3-2(k+1)^2x^2+4k(k-1)x+1+4k-2k^2]\ge 0,$$

that is equivalent to

$$(k+1)^2(x-1)^2f(x)\ge 0,$$

where $f(x)=x^2+2(2k+1)x+1+4k-2k^2$ is an increasing function. So, it suffices to show that

$$f\left(\frac{1}{\sqrt{k+1}}\right)\ge 0.$$

Since $2\sqrt{1+k}=(\sqrt{3}-1)k+\sqrt{3}$ and $2k^2-2(1+\sqrt{3})k-2-\sqrt{3}=0$, we have

$$f\left(\frac{1}{\sqrt{k+1}}\right)=\frac{1}{k+1}+\frac{2(2k+1)}{\sqrt{k+1}}+1+4k-2k^2$$

$$=\frac{2(2k+1)\sqrt{k+1}+2+5k+2k^2-2k^3}{\sqrt{k+1}}=\frac{(2k+1)[(\sqrt{3}-1)k+\sqrt{3}]+2+5k+2k^2-2k^3}{\sqrt{k+1}}$$

$$=\frac{2+\sqrt{3}+(4+3\sqrt{3})k+2\sqrt{3}k^2-2k^3}{\sqrt{k+1}}=\frac{(1+k)[2+\sqrt{3}+2(1+\sqrt{3})k-2k^2]}{\sqrt{k+1}}=0.$$

For $k=k_0$, the equality holds when $a=b=c=1$, and also when $a=b=\sqrt{3}$ and $c=0$ (or any cyclic permutation).

□

P 2.120. *If a,b,c are nonnegative real numbers such that $ab+bc+ca=3$, then*

$$\sqrt{\frac{2a+1}{3}}+\sqrt{\frac{2b+1}{3}}+\sqrt{\frac{2c+1}{3}}\ge 3.$$

(Vasile Cîrtoaje, 2008)

Solution. Assume that $a\ge b\ge c$. For fixed c, we may consider that a is a function of b. By differentiating the equality constraint, we get

$$(b+c)a'+a+c=0.$$

Writing the desired inequality as $E\ge 3$, we have

$$E'(b)=\frac{a'}{\sqrt{3(2a+1)}}+\frac{1}{\sqrt{3(2b+1)}}=\frac{-(a+c)}{(b+c)\sqrt{3(2a+1)}}+\frac{1}{\sqrt{3(2b+1)}}.$$

We show that $E'(b)\le 0$, i.e.

$$\frac{a+c}{b+c}\ge\sqrt{\frac{2a+1}{2b+1}}.$$

By Bernoulli's inequality, we have

$$\sqrt{\frac{2a+1}{2b+1}} = \sqrt{1+\frac{2(a-b)}{2b+1}} \le 1+\frac{a-b}{2b+1} = \frac{a+b+1}{2b+1},$$

So, it suffices to show that

$$\frac{a+c}{b+c} \ge \frac{a+b+1}{2b+1},$$

which is equivalent to the obvious inequality

$$(a-b)(1+b-c) \ge 0.$$

Since $E'(b) \le 0$, $E(b)$ is decreasing, hence it is minimum for $b=a$. Thus, we only need to prove the desired inequality for $a=b \ge c$. From $ab+bc+ca=3$, it follows that

$$c = \frac{3-b^2}{2b} \qquad 1 \le b \le \sqrt{3}.$$

Thus, the inequality can be written as

$$2\sqrt{1+2b} + \sqrt{1+\frac{3-b^2}{b}} \ge 3\sqrt{3}.$$

Substituting

$$t = \sqrt{\frac{1+2b}{3}}, \qquad 1 \le t \le \sqrt{\frac{1+2\sqrt{3}}{3}} < \frac{5}{4},$$

the inequality turns into

$$\sqrt{\frac{3+4t^2-3t^4}{2(3t^2-1)}} \ge 3-2t.$$

By squaring, we need to show that

$$7-8t-14t^2+24t^3-9t^4 \ge 0,$$

which is equivalent to

$$(1-t)^2(7+6t-9t^2) \ge 0.$$

This is true since

$$7+6t-9t^2 = 8-(3t-1)^2 > 8-\left(\frac{15}{4}-1\right)^2 = \frac{7}{16} > 0.$$

The equality occurs for $a=b=c=1$.

Remark. Similarly, we can prove the following generalization:

- *Let a,b,c be nonnegative real numbers such that $ab+bc+ca=3$. If*

$$k \ge k_0 = \frac{14\sqrt{3}-15}{24} \approx 0.38536,$$

then

$$\sqrt{\frac{a+k}{1+k}}+\sqrt{\frac{b+k}{1+k}}+\sqrt{\frac{c+k}{1+k}}\geq 3,$$

with equality for $a=b=c=1$. If $k=k_0$, then the equality also occurs for $a=b=\sqrt{3}$ and $c=0$ (or any cyclic permutation).

□

P 2.121. *If a,b,c are the lengths of the sides of a triangle such that $a+b+c=2$, then*

$$\sqrt{\frac{a}{a^2+bc}}+\sqrt{\frac{b}{b^2+ca}}+\sqrt{\frac{c}{c^2+ab}}\geq 2.$$

(Vasile Cîrtoaje, Math. Reflections, 2, 2024)

First Solution. Assume that $a\geq b\geq c$ and write the inequality in the homogeneous form

$$\sqrt{x}+\sqrt{y}+\sqrt{z}\geq 2,$$

where

$$x=\frac{a(a+b+c)}{2(a^2+bc)},\qquad y=\frac{b(a+b+c)}{2(b^2+ca)},\qquad z=\frac{c(a+b+c)}{2(c^2+ab)}.$$

Since

$$1-x=\frac{(a-b)(a-c)+bc}{2(a^2+bc}\geq 0,$$

$$1-y=\frac{c(2a-b)-b(a-b)}{2(b^2+ca)}\geq\frac{(a-b)(2a-b)-b(a-b)}{2(b^2+ca)}=\frac{2(a-b)^2}{2(b^2+ca)}\geq 0,$$

$$1-z=\frac{(a-c)(b-c)+ab}{2(c^2+ab)}>0,$$

we have $\sqrt{x}\geq x$, $\sqrt{y}\geq y$, $\sqrt{z}\geq z$, therefore it suffices to show that

$$x+y+z\geq 2,$$

that is

$$\frac{a}{a^2+bc}+\frac{b}{b^2+ca}+\frac{c}{c^2+ab}\geq\frac{4}{a+b+c}.$$

By the Cauchy-Schwarz inequality, we have

$$\frac{a}{a^2+bc}+\frac{b}{b^2+ca}+\frac{c}{c^2+ab}\geq\frac{(a+b+c)^2}{a(a^2+bc)+b(b^2+ca)+c(c^2+ab)}.$$

Thus, it is enough to prove that

$$(a+b+c)^3\geq 4(a^3+b^3+c^3)+12abc.$$

Substituting $a = y + z$, $b = z + x$ and $c = x + y$, where $x, y, z \ge 0$, we get

$$(a+b+c)^3 - 4(a^3+b^3+c^3) - 12abc = 24xyz \ge 0.$$

The equality occurs for a degenerate triangle with $a = b = 1$ and $c = 0$ (or any cyclic permutation).

Second Solution (by *Mohamed Amine*). From $b + c \ge a$, we get

$$2(b+c) \ge a+b+c = 2, \qquad b+c \ge 1.$$

Since

$$\sqrt{\frac{a}{a^2+bc}} \ge \sqrt{\frac{a}{(b+c)(a^2+bc)}} = \frac{a}{\sqrt{a(b+c)\cdot(a^2+bc)}} \ge \frac{2a}{a(b+c)+(a^2+bc)}$$

$$= \frac{a(a+b+c)}{a^2+ab+bc+ca} = 1 - \frac{bc}{a^2+ab+bc+ca} \ge 1 - \frac{bc}{ab+bc+ca} = \frac{a(b+c)}{ab+bc+ca},$$

we have

$$\sum \sqrt{\frac{a}{a^2+bc}} \ge \sum \frac{a(b+c)}{ab+bc+ca} = 2.$$

Third Solution (by *Theofilos Kanakaris*). Let $q = ab + bc + ca$. From $a \le b + c$, we get

$$a^2 \le a(b+c), \qquad a^2+bc \le q, \qquad \sqrt{\frac{a}{a^2+bc}} \ge \sqrt{\frac{a}{q}},$$

$$\sqrt{\frac{a}{a^2+bc}} + \sqrt{\frac{b}{b^2+ca}} + \sqrt{\frac{c}{c^2+ab}} \ge \frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{\sqrt{q}}.$$

Thus, it suffices to show that

$$\sqrt{a}+\sqrt{b}+\sqrt{c} \ge 2\sqrt{q},$$

i.e.

$$(\sqrt{a}+\sqrt{b}+\sqrt{c})^2 \ge 4q,$$

By Holder's inequality, we have

$$(\sqrt{a}+\sqrt{b}+\sqrt{c})(\sqrt{a}+\sqrt{b}+\sqrt{c})(a^2+b^2+c^2) \ge (a+b+c)^3 = 8,$$

i.e.

$$(\sqrt{a}+\sqrt{b}+\sqrt{c})^2 \ge \frac{8}{a^2+b^2+c^2} = \frac{4}{2-q}.$$

So,

$$(\sqrt{a}+\sqrt{b}+\sqrt{c})^2 - 4q \ge \frac{4}{2-q} - 4q = \frac{4(q-1)^2}{2-q} \ge 0.$$

Remark. Actually, the original inequality holds for any positive real numbers a, b, c such that $a + b + c = 2$ (*Pham Kim Hung*, 2005).

□

P 2.122. *For given $n \ge 3$, prove that $\dfrac{2n-1}{n-1}$ is the least positive value of the constant k such that*

$$\sum_{cyclic} \sqrt{\frac{a_2+a_3+\cdots+a_n}{ka_1+a_2+a_3+\cdots+a_n}} \ge n\sqrt{\frac{n-1}{k+n-1}}$$

holds for any nonnegative real numbers $a_1, a_2, \ldots, a_n$ with $a_1+a_2+\cdots+a_n > 0$.

(Vasile Cîrtoaje, Math. Reflections, 1, 2025)

Solution. For $a_2 = \cdots = a_n = 0$, the inequality becomes $n-1 \ge n\sqrt{\dfrac{n-1}{k+n-1}}$, which is equivalent to $k \ge \dfrac{2n-1}{n-1}$. To show that $\dfrac{2n-1}{n-1}$ is the least value of the constant k, we need to prove the inequality

$$\sum_{cyc} \sqrt{\frac{a_2+a_3+\cdots+a_n}{(2n-1)a_1+(n-1)(a_2+a_3+\cdots+a_n)}} \ge \sqrt{n-1}.$$

Due to homogeneity, we may assume that $a_1+a_2+\cdots+a_n = n$. The requested inequality becomes

$$\sum_{cyc} \sqrt{\frac{n-a_1}{a_1+n-1}} \ge \sqrt{n(n-1)}, \qquad \left(\sum_{cyc} \sqrt{\frac{n-a_1}{a_1+n-1}}\right)^2 \ge n(n-1).$$

By Horner's inequality, we have

$$\left(\sum_{cyc} \sqrt{\frac{n-a_1}{a_1+n-1}}\right)^2 \sum_{cyc}(n-a_1)^2(a_1+n-1) \ge \left[\sum_{cyc}(n-a_i)\right]^3 = n^3(n-1)^3.$$

So, it suffices to show that

$$n^2(n-1)^2 \ge \sum_{cyc}(n-a_1)^2(a_1+n-1),$$

that is

$$(n+1)\sum_{cyc} a_1^2 \ge \sum_{cyc} a_1^3 + n^2.$$

For $n = 3$, the inequality reduces to the well-known inequality

$$(a_1+a_2+a_3)(a_1a_2+a_2a_3+a_3a_1) \ge 9a_1a_2a_3.$$

For $n \ge 4$, assume that $a_1 = \max\{a_1, a_2, \ldots, a_n\}$ and write the inequality as follows:

$$\sum_{cyc}(-a_1^3+(n+1)a_1^2-n) \ge 0,$$

$$\sum_{cyc}[-a_1^3+(n+1)a_1^2-(2n-1)a_1+n-1]\geq 0,$$

$$\sum_{cyc}(1-a_1)^2(n-1-a_1)\geq 0.$$

For $a_1 \leq n-1$, the inequality is clearly true. For $a_1 \geq n-1$, from $a_1+a_2+\cdots+a_n=n$ it follows that $a_1 \in [n-1,n]$ and $a_2,\ldots,a_n \leq 1$. So,

$$(n+1)\sum_{cyc}a_1^2-\sum_{cyc}a_1^3-n^2 \geq (n+1)a_1^2-a_1^3-n^2+a_2^2(1-a_2)+\cdots+a_n^2(1-a_n)\geq (n+1)a_1^2-a_1^3-n^2$$

$$=(n-a_1)(a_1^2-a_1-n)=(n-a_1)[a_1(a_1-n+1)+(n-2)(a_1-2)+n-4]\geq 0.$$

The proof is completed. The equality occurs when $a_1=a_2=\cdots=a_n$, and also when $n-1$ numbers a_i are equal to 0.

□

Chapter 3

Symmetric Power-Exponential Inequalities

3.1 Applications

3.1. If a, b are positive real numbers such that $a+b=a^4+b^4$, then

$$a^a b^b \le 1 \le a^{a^3} b^{b^3}.$$

3.2. If a, b are positive real numbers, then

$$a^{2a}+b^{2b} \ge a^{a+b}+b^{a+b}.$$

3.3. If a, b are positive real numbers, then

$$a^a+b^b \ge a^b+b^a.$$

3.4. If a, b are positive real numbers, then

$$a^{2a}+b^{2b} \ge a^{2b}+b^{2a}.$$

3.5. If a, b are nonnegative real numbers such that $a+b=2$, then

(a) $$a^b+b^a \le 1+ab;$$

(b) $$a^{2b}+b^{2a} \le 1+ab.$$

3.6. If a, b are nonnegative real numbers such that $\frac{2}{3} \le a + b \le 2$, then

$$a^{2b} + b^{2a} \le 1 + ab.$$

3.7. If a, b are nonnegative real numbers such that $a^2 + b^2 = 2$, then

$$a^{2b} + b^{2a} \le 1 + ab.$$

3.8. If a, b are nonnegative real numbers such that $a^2 + b^2 = \frac{1}{4}$, then

$$a^{2b} + b^{2a} \le 1 + ab.$$

3.9. If a, b are positive real numbers, then

$$a^a b^b \le (a^2 - ab + b^2)^{(a+b)/2}.$$

3.10. If $a, b \in (0, 1]$, then

$$a^a b^b \le 1 - ab + a^2 b^2.$$

3.11. If a, b are positive real numbers such that $a + b \le 2$, then

$$\left(\frac{a}{b}\right)^b + \left(\frac{b}{a}\right)^a \le 2.$$

3.12. If a, b are positive real numbers such that $a + b = 2$, then

$$2a^a b^b \ge a^{2b} + b^{2a} + \frac{3}{4}(a - b)^2.$$

3.13. If $a, b \in (0, 1]$ or $a, b \in [1, \infty)$, then

$$2a^a b^b \ge a^2 + b^2.$$

3.14. If a, b are positive real numbers, then

$$2a^a b^b \ge a^2 + b^2.$$

3.15. If $a \ge 1 \ge b > 0$, then

$$2a^ab^b \ge a^{2b} + b^{2a}.$$

3.16. If $a \ge e \ge b > 0$, then

$$2a^ab^b \ge a^{2b} + b^{2a}.$$

3.17. If a, b are positive real numbers, then

$$a^ab^b \ge \left(\frac{a^2+b^2}{2}\right)^{(a+b)/2}.$$

3.18. If a, b are positive real numbers such that $a^2 + b^2 = 2$, then

$$2a^ab^b \ge a^{2b} + b^{2a} + \frac{1}{2}(a-b)^2.$$

3.19. If $a, b \in (0, 1]$, then

$$(a^2+b^2)\left(\frac{1}{a^{2a}} + \frac{1}{b^{2b}}\right) \le 4.$$

3.20. If a, b are positive real numbers such that $a + b = 2$, then

$$a^bb^a + 2 \ge 3ab.$$

3.21. Let a, b be positive real numbers such that $a + b = 2$. If $k \ge \frac{1}{2}$, then

$$a^{a^{kb}}b^{b^{ka}} \ge 1.$$

3.22. If a, b are positive real numbers such that $a + b = 2$, then

$$a^{\sqrt{a}}b^{\sqrt{b}} \ge 1.$$

3.23. If a, b are positive real numbers such that $a + b = 2$, then

$$a^{a+1}b^{b+1} \le 1 - \frac{1}{48}(a-b)^4.$$

3.24. If a, b are positive real numbers such that $a+b=2$, then

$$a^{-a}+b^{-b} \le 2.$$

3.25. If $a, b \in [0,1]$, then

$$a^{b-a}+b^{a-b}+(a-b)^2 \le 2.$$

3.26. If a, b are nonnegative real numbers such that $a+b \le 2$, then

$$a^{b-a}+b^{a-b}+\frac{7}{16}(a-b)^2 \le 2.$$

3.27. If a, b are nonnegative real numbers such that $a+b \le 4$, then

$$a^{b-a}+b^{a-b} \le 2.$$

3.28. If a, b are nonnegative real numbers such that $a+b=2$, then

$$a^{2b}+b^{2a} \ge a^b+b^a \ge a^2b^2+1.$$

3.29. If a, b are positive real numbers such that $a+b=2$, then

$$a^{3b}+b^{3a} \le 2.$$

3.30. If a, b are nonnegative real numbers such that $a+b=2$, then

$$a^{3b}+b^{3a}+\left(\frac{a-b}{2}\right)^4 \le 2.$$

3.31. If a, b are positive real numbers such that $a+b=2$, then

$$a^{\frac{2}{a}}+b^{\frac{2}{b}} \le 2.$$

3.32. If a, b are positive real numbers such that $a+b=2$, then

$$a^{\frac{3}{a}}+b^{\frac{3}{b}} \ge 2.$$

3.33. If a, b are positive real numbers such that $a + b = 2$, then

$$a^{5b^2} + b^{5a^2} \le 2.$$

3.34. If a, b are positive real numbers such that $a + b = 2$, then

$$a^{2\sqrt{b}} + b^{2\sqrt{a}} \le 2.$$

3.35. If a, b are nonnegative real numbers such that $a + b = 2$, then

$$\frac{ab(1-ab)^2}{2} \le a^{b+1} + b^{a+1} - 2 \le \frac{ab(1-ab)^2}{3}.$$

3.36. If a, b are nonnegative real numbers such that $a + b = 1$, then

$$a^{2b} + b^{2a} \le 1.$$

3.37. If a, b are positive real numbers such that $a + b = 1$, then

$$2a^a b^b \ge a^{2b} + b^{2a}.$$

3.38. If a, b are positive real numbers such that $a + b = 1$, then

$$a^{-2a} + b^{-2b} \le 4.$$

3.39. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 a_2 \cdots a_n = 1$, then

$$\left(1 - \frac{1}{n}\right)^{a_1} + \left(1 - \frac{1}{n}\right)^{a_2} + \cdots + \left(1 - \frac{1}{n}\right)^{a_n} \le n - 1.$$

3.2 Solutions

P 3.1. *If a, b are positive real numbers such that $a + b = a^4 + b^4$, then*

$$a^a b^b \le 1 \le a^{a^3} b^{b^3}.$$

(Vasile Cîrtoaje, 2008)

Solution. We will use the inequality

$$\ln x \le x - 1, \quad x > 0.$$

To prove this inequality, let us denote

$$f(x) = x - 1 - \ln x, \quad x > 0.$$

From

$$f'(x) = \frac{x-1}{x},$$

it follows that $f(x)$ is decreasing on $(0, 1]$ and increasing on $[1, \infty)$. Therefore,

$$f(x) \ge f(1) = 0.$$

Using this inequality, we have

$$\ln a^a b^b = a \ln a + b \ln b \le a(a-1) + b(b-1) = a^2 + b^2 - (a+b).$$

Therefore, the left inequality $a^a b^b \le 1$ is true if $a^2 + b^2 \le a + b$. We write this inequality in the homogeneous form

$$(a^2 + b^2)^3 \le (a+b)^2(a^4 + b^4),$$

which is equivalent to the obvious inequality

$$ab(a-b)(a^3 - b^3) \ge 0.$$

Taking now $x = \dfrac{1}{a}$ in the inequality $\ln x \le x - 1$ yields

$$a \ln a \ge a - 1.$$

Similarly,

$$b \ln b \ge b - 1,$$

hence

$$\ln a^{a^3} b^{b^3} = a^3 \ln a + b^3 \ln b \ge a^2(a-1) + b^2(b-1) = a^3 + b^3 - (a^2 + b^2).$$

Thus, to prove the right inequality $a^{a^3} b^{b^3} \ge 1$, it suffices to show that $a^3 + b^3 \ge a^2 + b^2$, which is equivalent to the homogeneous inequality

$$(a+b)(a^3 + b^3)^3 \ge (a^4 + b^4)(a^2 + b^2)^3.$$

We can write this inequality as

$$A-3B\ge 0,$$

where

$$A=(a+b)(a^9+b^9)-(a^4+b^4)(a^6+b^6),$$
$$B=a^2b^2(a^2+b^2)(a^4+b^4)-a^3b^3(a+b)(a^3+b^3).$$

Since

$$A=ab(a^3-b^3)(a^5-b^5),\quad B=a^2b^2(a-b)(a^5-b^5),$$

we get

$$A-3B=ab(a-b)^3(a^5-b^5)\ge 0.$$

Both inequalities become equalities for $a=b=1$.

□

P 3.2. *If a,b are positive real numbers, then*

$$a^{2a}+b^{2b}\ge a^{a+b}+b^{a+b}.$$

(Vasile Cîrtoaje, 2010)

Solution. Assume that $a\ge b$ and consider the following two cases.

Case 1: $a\ge 1$. Write the inequality as

$$a^{a+b}(a^{a-b}-1)\ge b^{2b}(b^{a-b}-1).$$

For $b\le 1$, we have

$$a^{a+b}(a^{a-b}-1)\ge 0\ge b^{2b}(b^{a-b}-1).$$

For $b\ge 1$, the inequality is also true since

$$a^{a+b}\ge a^{2b}\ge b^{2b},\quad a^{a-b}-1\ge b^{a-b}-1\ge 0.$$

Case 2: $a\le 1$. Since

$$a^{2a}+b^{2b}\ge 2a^ab^b,$$

it suffices to show that

$$2a^ab^b\ge a^{a+b}+b^{a+b},$$

which can be written as

$$\left(\frac{a}{b}\right)^b+\left(\frac{b}{a}\right)^a\le 2.$$

By Bernoulli's inequality, we get

$$\left(\frac{a}{b}\right)^b+\left(\frac{b}{a}\right)^a=\left(1+\frac{a-b}{b}\right)^b+\left(1+\frac{b-a}{a}\right)^a\le 1+\frac{b(a-b)}{b}+1+\frac{a(b-a)}{a}=2.$$

The equality holds for $a = b$.

Conjecture 1. *If a, b are positive real numbers, then*

$$a^{4a} + b^{4b} \geq a^{2a+2b} + b^{2a+2b}.$$

Conjecture 2. *If a, b, c are positive real numbers, then*

$$a^{3a} + b^{3b} + c^{3c} \geq a^{a+b+c} + b^{a+b+c} + c^{a+b+c}.$$

Conjecture 3. *If a, b, c, d are positive real numbers, then*

$$a^{4a} + b^{4b} + c^{4c} + d^{4d} \geq a^{a+b+c+d} + b^{a+b+c+d} + c^{a+b+c+d} + d^{a+b+c+d}.$$

□

P 3.3. *If a, b are positive real numbers, then*

$$a^a + b^b \geq a^b + b^a.$$

(M. Laub, Israel, 1985, AMM)

Solution. Assume that $a \geq b$. We will show that if $a \geq 1$, then the inequality is true. From

$$a^{a-b} \geq b^{a-b},$$

we get

$$b^b \geq \frac{a^b b^a}{a^a}.$$

Therefore,

$$a^a + b^b - a^b - b^a \geq a^a + \frac{a^b b^a}{a^a} - a^b - b^a = \frac{(a^a - a^b)(a^a - b^a)}{a^a} \geq 0.$$

Consider further the case $0 < b \leq a < 1$.

First Solution. Denoting

$$c = a^b, \quad d = b^b, \quad k = \frac{a}{b},$$

where $c \geq d$ and $k \geq 1$, the inequality becomes

$$c^k - d^k \geq c - d.$$

Since the function $f(x) = x^k$ is convex for $x \geq 0$, from the well-known inequality

$$f(c) - f(d) \geq f'(d)(c - d),$$

we get

$$c^k - d^k \geq k d^{k-1}(c - d).$$

Thus, it suffices to show that

$$kd^{k-1} \geq 1,$$

which is equivalent to

$$b^{1-a+b} \leq a.$$

Indeed, since $0 < 1 - a + b \leq 1$, by Bernoulli's inequality, we get

$$b^{1-a+b} = [1 + (b-1)]^{1-a+b} \leq 1 + (1 - a + b)(b - 1) = a - b(a - b) \leq a.$$

The equality holds for $a = b$.

Second Solution. Denoting

$$c = \frac{b^a}{a^b + b^a}, \quad d = \frac{a^b}{a^b + b^a}, \quad k = \frac{a}{b},$$

where $c + d = 1$ and $k \geq 1$, the inequality becomes

$$ck^a + dk^{-b} \geq 1.$$

By the weighted AM-GM inequality, we have

$$ck^a + dk^{-b} \geq k^{ac} \cdot k^{-bd} = k^{ac-bd}.$$

Thus, it suffices to show that $ac \geq bd$; that is,

$$a^{1-b} \geq b^{1-a},$$

which is equivalent to $f(a) \geq f(b)$, where

$$f(x) = \frac{\ln x}{1 - x}.$$

It is enough to prove that $f(x)$ is an increasing function. Since

$$f'(x) = \frac{g(x)}{(1-x)^2}, \quad g(x) = \frac{1}{x} - 1 + \ln x.$$

we need to show that $g(x) \geq 0$ for $x \in (0, 1)$. Indeed, from

$$g'(x) = \frac{x - 1}{x^2} < 0,$$

it follows that $g(x)$ is strictly decreasing, hence $g(x) > g(1) = 0$.

□

P 3.4. *If a, b are positive real numbers, then*

$$a^{2a} + b^{2b} \geq a^{2b} + b^{2a}.$$

Solution. Without loss of generality, assume that $a > b$. We have two cases to consider: $a \ge 1$ and $0 < b < a < 1$.

Case 1: $a \ge 1$. From

$$a^{2(a-b)} \ge b^{2(a-b)},$$

we get

$$b^{2b} \ge \frac{a^{2b}b^{2a}}{a^{2a}}.$$

Therefore,

$$a^{2a} + b^{2b} - a^{2b} - b^{2a} \ge a^{2a} + \frac{a^{2b}b^{2a}}{a^{2a}} - a^{2b} - b^{2a} = \frac{(a^{2a} - a^{2b})(a^{2a} - b^{2a})}{a^{2a}} \ge 0$$

because $a^{2a} \ge a^{2b}$ and $a^{2a} \ge b^{2a}$.

Case 2: $0 < b < a < 1$. Denoting

$$c = a^b, \quad d = b^b, \quad k = \frac{a}{b},$$

where $c > d$ and $k > 1$, the inequality becomes

$$c^{2k} - d^{2k} \ge c^2 - d^2.$$

We will show that

$$c^{2k} - d^{2k} > k(cd)^{k-1}(c^2 - d^2) > c^2 - d^2.$$

The left inequality follows from Lemma below for $x = (c/d)^2$. The right inequality is equivalent to

$$k(cd)^{k-1} > 1,$$

$$(ab)^{a-b} > \frac{b}{a},$$

$$\frac{1+a-b}{1-a+b}\ln a > \ln b.$$

For fixed a, let us define

$$f(b) = \frac{1+a-b}{1-a+b}\ln a - \ln b.$$

If $f'(b) < 0$, then $f(b)$ is strictly decreasing, and hence $f(b) > f(a) = 0$. Since

$$f'(b) = \frac{-2}{(1-a+b)^2}\ln a - \frac{1}{b},$$

we need to show that $g(a) > 0$, where

$$g(a) = 2\ln a + \frac{(1-a+b)^2}{b}.$$

From
$$g'(a)=\frac{2}{a}-\frac{2(1-a+b)}{b}=\frac{2(a-b)(a-1)}{ab}<0,$$
it follows that $g(a)$ is strictly decreasing, therefore $g(a)>g(1)=b>0$. This completes the proof. The equality holds for $a=b$.

Lemma. *Let k and x be positive real numbers. If either $k>1$ and $x\ge 1$, or $0<k<1$ and $0<x\le 1$, then*
$$x^k-1\ge kx^{\frac{k-1}{2}}(x-1).$$

Proof. We need to show that $f(x)\ge 0$, where
$$f(x)=x^k-1-kx^{\frac{k-1}{2}}(x-1).$$

We have
$$f'(x)=\frac{1}{2}kx^{\frac{k-3}{2}}g(x),\quad g(x)=2x^{\frac{k+1}{2}}-(k+1)x+k-1.$$

Since
$$g'(x)=(k+1)\left(x^{\frac{k-1}{2}}-1\right)\ge 0,$$
$g(x)$ is increasing. If $x\ge 1$, then $g(x)\ge g(1)=0$, $f(x)$ is increasing, hence $f(x)\ge f(1)=0$. If $0<x\le 1$, then $g(x)\le g(1)=0$, $f(x)$ is decreasing, hence $f(x)\ge f(1)=0$. The equality holds for $x=1$.

Remark. The following more general results are valid (*Vasile Cîrtoaje*, 2006):

- *Let $0<k\le e$.*
  (a) *If $a,b>0$, then*
  $$a^{ka}+b^{kb}\ge a^{kb}+b^{ka};$$
  (b) *If $a,b\in(0,1]$, then*
  $$2\sqrt{a^{ka}b^{kb}}\ge a^{kb}+b^{ka}.$$

Notice that these inequalities are known as the first and the second *Vasc's power exponential inequalities.*

Conjecture 1. *If $0<k\le e$ and either $a,b\in(0,4]$ or $0<a\le 1\le b$, then*
$$2\sqrt{a^{ka}b^{kb}}\ge a^{kb}+b^{ka}.$$

Conjecture 2. *If $0<a\le 1\le b$, then*
$$2\sqrt{a^{3a}b^{3b}}\ge a^{3b}+b^{3a}.$$

Conjecture 3. *If $a,b\in(0,5]$, then*
$$2a^ab^b\ge a^{2b}+b^{2a}.$$

Conjecture 4. *If $a,b\in[0,5]$, then*
$$\left(\frac{a^2+b^2}{2}\right)^{(a+b)/2}\ge a^{2b}+b^{2a}.$$

□

P 3.5. *If a, b are nonnegative real numbers such that $a + b = 2$, then*

(a) $$a^b + b^a \le 1 + ab;$$

(b) $$a^{2b} + b^{2a} \le 1 + ab.$$

Solution. Without loss of generality, assume that $a \ge b$. Since

$$0 \le b \le 1, \qquad 0 \le a - 1 \le 1,$$

by Bernoulli's inequality, we have

$$a^b \le 1 + b(a-1) = 1 + b - b^2$$

and

$$b^a = b \cdot b^{a-1} \le b[1 + (a-1)(b-1)] = b^2(2-b).$$

(a) We have

$$a^b + b^a - 1 - ab \le (1 + b - b^2) + b^2(2-b) - 1 - (2-b)b = -b(b-1)^2 \le 0.$$

The equality holds for $a = b = 1$, for $a = 2$ and $b = 0$, and for $a = 0$ and $b = 2$.

(b) We have

$$\begin{aligned} a^{2b} + b^{2a} - 1 - ab &\le (1 + b - b^2)^2 + b^4(2-b)^2 - 1 - (2-b)b \\ &= b^3(b-1)^2(b-2) = -ab^3(b-1)^2 \le 0. \end{aligned}$$

The equality holds for $a = b = 1$, for $a = 2$ and $b = 0$, and for $a = 0$ and $b = 2$.

□

P 3.6. *If a, b are nonnegative real numbers such that $\dfrac{2}{3} \le a + b \le 2$, then*

$$a^{2b} + b^{2a} \le 1 + ab.$$

(Vasile Cîrtoaje, 2007)

Solution. Assume that

$$a \ge b.$$

From $2\sqrt{ab} \le a + b \le 2$, we get

$$ab \le 1.$$

There are two cases to consider: $a + b \le 1$ and $a + b \ge 1$.

Case 1: $\dfrac{2}{3} \le a + b \le 1$. Since $2b \le 1$, by Bernoulli's inequality, we have

$$a^{2b} \le 1 + 2b(a-1) = 1 + 2ab - 2b.$$

Therefore, it suffices to show that

$$(1+2ab-2b)+b^{2a} \le 1+ab,$$

which is equivalent to

$$ab+b^{2a} \le 2b.$$

For $2a \ge 1$, this inequality is true since

$$ab \le b, \qquad b^{2a} \le b.$$

For $2a \le 1$, by Bernoulli's inequality, we have

$$b^{2a} \le 1+2a(b-1) = 1+2ab-2a.$$

Therefore, it suffices to show that

$$(1+2ab-2b)+(1+2ab-2a) \le 1+ab,$$

which is equivalent to

$$1+3ab \le 2(a+b).$$

Indeed, we have

$$\begin{aligned} 4+12ab-8(a+b) &\le 4+3(a+b)^2-8(a+b) \\ &= (a+b-2)[3(a+b)-2] \le 0. \end{aligned}$$

Case 2: $1 \le a+b \le 2$. For $a, b \le 1$, by Bernoulli's inequality, we have

$$a^{2b} = (a^2)^b \le 1+b(a^2-1) = 1-b+a^2b,$$

$$b^{2a} = (b^2)^a \le 1+a(b^2-1) = 1-a+ab^2,$$

hence

$$\begin{aligned} a^{2b}+b^{2a}-1-ab &\le (1-b+a^2b)+(1-a+ab^2)-1-ab \\ &= (1-ab)(1-a-b) \le 0. \end{aligned}$$

Consider further that $a \ge 1 \ge b$. By Bernoulli's inequality, we have

$$a^b \le 1+b(a-1) = ab+1-b,$$

$$\begin{aligned} b^{2a} = b^{a-1} \cdot b^{a+1} \le b^{a+1} = b^2 \cdot b^{a-1} &\le b^2[1+(a-1)(b-1)] \\ &= b^2(ab+2-a-b). \end{aligned}$$

Therefore, it suffices to show that

$$(ab+1-b)^2+b^2(ab+2-a-b) \le 1+ab,$$

which can be written as

$$1 + ab - (ab + 1 - b)^2 \ge b^2(ab + 2 - a - b).$$

Since

$$1 + ab - (ab + 1 - b)^2 = bB,$$

where

$$B = (2 - a - b) + 2ab - a^2b \ge 2ab - a^2b = ab(2 - a),$$

it is enough to prove that

$$ab^2(2 - a) \ge b^2(ab + 2 - a - b),$$

which is equivalent to the obvious inequality

$$b^2(a - 1)(2 - a - b) \ge 0.$$

The equality holds for $a = 0$ or $b = 0$. If $a+b = 2$, then the equality holds also for $a = b = 1$.

Remark. Actually, the following extension is valid:

- *If a, b are nonnegative real numbers such that*

$$\frac{1}{2} \le a + b \le 2,$$

then

$$a^{2b} + b^{2a} \le 1 + ab.$$

□

P 3.7. *If a, b are nonnegative real numbers such that $a^2 + b^2 = 2$, then*

$$a^{2b} + b^{2a} \le 1 + ab.$$

(Vasile Cîrtoaje, 2007)

Solution. Without loss of generality, assume that $a \ge 1 \ge b$. Applying Bernoulli's inequality gives

$$a^b \le 1 + b(a - 1),$$

hence

$$a^{2b} \le (1 + ab - b)^2.$$

Also, since $0 \le b \le 1$ and $2a \ge 2$, we have

$$b^{2a} \le b^2.$$

Therefore, it suffices to show that

$$(1+ab-b)^2+b^2 \le 1+ab,$$

which can be written as

$$b(2+2ab-a-2b-a^2b) \ge 0.$$

So, we need to show that

$$2+2ab-a-2b-a^2b \ge 0,$$

which is equivalent to

$$4(1-a)(1-b)+a(2-2ab) \ge 0,$$

$$4(1-a)(1-b)+a(a-b)^2 \ge 0.$$

Since $a \ge 1$, it suffices to show that

$$4(1-a)(1-b)+(a-b)^2 \ge 0.$$

Indeed,

$$\begin{aligned} 4(1-a)(1-b)+(a-b)^2 &= -4(a-1)(1-b)+[(a-1)+(1-b)]^2 \\ &= [(a-1)-(1-b)]^2 = (a+b-2)^2 \ge 0. \end{aligned}$$

The equality holds for $a=b=1$, for $a=\sqrt{2}$ and $b=0$, and for $a=0$ and $b=\sqrt{2}$.

□

P 3.8. *If a,b are nonnegative real numbers such that $a^2+b^2=\frac{1}{4}$, then*

$$a^{2b}+b^{2a} \le 1+ab.$$

(Vasile Cîrtoaje, 2007)

Solution. From $a^2+b^2=\frac{1}{4}$, it follows that

$$a,b \le \frac{1}{2},$$

$$ab = \frac{1}{2}(a+b)^2-\frac{1}{8},$$

$$a+b \ge \sqrt{a^2+b^2} = \frac{1}{2},$$

$$a+b \le \sqrt{2(a^2+b^2)} = \frac{1}{\sqrt{2}}.$$

Applying Bernoulli's inequality gives

$$a^{2b} \le 1+2b(a-1) = 1-2b+2ab,$$

$$b^{2a} \le 1 + 2a(b-1) = 1 - 2a + 2ab.$$

Thus, it suffices to show that

$$(1 - 2b + 2ab) + (1 - 2a + 2ab) \le 1 + ab,$$

$$1 + 3ab \le 2(a+b),$$

$$1 + \frac{3}{2}(a+b)^2 - \frac{3}{8} \le 2(a+b),$$

$$\left(a + b - \frac{1}{2}\right)\left(a + b - \frac{5}{6}\right) \le 0.$$

The left inequality is true since

$$a + b \le \frac{1}{\sqrt{2}} < \frac{5}{6}.$$

The equality holds for $a = 0$ and $b = \frac{1}{2}$, and for $a = \frac{1}{2}$ and $b = 0$.

Remark. Actually, the following extended result is valid:

- *If a, b are nonnegative real numbers such that*

$$\frac{1}{4} \le a^2 + b^2 \le 2,$$

then

$$a^{2b} + b^{2a} \le 1 + ab.$$

This inequality is a consequence of Remark from P 3.6 (since $\frac{1}{4} \le a^2 + b^2 \le 2$ involves $\frac{1}{2} \le a + b \le 2$).

□

P 3.9. *If a, b are positive real numbers, then*

$$a^a b^b \le (a^2 - ab + b^2)^{(a+b)/2}.$$

Solution. By the weighted AM-GM inequality, we have

$$a \cdot a + b \cdot b \ge (a+b) a^{\frac{a}{a+b}} b^{\frac{b}{a+b}},$$

$$\left(\frac{a^2 + b^2}{a+b}\right)^{a+b} \ge a^a b^b.$$

Thus, it suffices to show that

$$a^2 - ab + b^2 \ge \left(\frac{a^2 + b^2}{a+b}\right)^2,$$

which is equivalent to

$$(a+b)(a^3+b^3) \ge (a^2+b^2)^2,$$

$$ab(a-b)^2 \ge 0.$$

The equality holds for $a = b$.

□

P 3.10. *If $a, b \in (0,1]$, then*

$$a^a b^b \le 1 - ab + a^2b^2.$$

(Vasile Cîrtoaje, 2010)

Solution. We claim that

$$x^x \le 1 - x + x^2$$

for all $x \in (0,1]$. If this is true, then

$$\begin{aligned} 1 - ab + a^2b^2 - a^a b^b &\ge 1 - ab + a^2b^2 - (1-a+a^2)(1-b+b^2) \\ &= (a+b)(1-a)(1-b) \ge 0. \end{aligned}$$

Thus, it suffices to show that $f(x) \le 0$ for $x \in (0,1]$, where

$$f(x) = x\ln x - \ln(x^2 - x + 1).$$

We have

$$f'(x) = \ln x + 1 - \frac{2x-1}{x^2-x+1},$$

$$f''(x) = \frac{(1-x)(1-2x-x^2-x^4)}{x(x^2-x+1)^2}.$$

Let $x_1 \in (0,1)$ be the positive root of the equation $x^4 + x^2 + 2x = 1$. Then, $f''(x) > 0$ for $x \in (0, x_1)$ and $f''(x) < 0$ for $x \in (x_1, 1)$, hence f' is strictly increasing on $(0, x_1]$ and strictly decreasing on $[x_1, 1]$. Since $\lim_{x\to 0} f'(x) = -\infty$ and $f'(1) = 0$, there is $x_2 \in (0, x_1)$ such that $f'(x_2) = 0$, $f'(x) < 0$ for $x \in (0, x_2)$ and $f'(x) > 0$ for $x \in (x_2, 1)$. Therefore, f is decreasing on $(0, x_2]$ and increasing on $[x_2, 1]$. Since $\lim_{x\to 0} f(x) = 0$ and $f(1) = 0$, it follows that $f(x) \le 0$ for $x \in (0,1]$. The proof is completed. The equality holds for $a = b = 1$.

□

P 3.11. *If a, b are positive real numbers such that $a + b \le 2$, then*

$$\left(\frac{a}{b}\right)^b + \left(\frac{b}{a}\right)^a \le 2.$$

(Vasile Cîrtoaje, 2010)

Solution. Using the substitution $a = tc$ and $b = td$, where c, d, t are positive real numbers such that $c + d = 2$ and $t \le 1$, we need to show that

$$\left(\frac{c}{d}\right)^{td} + \left(\frac{d}{c}\right)^{tc} \le 2.$$

Write this inequality as

$$f(t) \le 2,$$

where

$$f(t) = A^t + B^t, \qquad A = \left(\frac{c}{d}\right)^{d}, \quad B = \left(\frac{d}{c}\right)^{c}.$$

Since $f(t)$ is a convex function, we have

$$f(t) \le \max\{f(0), f(1)\} = \max\{2, f(1)\}.$$

Therefore, we only need to show that $f(1) \le 2$; that is,

$$2c^c d^d \ge c^2 + d^2.$$

Setting $c = 1 + x$ and $d = 1 - x$, where $0 \le x < 1$, this inequality turns into

$$(1+x)^{1+x}(1-x)^{1-x} \ge 1 + x^2,$$

which is equivalent to $f(x) \ge 0$, where

$$f(x) = (1+x)\ln(1+x) + (1-x)\ln(1-x) - \ln(1+x^2).$$

We have

$$f'(x) = \ln(1+x) - \ln(1-x) - \frac{2x}{1+x^2},$$

$$f''(x) = \frac{1}{1+x} + \frac{1}{1-x} - \frac{2(1-x^2)}{(1+x^2)^2} = \frac{8x^2}{(1-x^2)(1+x^2)^2}.$$

Since $f''(x) \ge 0$ for $x \in [0, 1)$, it follows that f' is increasing, $f'(x) \ge f'(0) = 0$, $f(x)$ is increasing, hence $f(x) \ge f(0) = 0$. The proof is completed. The equality holds for $a = b$. □

P 3.12. *If a, b are positive real numbers such that $a + b = 2$, then*

$$2a^a b^b \ge a^{2b} + b^{2a} + \frac{3}{4}(a-b)^2.$$

(Vasile Cîrtoaje, 2010)

Solution. According to the inequalities in P 3.5-(b) and P 3.11 (for $a+b=2$), we have

$$a^{2b}+b^{2a}\le 1+ab$$

and

$$2a^ab^b\ge a^2+b^2.$$

Therefore, it suffices to show that

$$a^2+b^2\ge 1+ab+\frac{3}{4}(a-b)^2.$$

which is an identity. The equality holds for $a=b=1$.

□

P 3.13. *If $a,b\in(0,1]$ or $a,b\in[1,\infty)$, then*

$$2a^ab^b\ge a^2+b^2.$$

Solution. For $a=x$ and $b=1$, the desired inequality becomes

$$2x^x\ge x^2+1,\quad x>0.$$

If this inequality is true, then

$$4a^ab^b-2(a^2+b^2)\ge(a^2+1)(b^2+1)-2(a^2+b^2)=(a^2-1)(b^2-1)\ge 0.$$

To prove the inequality $2x^x\ge x^2+1$, we show that $f(x)\ge 0$, where

$$f(x)=\ln 2+x\ln x-\ln(x^2+1).$$

We have

$$f'(x)=\ln x+1-\frac{2x}{x^2+1},$$

$$f''(x)=\frac{x^2(x+1)^2+(x-1)^2}{x(x^2+1)^2}.$$

Since $f''(x)>0$ for $x>0$, f' is strictly increasing. Since $f'(1)=0$, it follows that $f'(x)<0$ for $x\in(0,1)$ and $f'(x)>0$ for $x\in(1,\infty)$. Therefore, f is decreasing on $(0,1]$ and increasing on $[1,\infty)$, hence $f(x)\ge f(1)=0$ for $x>0$. This completes the proof. The equality holds for $a=b=1$.

□

P 3.14. *If a,b are positive real numbers, then*

$$2a^ab^b\ge a^2+b^2.$$

(Vasile Cîrtoaje, 2014)

Solution. By Lemma below, it suffices to show that

$$(a^4 - 2a^3 + 4a^2 - 2a + 3)(b^4 - 2b^3 + 4b^2 - 2b + 3) \geq 8(a^2 + b^2),$$

which is equivalent to $A \geq 0$, where

$$\begin{aligned} A =& a^4b^4 - 2a^3b^3(a+b) + 4a^2b^2(a^2+b^2+ab) - [2ab(a^3+b^3) + 8a^2b^2(a+b)] \\ &+ [3(a^4+b^4) + 4ab(a^2+b^2) + 16a^2b^2] - [6(a^3+b^3) + 8ab(a+b)] \\ &+ 4(a^2+b^2+ab) - 6(a+b) + 9. \end{aligned}$$

We can check that

$$A = [a^2b^2 - ab(a+b) + a^2 + b^2 - 1]^2 + B,$$

where

$$\begin{aligned} B =& a^2b^2(a+b)^2 - 6a^2b^2(a+b) + [2(a^4+b^4) + 4ab(a^2+b^2) + 16a^2b^2] \\ &- [6(a^3+b^3) + 10ab(a+b)] + [6(a^2+b^2) + 4ab] - 6(a+b) + 8. \end{aligned}$$

Also, we have

$$B = [ab(a+b) - 3ab + 1]^2 + C,$$

where

$$\begin{aligned} C =& [2(a^4+b^4) + 4ab(a^2+b^2) + 7a^2b^2] - [6(a^3+b^3) + 12ab(a+b)] \\ &+ [6(a^2+b^2) + 10ab] - 6(a+b) + 7, \end{aligned}$$

and

$$C = (ab-1)^2 + 2D,$$

where

$$\begin{aligned} D =& [a^4+b^4 + 2ab(a^2+b^2) + 3a^2b^2] - [3(a^3+b^3) + 6ab(a+b)] \\ &+ 3(a+b)^2 - 3(a+b) + 3, \end{aligned}$$

It suffices to show that $D \geq 0$. Indeed,

$$\begin{aligned} D =& [(a+b)^4 - 2ab(a+b)^2 + a^2b^2] - 3[(a+b)^3 - ab(a+b)] \\ &+ 3(a+b)^2 - 3(a+b) + 3 \\ =& [(a+b)^2 - ab]^2 - 3(a+b)[(a+b)^2 - ab] + 3(a+b)^2 - 3(a+b) + 3 \\ =& \left[(a+b)^2 - ab - \frac{3}{2}(a+b)\right]^2 + 3\left(\frac{a+b}{2} - 1\right)^2 \geq 0. \end{aligned}$$

This completes the proof. The equality holds for $a = b = 1$.

Lemma. *If $x > 0$, then*

$$x^x \geq x + \frac{1}{4}(x-1)^2(x^2+3).$$

Proof. We need to show that $f(x) \ge 0$ for $x > 0$, where

$$f(x) = \ln 4 + x \ln x - \ln g(x), \quad g(x) = x^4 - 2x^3 + 4x^2 - 2x + 3.$$

We have

$$f'(x) = 1 + \ln x - \frac{2(2x^3 - 3x^2 + 4x - 1)}{g(x)},$$

$$f''(x) = \frac{x^8 + 6x^4 - 32x^3 + 48x^2 - 32x + 9}{g^2(x)} = \frac{(x-1)^2 h(x)}{g^2(x)},$$

where

$$h(x) = x^6 + 2x^5 + 3x^4 + 4x^3 + 11x^2 - 14x + 9.$$

Since

$$h(x) > 7x^2 - 14x + 7 = 7(x-1)^2 \ge 0,$$

we have $f''(x) \ge 0$, hence f' is strictly increasing on $(0, \infty)$. Since $f'(1) = 0$, it follows that $f'(x) < 0$ for $x \in (0, 1)$ and $f'(x) > 0$ for $x \in (1, \infty)$. Therefore, f is decreasing on $(0, 1]$ and increasing on $[1, \infty)$, hence $f(x) \ge f(1) = 0$ for $x > 0$.

□

P 3.15. *If $a \ge 1 \ge b > 0$, then*

$$2a^a b^b \ge a^{2b} + b^{2a}.$$

Solution. Taking into account the inequality $2a^a b^b \ge a^2 + b^2$ from the preceding P 3.14, it suffices to show that

$$a^2 + b^2 \ge a^{2b} + b^{2a}.$$

This inequality follows immediately from $a^2 \ge a^{2b}$ and $b^2 \ge b^{2a}$. The equality holds for $a = b = 1$.

□

P 3.16. *If $a \ge e \ge b > 0$, then*

$$2a^a b^b \ge a^{2b} + b^{2a}.$$

Solution. It suffices to show that $a^a b^b \ge a^{2b}$ and $a^a b^b \ge b^{2a}$. Write the first inequality as

$$a^{a-b} \ge \left(\frac{a}{b}\right)^b,$$

$$a^{x-1} \ge x, \qquad x = \frac{a}{b} \ge 1.$$

Since $a^{x-1} \ge e^{x-1}$, we only need to show that

$$e^{x-1} \ge x,$$

which is equivalent to $f(x) \geq 0$ for $x \geq 1$, where

$$f(x) = x - 1 - \ln x.$$

From

$$f'(x) = 1 - \frac{1}{x} \geq 0,$$

it follows that f is increasing on $[1, \infty)$, therefore $f(x) \geq f(1) = 0$.

Write the second inequality as

$$\left(\frac{b}{a}\right)^a b^{a-b} \leq 1,$$

$$xb^{1-x} \leq 1, \qquad x = \frac{b}{a} \leq 1.$$

Since $b^{1-x} \leq e^{1-x}$, we only need to show that

$$xe^{1-x} \leq 1,$$

which is equivalent to $f(x) \leq 0$ for $x \leq 1$, where

$$f(x) = \ln x + 1 - x.$$

Since

$$f'(x) = \frac{1}{x} - 1 \geq 0,$$

f is increasing on $(0, 1]$, therefore $f(x) \leq f(1) = 0$. This completes the proof. The equality holds for $a = b = e$.

□

P 3.17. *If a, b are positive real numbers, then*

$$a^a b^b \geq \left(\frac{a^2 + b^2}{2}\right)^{(a+b)/2}.$$

First Solution. Using the substitution $a = bx$, where $x > 0$, the inequality becomes as follows:

$$(bx)^{bx} b^b \geq \left(\frac{b^2x^2 + b^2}{2}\right)^{\frac{bx+b}{2}},$$

$$(bx)^x b \geq \left(\frac{b^2x^2 + b^2}{2}\right)^{\frac{x+1}{2}},$$

$$b^{x+1} x^x \geq b^{x+1} \left(\frac{x^2 + 1}{2}\right)^{\frac{x+1}{2}},$$

$$x^x \geq \left(\frac{x^2+1}{2}\right)^{\frac{x+1}{2}}.$$

It is true if $f(x) \geq 0$ for all $x > 0$, where

$$f(x) = \frac{x}{x+1}\ln x - \frac{1}{2}\ln\frac{x^2+1}{2}.$$

We have

$$f'(x) = \frac{1}{(x+1)^2}\ln x + \frac{1}{x+1} - \frac{x}{x^2+1} = \frac{g(x)}{(x+1)^2},$$

where

$$g(x) = \ln x - \frac{x^2-1}{x^2+1}.$$

Since

$$g'(x) = \frac{(x^2-1)^2}{x(x^2+1)^2} \geq 0,$$

g is strictly increasing, therefore $g(x) < 0$ for $x \in (0,1)$, $g(1) = 0$, $g(x) > 0$ for $x \in (1,\infty)$. Thus, f is decreasing on $(0,1]$ and increasing on $[1,\infty)$, hence $f(x) \geq f(1) = 0$. This completes the proof. The equality holds for $a = b$.

Second Solution. Write the inequality in the form

$$a\ln a + b\ln b \geq \frac{a+b}{2}\ln\frac{a^2+b^2}{2}.$$

Without loss of generality, consider $a+b = 2k$, $k > 0$, and denote

$$a = k+x, \quad b = k-x, \quad 0 \leq x < k.$$

We need to show that $f(x) \geq 0$, where

$$f(x) = (k+x)\ln(k+x) + (k-x)\ln(k-x) - k\ln(x^2+k^2).$$

We have

$$f'(x) = \ln(k+x) - \ln(k-x) - \frac{2kx}{x^2+k^2},$$

$$\begin{aligned} f''(x) &= \frac{1}{k+x} + \frac{1}{k-x} + \frac{2k(x^2-k^2)}{(x^2+k^2)^2} \\ &= \frac{8k^2x^2}{(k^2-x^2)(x^2+k^2)^2}. \end{aligned}$$

Since $f''(x) \geq 0$ for $x \geq 0$, f' is increasing, hence $f'(x) \geq f'(0) = 0$. Therefore, f is increasing on $[0,k)$, hence $f(x) \geq f(0) = 0$.

Remark. For $a+b = 2$, this inequality can be rewritten as

$$2a^ab^b \geq a^2+b^2,$$

$$2 \ge \left(\frac{a}{b}\right)^b + \left(\frac{b}{a}\right)^a.$$

Also, for $a + b = 1$, the inequality becomes

$$2a^{2a}b^{2b} \ge a^2 + b^2,$$

$$2 \ge \left(\frac{a}{b}\right)^{2b} + \left(\frac{b}{a}\right)^{2a}.$$

□

P 3.18. *If a, b are positive real numbers such that $a^2 + b^2 = 2$, then*

$$2a^a b^b \ge a^{2b} + b^{2a} + \frac{1}{2}(a - b)^2.$$

(Vasile Cîrtoaje, 2010)

Solution. According to the inequalities in P 3.7 and P 3.17, we have

$$a^{2b} + b^{2a} \le 1 + ab$$

and

$$a^a b^b \ge 1.$$

Therefore, it suffices to show that

$$2 \ge 1 + ab + \frac{1}{2}(a - b)^2,$$

which is an identity. The equality holds for $a = b = 1$.

□

P 3.19. *If $a, b \in (0, 1]$, then*

$$(a^2 + b^2)\left(\frac{1}{a^{2a}} + \frac{1}{b^{2b}}\right) \le 4.$$

(Vasile Cîrtoaje, 2014)

Solution. For $a = x$ and $b = 1$, the desired inequality becomes

$$x^{2x} \ge \frac{1 + x^2}{3 - x^2}, \quad x \in (0, 1].$$

If this inequality is true, it suffices to show that

$$(a^2 + b^2)\left(\frac{3 - a^2}{1 + a^2} + \frac{3 - b^2}{1 + b^2}\right) \le 4,$$

which is equivalent to

$$a^2b^2(2+a^2+b^2)+2-(a^2+b^2)-(a^2+b^2)^2 \geq 0,$$

$$(2+a^2+b^2)(1-a^2)(1-b^2) \geq 0.$$

To prove the inequality $x^{2x} \geq \dfrac{1+x^2}{3-x^2}$, we show that $f(x) \geq 0$, where

$$f(x) = x\ln x + \frac{1}{2}\ln(3-x^2) - \frac{1}{2}\ln(1+x^2), \qquad x \in (0,1].$$

We have

$$f'(x) = 1 + \ln x - \frac{x}{3-x^2} - \frac{x}{1+x^2},$$

$$\begin{aligned} f''(x) &= \frac{1}{x} - \frac{3+x^2}{(3-x^2)^2} - \frac{1-x^2}{(1+x^2)^2} \\ &= \frac{(1-x)(9+6x-x^3)}{x(3-x)^2} - \frac{1-x^2}{(1+x^2)^2}. \end{aligned}$$

We will show that $f''(x) > 0$ for $0 < x < 1$. This is true if

$$\frac{9+6x-x^3}{x(3-x)^2} - \frac{1+x}{(1+x^2)^2} > 0.$$

Indeed,

$$\frac{9+6x-x^3}{x(3-x)^2} - \frac{1+x}{(1+x^2)^2} > \frac{9}{9x} - \frac{1+x}{x(1+x)^2} = \frac{1}{1+x} > 0.$$

Since $f''(x) > 0$, f' is strictly increasing on $(0,1]$. Since $f'(1) = 0$, it follows that $f'(x) < 0$ for $x \in (0,1)$, f is strictly decreasing on $(0,1]$, hence $f(x) \geq f(1) = 0$. This completes the proof. The equality holds for $a = b = 1$.

□

P 3.20. *If a, b are positive real numbers such that $a+b=2$, then*

$$a^b b^a + 2 \geq 3ab.$$

(Vasile Cîrtoaje, 2010)

Solution. Setting

$$a = 1+x, \qquad b = 1-x, \qquad 0 \leq x < 1,$$

the inequality is equivalent to

$$(1+x)^{1-x}(1-x)^{1+x} \geq 1-3x^2.$$

Consider further the nontrivial case $0 \le x < \frac{1}{\sqrt{3}}$, and write the desired inequality as $f(x) \ge 0$, where

$$f(x) = (1-x)\ln(1+x) + (1+x)\ln(1-x) - \ln(1-3x^2).$$

We have

$$f'(x) = -\ln(1+x) + \ln(1-x) + \frac{1-x}{1+x} - \frac{1+x}{1-x} + \frac{6x}{1-3x^2},$$

$$\frac{1}{2}f''(x) = \frac{-1}{1-x^2} - \frac{2(x^2+1)}{(1-x^2)^2} + \frac{3(3x^2+1)}{(1-3x^2)^2}.$$

Making the substitution

$$t = x^2, \qquad 0 \le t < \frac{1}{3},$$

we get

$$\frac{1}{2}f''(x) = \frac{3(3t+1)}{(3t-1)^2} - \frac{t+3}{(t-1)^2} = \frac{4t(5-9t)}{(t-1)^2(3t-1)^2} > 0.$$

Therefore, $f'(x)$ is strictly increasing, $f'(x) \ge f'(0) = 0$, $f(x)$ is strictly increasing, hence $f(x) \ge f(0) = 0$. This completes the proof. The equality holds for $a = b = 1$.

□

P 3.21. *Let a, b be positive real numbers such that $a + b = 2$. If $k \ge \frac{1}{2}$, then*

$$a^{a^{kb}} b^{b^{ka}} \ge 1.$$

(Vasile Cîrtoaje, 2010)

Solution. Setting

$$a = 1+x, \qquad b = 1-x, \qquad 0 \le x < 1,$$

the inequality can be written as

$$(1+x)^{k(1-x)}\ln(1+x) + (1-x)^{k(1+x)}\ln(1-x) \ge 0.$$

Consider further the nontrivial case $0 < x < 1$, and write the desired inequality as $f(x) \ge 0$, where

$$f(x) = k(1-x)\ln(1+x) - k(1+x)\ln(1-x) + \ln\ln(1+x) - \ln(-\ln(1-x)).$$

It suffices to show that $f'(x) > 0$. Indeed, if this is true, then $f(x)$ is strictly increasing, hence

$$f(x) > \lim_{x \to 0} f(x) = 0.$$

We have

$$\begin{aligned} f'(x) &= \frac{2k(1+x^2)}{1-x^2} - k\ln(1-x^2) + \frac{1}{(1+x)\ln(1+x)} + \frac{1}{(1-x)\ln(1-x)} \\ &> \frac{2k}{1-x^2} + \frac{1}{(1+x)\ln(1+x)} + \frac{1}{(1-x)\ln(1-x)} \\ &\ge \frac{1}{1-x^2} + \frac{1}{(1+x)\ln(1+x)} + \frac{1}{(1-x)\ln(1-x)} \\ &= \frac{g(x)}{(1-x^2)\ln(1+x)\ln(1-x)}, \end{aligned}$$

where

$$g(x) = \ln(1+x)\ln(1-x) + (1+x)\ln(1+x) + (1-x)\ln(1-x).$$

It is enough to how that $g(x) < 0$. We have

$$g'(x) = \frac{-x}{1-x^2}h(x),$$

where

$$h(x) = (1+x)\ln(1+x) + (1-x)\ln(1-x).$$

Since

$$h'(x) = \ln\frac{1+x}{1-x} > 0,$$

$h(x)$ is strictly increasing, $h(x) > h(0) = 0$, $g'(x) < 0$, $g(x)$ is strictly decreasing, and hence $g(x) < g(0) = 0$. This completes the proof. The equality holds for $a = b = 1$.

□

P 3.22. *If a, b are positive real numbers such that $a + b = 2$, then*

$$a^{\sqrt{a}} b^{\sqrt{b}} \ge 1.$$

(Vasile Cîrtoaje, 2010)

Solution. Assume that $a > 1 > b$. Taking logarithms of both sides, the inequality becomes in succession:

$$\sqrt{a}\ln a + \sqrt{b}\ln b \ge 0,$$

$$\sqrt{a}\ln a \ge \sqrt{b}(-\ln b),$$

$$\frac{1}{2}\ln a + \ln\ln a \ge \frac{1}{2}\ln b + \ln(-\ln b).$$

Substituting

$$a = 1+x, \quad b = 1-x, \quad 0 < x < 1,$$

we need to show that $f(x) \geq 0$, where

$$f(x) = \frac{1}{2}\ln(1+x) - \frac{1}{2}\ln(1-x) + \ln\ln(1+x) - \ln(-\ln(1-x)).$$

We have

$$f'(x) = \frac{1}{1-x^2} + \frac{1}{(1+x)\ln(1+x)} + \frac{1}{(1-x)\ln(1-x)}.$$

As shown in the proof of the preceding P 3.21, we have $f'(x) > 0$. Therefore, $f(x)$ is strictly increasing, therefore

$$f(x) > \lim_{x\to 0} f(x) = 0.$$

The equality holds for $a = b = 1$.

□

P 3.23. *If a, b are positive real numbers such that $a + b = 2$, then*

$$a^{a+1}b^{b+1} \leq 1 - \frac{1}{48}(a-b)^4.$$

(Vasile Cîrtoaje, 2010)

Solution. Putting

$$a = 1 + x, \quad b = 1 - x, \quad 0 \leq x < 1,$$

the inequality becomes

$$(1+x)^{2+x}(1-x)^{2-x} \leq 1 - \frac{1}{3}x^4.$$

Write this inequality as $f(x) \leq 0$, where

$$f(x) = (2+x)\ln(1+x) + (2-x)\ln(1-x) - \ln\left(1 - \frac{1}{3}x^4\right).$$

We have

$$f'(x) = \ln(1+x) - \ln(1-x) - \frac{2x}{1-x^2} + \frac{4x^3}{3-x^4},$$

$$f''(x) = \frac{2}{1-x^2} - \frac{2(1+x^2)}{(1-x^2)^2} + \frac{4x^2(x^4+9)}{(3-x^4)^2}$$

$$= \frac{-4x^2}{(1-x^2)^2} + \frac{4x^2(x^4+9)}{(3-x^4)^2} = \frac{-8x^4[x^4+1+8(1-x^2)]}{(1-x^2)^2(3-x^4)^2} \leq 0.$$

Therefore, $f'(x)$ is decreasing, $f'(x) \leq f'(0) = 0$, $f(x)$ is decreasing, $f(x) \leq f(0) = 0$. The equality holds for $a = b = 1$.

□

P 3.24. *If a, b are positive real numbers such that $a+b=2$, then*

$$a^{-a}+b^{-b} \le 2.$$

(Vasile Cîrtoaje, 2010)

Solution. Consider $a \ge b$, when we have

$$0 < b \le 1 \le a < 2,$$

and write the inequality as

$$\frac{a^a-1}{a^a}+\frac{b^b-1}{b^b} \ge 0.$$

According to Lemma from the proof of P 3.4, we have

$$a^a-1 \ge a^{\frac{a+1}{2}}(a-1), \qquad b^b-1 \ge b^{\frac{b+1}{2}}(b-1).$$

Therefore, it suffices to show that

$$a^{\frac{1-a}{2}}(a-1)+b^{\frac{1-b}{2}}(b-1) \ge 0,$$

which is equivalent to

$$a^{\frac{1-a}{2}} \ge b^{\frac{1-b}{2}},$$

$$(ab)^{\frac{1-b}{2}} \le 1,$$

$$ab \le 1,$$

$$(a-b)^2 \ge 0.$$

The equality holds for $a=b=1$.

□

P 3.25. *If $a, b \in [0,1]$, then*

$$a^{b-a}+b^{a-b}+(a-b)^2 \le 2.$$

(Vasile Cîrtoaje, 2010)

Solution *(by Vo Quoc Ba Can).* Without loss of generality, assume that $a \ge b$. Using the substitution

$$c = a-b,$$

we need to show that

$$(b+c)^{-c}+b^c+c^2 \le 2$$

for

$$0 \le b \le 1-c, \quad 0 \le c \le 1.$$

If $c = 1$, then $b = 0$, and the inequality is an equality. Also, for $c = 0$, the inequality is an equality. Consider further that
$$0 < c < 1.$$
We need to show that $f(x) \le 0$, where
$$f(x) = (x+c)^{-c} + x^c + c^2 - 2, \qquad x \in [0, 1-c].$$
We claim that $f'(x) > 0$ for $x > 0$. On this assumption, $f(x)$ is strictly increasing on $[0, 1-c]$, hence
$$f(x) \le f(1-c) = (1-c)^c - (1-c^2).$$
By Bernoulli's inequality, we have
$$f(x) \le 1 + c(-c) - (1-c^2) = 0.$$
Since
$$f'(x) = \frac{c[(x+c)^{1+c} - x^{1-c}]}{(x+c)^{1+c}x^{1-c}},$$
the inequality $f'(x) > 0$ holds for $x > 0$ if and only if
$$x + c > x^{\frac{1-c}{1+c}}.$$
For any $d > 0$, using the weighted AM-GM inequality yields
$$x + c = x + d \cdot \frac{c}{d} \ge (1+d)x^{\frac{1}{1+d}} \left(\frac{c}{d}\right)^{\frac{d}{1+d}}.$$
Choosing
$$d = \frac{2c}{1-c},$$
we get
$$x + c \ge \frac{1+c}{2}\left(\frac{1-c}{2}\right)^{\frac{c-1}{1+c}} x^{\frac{1-c}{1+c}}.$$
Thus, it suffices to show that
$$\frac{1+c}{2} \ge \left(\frac{1-c}{2}\right)^{\frac{1-c}{1+c}}.$$
Indeed, using Bernoulli's inequality, we get
$$\left(\frac{1-c}{2}\right)^{\frac{1-c}{1+c}} = \left(1 - \frac{1+c}{2}\right)^{\frac{1-c}{1+c}} \le 1 - \frac{1-c}{1+c} \cdot \frac{1+c}{2} = \frac{1+c}{2}.$$
The equality holds for $a = b$, for $a = 1$ and $b = 0$, and for $a = 0$ and $b = 1$.

□

P 3.26. *If a, b are nonnegative real numbers such that $a + b \le 2$, then*

$$a^{b-a} + b^{a-b} + \frac{7}{16}(a-b)^2 \le 2.$$

(Vasile Cîrtoaje, 2010)

***Solution*.** Assume that $a \ge b$. Using the substitution

$$c = a - b,$$

we need to show that

$$a^{-c} + (a-c)^c + \frac{7}{16}c^2 \le 2$$

for

$$0 \le c \le 2, \quad c \le a \le 1 + \frac{c}{2}.$$

For $c = 0$ and $c = 2$ (which involves $a = 2$), the inequality is an equality. Therefore, we only need to show that $f(x) \le 0$ for $0 < c < 2$, where

$$f(x) = x^{-c} + (x-c)^c + \frac{7}{16}c^2 - 2, \qquad x \in \left[c, 1 + \frac{c}{2}\right].$$

In the case $c = 1$, we need to show that $f(x) \le 0$ for $x \in \left[1, \frac{3}{2}\right]$; indeed, we have

$$f(x) = \frac{1}{x} + x - \frac{41}{16} \le \frac{2}{3} + \frac{3}{2} - \frac{41}{16} = \frac{-19}{48} < 0.$$

Consider next that

$$c \in (0,1) \cup (1,2).$$

The derivative

$$f'(x) = \frac{c[x^{1+c} - (x-c)^{1-c}]}{x^{1+c}(x-c)^{1-c}}$$

has the same sign as

$$g(x) = (1+c)\ln x - (1-c)\ln(x-c).$$

We have

$$g'(x) = \frac{c(2x-1-c)}{x(x-c)}.$$

Case 1: $0 < c < 1$. We claim that $g(x) > 0$ for $x \in \left(c, 1 + \frac{c}{2}\right]$. On this assumption, f is strictly increasing on $\left[c, 1 + \frac{c}{2}\right]$, hence

$$f(x) \le f\left(1 + \frac{c}{2}\right).$$

Thus, we need to show that $f\left(1 + \frac{c}{2}\right) \le 0$, which is just the inequality in Lemma 4 below.

From the expression of $g'(x)$, it follows that $g(x)$ is decreasing on $\left(c, \dfrac{1+c}{2}\right]$, and increasing on $\left[\dfrac{1+c}{2}, 1+\dfrac{c}{2}\right]$. Then, to show that $g(x) > 0$ for $x \in \left(c, 1+\dfrac{c}{2}\right]$, it suffices to prove that

$$g\left(\frac{1+c}{2}\right) > 0,$$

which is equivalent to

$$\left(\frac{1+c}{2}\right)^{1+c} > \left(\frac{1-c}{2}\right)^{1-c}.$$

This inequality follows from Bernoulli's inequality, as follows:

$$\left(\frac{1+c}{2}\right)^{1+c} = \left(1-\frac{1-c}{2}\right)^{1+c} > 1-\frac{(1+c)(1-c)}{2} = \frac{1+c^2}{2}$$

and

$$\left(\frac{1-c}{2}\right)^{1-c} = \left(1-\frac{1+c}{2}\right)^{1-c} < 1-\frac{(1-c)(1+c)}{2} = \frac{1+c^2}{2}.$$

Case 2: $1 < c < 2$. Since

$$2x-1-c \geq 2c-1-c = c-1 > 0,$$

it follows that $g'(x) > 0$, hence $g(x)$ is strictly increasing. For $x \to c$, we have $g(x) \to -\infty$. If $g(1+c/2) \leq 0$, then $g(x) \leq 0$, hence f is decreasing. If $g(1+c/2) > 0$, then there exists $x_1 \in (c, 1+c/2)$ such that $g(x_1) = 0$, $g(x) < 0$ for $x \in [c, x_1)$ and $g(x) > 0$ for $x \in (x_1, 1+c/2]$, hence f is decreasing on $[c, x_1]$ and increasing on $[x_1, 1+c/2]$. Therefore, it suffices to show that $f(c) \leq 0$ and $f\left(1+\dfrac{c}{2}\right) \leq 0$. These inequalities follow respectively from Lemma 1 and Lemma 4 below.

The proof is completed. The equality holds for $a = b$, for $a = 2$ and $b = 0$, and for $a = 0$ and $b = 2$.

Lemma 1. *If* $1 \leq c \leq 2$*, then*

$$c^{-c} + \frac{7}{16}c^2 \leq 2,$$

with equality for $c = 2$.

Proof. The desired inequality is equivalent to $h(c) \geq 0$, where

$$h(c) = c\ln c + \ln\left(2-\frac{7}{16}c^2\right), \qquad c \in [1,2].$$

We have

$$h'(c) = 1 + \ln c - \frac{14c}{32-7c^2},$$

$$h''(c) = \frac{1}{c} - \frac{14(32+7c^2)}{(32-7c^2)^2}.$$

Since h'' is strictly decreasing, $h''(1) = 79/625$ and $h''(2) = -52$, there exists $c_1 \in (1,2)$ such that $h''(c_1) = 0$, $h''(c) > 0$ for $c \in [1, c_1)$ and $h''(c) < 0$ for $c \in (c_1, 2]$, hence h' is strictly increasing on $[1, c_1]$ and strictly decreasing on $[c_1, 2]$. Since $h'(1) = 11/25$ and $h'(2) = \ln 2 - 6 < 0$, there exists $c_2 \in (1,2)$ such that $h'(c_2) = 0$, $h'(c) > 0$ for $c \in [1, c_2)$ and $h'(c) < 0$ for $c \in (c_2, 2]$, hence h is strictly increasing on $[1, c_2]$ and strictly decreasing on $[c_2, 2]$. Thus, it suffices to show that $h(1) \ge 0$ and $h(2) \ge 0$. Indeed, $h(1) = \ln 25 - \ln 16 > 0$ and $h(2) = 0$.

Lemma 2. *If $0 \le x \le 2$, then*

$$\left(1 + \frac{x}{2}\right)^{-x} + \frac{3}{16}x^2 \le 1,$$

with equality for $x = 0$ and $x = 2$.

Proof. We need to show that $f(x) \le 0$, where

$$f(x) = -x \ln\left(1 + \frac{x}{2}\right) - \ln\left(1 - \frac{3}{16}x^2\right), \qquad x \in [0,2].$$

We have

$$f'(x) = -\ln\left(1 + \frac{x}{2}\right) + \frac{x(3x^2 + 6x - 4)}{(2+x)(16 - 3x^2)},$$

$$f''(x) = \frac{g(x)}{(2+x)^2(16-3x^2)^2},$$

where

$$g(x) = -9x^5 - 18x^4 + 168x^3 + 552x^2 + 128x - 640.$$

Since $g(x_1) = 0$ for $x_1 \approx 0,88067$, $g(x) < 0$ for $x \in [0, x_1)$ and $g(x) > 0$ for $x \in (x_1, 2]$, f' is strictly decreasing on $[0, x_1]$ and strictly increasing on $[x_1, 2]$. Since $f'(0) = 0$ and $f'(2) = -\ln 2 + \dfrac{5}{2} > 0$, there is $x_2 \in (x_1, 2)$ such that $f'(x_2) = 0$, $f'(x) < 0$ for $x \in (0, x_2)$, and $f'(x) > 0$ for $x \in (x_2, 2]$. Therefore, f is decreasing on $[0, x_2]$ and increasing on $[x_2, 2]$. Since $f(0) = f(2) = 0$, it follows that $f(x) \le 0$ for $x \in [0,2]$.

Lemma 3. *If $0 \le x \le 2$, then*

$$\left(1 - \frac{x}{2}\right)^{x} + \frac{1}{4}x^2 \le 1,$$

with equality for $x = 0$ and $x = 2$.

Proof. We need to show that $f(x) \le 0$, where

$$f(x) = x \ln\left(1 - \frac{x}{2}\right) - \ln\left(1 - \frac{1}{4}x^2\right), \qquad x \in [0,2).$$

We have

$$f'(x) = \ln\left(1 - \frac{x}{2}\right) - \frac{x^2}{4 - x^2},$$

$$f''(x) = \frac{-1}{2-x} - \frac{8x}{(4-x^2)^2}.$$

Since $f'' < 0$ for $x \in [0,2)$, f' is strictly decreasing, hence $f'(x) \le f'(0) = 0$, f is strictly decreasing, therefore $f(x) \le f(0) = 0$ for $x \in [0,2)$.

Lemma 4. *If $0 \le x \le 2$, then*

$$\left(1+\frac{x}{2}\right)^{-x} + \left(1-\frac{x}{2}\right)^{x} + \frac{7}{16}x^2 \le 2,$$

with equality for $x = 0$ and $x = 2$.

Proof. By Lemma 2 and Lemma 3, we have

$$\left(1+\frac{x}{2}\right)^{-x} + \frac{3}{16}x^2 \le 1$$

and

$$\left(1-\frac{x}{2}\right)^{x} + \frac{1}{4}x^2 \le 1.$$

The desired inequality follows by adding up these inequalities.

Conjecture. *If a, b are nonnegative real numbers such that $a + b = \frac{1}{4}$, then*

$$a^{2(b-a)} + b^{2(a-b)} \le 2.$$

□

P 3.27. *If a, b are nonnegative real numbers such that $a + b \le 4$, then*

$$a^{b-a} + b^{a-b} \le 2.$$

(Vasile Cîrtoaje, 2010)

Solution. Without loss of generality, assume that $a \ge b$. Consider first that $a - b \ge 2$. We have

$$a \ge a - b \ge 2,$$

and from

$$4 \ge a + b = (a - b) + 2b \ge 2 + 2b,$$

we get $b \le 1$. Clearly, the desired inequality is true because

$$a^{b-a} < 1, \quad b^{a-b} \le 1.$$

Since the case $a - b = 0$ is trivial, consider further that $0 < a - b < 2$ and use the substitution

$$c = a - b.$$

So, we need to show that

$$a^{-c} + (a - c)^c \le 2$$

for
$$0 < c < 2, \quad c \le a \le 2 + \frac{c}{2}.$$
Equivalently, we need to show that $f(x) \le 0$ for $0 < c < 2$, where
$$f(x) = x^{-c} + (x-c)^c - 2, \qquad x \in \left[c, 2 + \frac{c}{2}\right].$$
The derivative
$$f'(x) = \frac{c[x^{1+c} - (x-c)^{1-c}]}{x^{1+c}(x-c)^{1-c}}$$
has the same sign as
$$g(x) = (1+c)\ln x - (1-c)\ln(x-c).$$
We have
$$g'(x) = \frac{c(2x-1-c)}{x(x-c)}.$$

Case 1: $c = 1$. We need to show that $x^2 - 3x + 1 \le 0$ for $x \in \left[1, \frac{5}{2}\right]$. Indeed, we have
$$2(x^2 - 3x + 1) = (x-1)(2x-5) + (x-3) < 0.$$

Case 2: $0 < c < 1$. We will show that $g(x) > 0$ for $x \in \left(c, 2 + \frac{c}{2}\right]$. From
$$g'(x) = \frac{c(2x-1-c)}{x(x-c)},$$
it follows that $g(x)$ is decreasing on $\left(c, \frac{1+c}{2}\right]$ and increasing on $\left[\frac{1+c}{2}, 2 + \frac{c}{2}\right]$. Then, to show that $g(x) > 0$ for $x \in \left(c, 1 + \frac{c}{2}\right]$, it suffices to prove that
$$g\left(\frac{1+c}{2}\right) > 0,$$
which is equivalent to
$$\left(\frac{1+c}{2}\right)^{1+c} > \left(\frac{1-c}{2}\right)^{1-c}.$$
This inequality follows from Bernoulli's inequality, as follows:
$$\left(\frac{1+c}{2}\right)^{1+c} = \left(1 - \frac{1-c}{2}\right)^{1+c} > 1 - \frac{(1+c)(1-c)}{2} = \frac{1+c^2}{2}$$
and
$$\left(\frac{1-c}{2}\right)^{1-c} = \left(1 - \frac{1+c}{2}\right)^{1-c} < 1 - \frac{(1-c)(1+c)}{2} = \frac{1+c^2}{2}.$$

Since $g(x) > 0$ involves $f'(x) > 0$, it follows that $f(x)$ is strictly increasing on $\left[c, 2+\frac{c}{2}\right]$, and hence

$$f(x) \le f\left(2+\frac{c}{2}\right).$$

So, we need to show that $f\left(2+\frac{c}{2}\right) \le 0$ for $0 < c < 1$, which follows immediately from Lemma 3 below.

Case 3: $1 < c < 2$. Since

$$2x - 1 - c \ge 2c - 1 - c > 0,$$

we have $g'(x) > 0$, hence $g(x)$ is strictly increasing. Since $g(x) \to -\infty$ when $x \to c$ and

$$\begin{aligned} g\left(2+\frac{c}{2}\right) &= (1+c)\ln\left(2+\frac{c}{2}\right) + (c-1)\ln\left(2-\frac{c}{2}\right) \\ &> (c-1)\ln\left(2-\frac{c}{2}\right) > 0, \end{aligned}$$

there exists $x_1 \in \left(c, 2+\frac{c}{2}\right)$ such that $g(x_1) = 0$, $g(x) < 0$ for $x \in (c, x_1)$ and $g(x) > 0$ for $x \in \left(x_1, 2+\frac{c}{2}\right)$. Thus, $f(x)$ is decreasing on $[c, x_1]$ and increasing on $\left[x_1, 2+\frac{c}{2}\right]$. Then, it suffices to show that $f(c) \le 0$ and $f\left(2+\frac{c}{2}\right) \le 0$. The first inequality is true because

$$f(c) = c^{-c} - 2 < 1 - 2 < 0,$$

while the second inequality follows immediately from Lemma 3 below.

The proof is completed. The equality holds for $a = b$.

Lemma 1. *If $x < 4$, then*

$$xh(x) \le 0,$$

where

$$h(x) = \ln\left(2-\frac{x}{2}\right) - \left(\ln 2 - \frac{x}{4} - \frac{1}{32}x^2\right).$$

Proof. From

$$h'(x) = \frac{-x^2}{16(4-x)} \le 0,$$

it follows that $h(x)$ is decreasing. Since $h(0) = 0$, we have $h(x) \ge 0$ for $x \le 0$, and $h(x) \le 0$ for $x \in [0, 4)$; that is, $xh(x) \le 0$ for $x < 4$.

Lemma 2. *If*

$$-2 \le x \le 2,$$

then

$$\left(2-\frac{x}{2}\right)^x \le 1 + x\ln 2 - \frac{x^3}{9}.$$

Proof. We have

$$\ln 2 \approx 0.693 < 7/9.$$

If $x \in [0,2]$, then

$$1 + x\ln 2 - \frac{x^3}{9} \ge 1 - \frac{x^3}{9} \ge 1 - \frac{8}{9} > 0.$$

Also, if $x \in [-2,0]$, then

$$\begin{aligned}1 + x\ln 2 - \frac{x^3}{9} &\ge 1 + \frac{7x}{9} - \frac{x^3}{9} > \frac{8+7x-x^3}{9}\\ &= \frac{2(x+2)^2 + (-x)(x+1)^2}{9} > 0.\end{aligned}$$

So, we can write the desired inequality as $f(x) \ge 0$, where

$$f(x) = \ln\left(1 + dx - \frac{x^3}{9}\right) - x\ln\left(2 - \frac{x}{2}\right), \quad d = \ln 2.$$

We have

$$f'(x) = \frac{9d - 3x^2}{9 + 9dx - x^3} + \frac{x}{4-x} - \ln\left(2 - \frac{x}{2}\right).$$

Since $f(0) = 0$, it suffices to show that $f'(x) \le 0$ for $x \in [-2,0]$, and $f'(x) \ge 0$ for $x \in [0,2]$; that is, $xf'(x) \ge 0$ for $x \in [-2,2]$. We have

$$f'(x) = g(x) - h(x),$$

where

$$g(x) = \frac{9d - 3x^2}{9 + 9dx - x^3} + \frac{x}{4-x} - \left(d - \frac{x}{4} - \frac{1}{32}x^2\right),$$

$$h(x) = \ln\left(2 - \frac{x}{2}\right) - \left(d - \frac{x}{4} - \frac{1}{32}x^2\right).$$

According to Lemma 1,

$$xf'(x) = xg(x) - xh(x) \ge xg(x).$$

Therefore, to show that $xf'(x) \ge 0$, it suffices to prove that $xg(x) \ge 0$. We have

$$\begin{aligned}g(x) &= \left(\frac{9d - 3x^2}{9 + 9dx - x^3} - d\right) + \left(\frac{x}{4-x} + \frac{x}{4} + \frac{1}{32}x^2\right)\\ &= x\left[\frac{dx^2 - 3x - 9d^2}{9 + 9dx - x^3} + \frac{64 - 4x - x^2}{32(4-x)}\right],\end{aligned}$$

hence

$$xg(x) = \frac{x^2 g_1(x)}{32(4-x)(9 + 9dx - x^3)},$$

where

$$\begin{aligned}g_1(x) =& 32(4-x)(dx^2 - 3x - 9d^2) + (64 - 4x - x^2)(9 + 9dx - x^3)\\ =& x^5 + 4x^4 - (64 + 41d)x^3 + (87 + 92d)x^2 + 12(24d^2 + 48d - 35)x\\ &+ 576(1 - 2d^2).\end{aligned}$$

Since $g_1(x) \ge 0$ for $x \in [a_1, b_1]$, where $a_1 \approx -12.384$ and $b_1 =\approx 2.652$, we have $g_1(x) \ge 0$ for $x \in [-2, 2]$.

Lemma 3. *If $0 \le c \le 2$, then*

$$\left(2+\frac{c}{2}\right)^{-c} + \left(2-\frac{c}{2}\right)^{c} \le 2.$$

Proof. According to Lemma 2, the following inequalities hold for $c \in [0, 2]$:

$$\left(2+\frac{c}{2}\right)^{-c} \le 1 - c\ln 2 + \frac{c^3}{9},$$

$$\left(2-\frac{c}{2}\right)^{c} \le 1 + c\ln 2 - \frac{c^3}{9}.$$

Summing these inequalities, the desired inequality follows.

□

P 3.28. *If a, b are nonnegative real numbers such that $a + b = 2$, then*

$$a^{2b} + b^{2a} \ge a^b + b^a \ge a^2b^2 + 1.$$

(Vasile Cîrtoaje, 2010)

Solution. Since $a, b \in [0, 2]$ and

$$(1-a)(1-b) = -(1-a)^2 \le 0,$$

from Lemma below, we have

$$a^b - 1 \ge \frac{b(ab+3-a-b)(a-1)}{2} = \frac{b(ab+1)(a-1)}{2}$$

and

$$b^a - 1 \ge \frac{a(ab+1)(b-1)}{2}.$$

Based on these inequalities, we get

$$\begin{aligned} a^b + b^a - a^2b^2 - 1 &= (a^b - 1) + (b^a - 1) + 1 - a^2b^2 \\ &\ge \frac{b(ab+1)(a-1)}{2} + \frac{a(ab+1)(b-1)}{2} + 1 - a^2b^2 \\ &= (ab+1)(ab-1) + 1 - a^2b^2 = 0 \end{aligned}$$

and

$$\begin{aligned} a^{2b}+b^{2a}-a^b-b^a &= a^b(a^b-1)+b^a(b^a-1) \\ &\ge \frac{a^b b(ab+1)(a-1)}{2}+\frac{b^a a(ab+1)(b-1)}{2} \\ &= \frac{ab(ab+1)(a-b)(a^{b-1}-b^{a-1})}{4}. \end{aligned}$$

Under the assumption that $a \ge b$, we only need to show that $a^{b-1} \ge b^{a-1}$, which is equivalent to

$$a^{\frac{b-a}{2}} \ge b^{\frac{a-b}{2}}, \quad 1 \ge (ab)^{\frac{a-b}{2}}, \quad 1 \ge ab, \quad (a-b)^2 \ge 0.$$

For both inequalities, the equality holds when $a=b=1$, when $a=0$ and $b=2$, and when $a=2$ and $b=0$.

Lemma. *If $x, y \in [0,2]$ such that $(1-x)(1-y) \le 0$, then*

$$x^y - 1 \ge \frac{y(xy+3-x-y)(x-1)}{2},$$

with equality for $x=1$, and also for $y=0$, $y=1$ and $y=2$.

Proof. For $y=0$, $y=1$ and $y=2$, the inequality is an identity. For fixed

$$y \in (0,1) \cup (1,2),$$

let us define

$$f(x) = x^y - 1 - \frac{y(xy+3-x-y)(x-1)}{2}.$$

We have

$$f'(x) = y\left[x^{y-1} - \frac{xy+3-x-y}{2} - \frac{(x-1)(y-1)}{2}\right],$$

$$f''(x) = y(y-1)(x^{y-2}-1).$$

Since $x^{y-2}-1$ has the same sign as $1-x$, it follows that $f''(x) \ge 0$ for $x \in (0,2]$, therefore f' is increasing. There are two cases to consider.

Case 1: $x \ge 1 > y$. We have $f'(x) \ge f'(1) = 0$, $f(x)$ is increasing, hence

$$f(x) \ge f(1) = 0.$$

Case 2: $y > 1 \ge x$. We have $f'(x) \le f'(1) = 0$, $f(x)$ is decreasing, hence

$$f(x) \ge f(1) = 0.$$

□

P 3.29. *If a, b are positive real numbers such that $a+b=2$, then*

$$a^{3b}+b^{3a}\le 2.$$

(Vasile Cîrtoaje, 2007)

Solution. Without loss of generality, assume that $a \ge b$. Using the substitution

$$a=1+x,\qquad b=1-x,\qquad 0\le x<1,$$

we can write the inequality as

$$e^{3(1-x)\ln(1+x)}+e^{3(1+x)\ln(1-x)}\le 2.$$

Applying Lemma below, it suffices to show that $f(x)\le 2$, where

$$f(x)=e^{3(1-x)\left(x-\frac{x^2}{2}+\frac{x^3}{3}\right)}+e^{-3(1+x)\left(x+\frac{x^2}{2}+\frac{x^3}{3}\right)}.$$

Since $f(0)=2$, it suffices to show that $f'(x)\le 0$ for $x\in[0,1)$. From

$$f'(x)=\left(3-9x+\frac{15}{2}x^2-4x^3\right)e^{3x-\frac{9x^2}{2}+\frac{5x^3}{2}-x^4}$$
$$-\left(3+9x+\frac{15}{2}x^2+4x^3\right)e^{-3x-\frac{9x^2}{2}-\frac{5x^3}{2}-x^4},$$

it follows that $f'(x)\le 0$ is equivalent to

$$e^{-6x-5x^3}\ge\frac{6-18x+15x^2-8x^3}{6+18x+15x^2+8x^3}.$$

For the nontrivial case $6-18x+15x^2-8x^3>0$, we rewrite this inequality as $g(x)\ge 0$, where

$$g(x)=-6x-5x^3-\ln(6-18x+15x^2-8x^3)+\ln(6+18x+15x^2+8x^3).$$

Since $g(0)=0$, it suffices to show that $g'(x)\ge 0$ for $x\in[0,1)$. From

$$\frac{1}{3}g'(x)=-2-5x^2+\frac{(6+8x^2)-10x}{6+15x^2-(18x+8x^3)}+\frac{(6+8x^2)+10x}{6+15x^2+(18x+8x^3)},$$

it follows that $g'(x)\ge 0$ is equivalent to

$$2(6+8x^2)(6+15x^2)-20x(18x+8x^3)\ge(2+5x^2)[(6+15x^2)^2-(18x+8x^3)^2].$$

Since

$$(6+15x^2)^2-(18x+8x^3)^2\le(6+15x^2)^2-324x^2-288x^4\le 4(9-36x^2),$$

it suffices to show that

$$(3+4x^2)(6+15x^2)-5x(18x+8x^3) \ge (2+5x^2)(9-36x^2).$$

This reduces to $6x^2+200x^4 \ge 0$, which is clearly true. The equality holds for $a=b=1$.

Lemma. *If $t>-1$, then*

$$\ln(1+t) \le t-\frac{t^2}{2}+\frac{t^3}{3}.$$

Proof. We need to prove that $f(t) \ge 0$, where

$$f(t)=t-\frac{t^2}{2}+\frac{t^3}{3}-\ln(1+t).$$

Since

$$f'(t)=\frac{t^3}{t+1},$$

$f(t)$ is decreasing on $(-1,0]$ and increasing on $[0,\infty)$. Therefore,

$$f(t) \ge f(0)=0.$$

□

P 3.30. *If a,b are nonnegative real numbers such that $a+b=2$, then*

$$a^{3b}+b^{3a}+\left(\frac{a-b}{2}\right)^4 \le 2.$$

(Vasile Cîrtoaje, 2007)

Solution *(by M. Miyagi and Y. Nishizawa).* Using the substitution

$$a=1+x, \qquad b=1-x, \qquad 0 \le x \le 1,$$

we can write the inequality as

$$(1+x)^{3(1-x)}+(1-x)^{3(1+x)}+x^4 \le 2.$$

By Lemma below, we have

$$(1+x)^{1-x} \le \frac{1}{4}(1+x)^2(2-x^2)(2-2x+x^2),$$

$$(1-x)^{1+x} \le \frac{1}{4}(1-x)^2(2-x^2)(2+2x+x^2).$$

Therefore, it suffices to show that

$$(1+x)^6(2-x^2)^3(2-2x+x^2)^3+(1-x)^6(2-x^2)^3(2+2x+x^2)^3+64x^4 \le 128,$$

which is equivalent to

$$x^4(1-x^2)[x^6(x^6-8x^4+31x^2-34)-2(17x^6-38x^4+16x^2+8)] \le 0.$$

Thus, it suffices to show that

$$t^3-8t^2+31t-34<0$$

and

$$17t^3-38t^2+16t+8>0$$

for all $t \in [0,1]$. Indeed, we have

$$t^3-8t^2+31t-34<t^3-8t^2+31t-24=(t-1)(t^2-7t+24) \le 0,$$

$$17t^3-38t^2+16t+8=17t(t-1)^2+(-4t^2-t+8)>0.$$

Lemma. *If $-1 \le t \le 1$, then*

$$(1+t)^{1-t} \le \frac{1}{4}(1+t)^2(2-t^2)(2-2t+t^2),$$

with equality for $t=-1$, $t=0$ and $t=1$.
Proof. It suffices to consider that

$$-1<t \le 1.$$

Rewrite the inequality as

$$(1+t)^{1+t}(2-t^2)(2-2t+t^2) \ge 4,$$

which is equivalent to $f(t) \ge 0$, where

$$f(t)=(1+t)\ln(1+t)+\ln(2-t^2)+\ln(2-2t+t^2)-\ln 4.$$

We have

$$f'(t)=1+\ln(1+t)-\frac{2t}{2-t^2}+\frac{2(t-1)}{2-2t+t^2},$$

$$f''(t)=\frac{t^2g(t)}{(1+t)(2-t^2)^2(2-2t+t^2)^2},$$

where

$$g(t)=t^6-8t^5+12t^4+8t^3-20t^2-16t+16.$$

Case 1: $0 \le t \le 1$. From

$$\begin{aligned} g'(t) &= 6t^5-40t^4+48t^3+24t^2-40t-16 \\ &= 6t^5-8t-16-8t(5t^3-6t^2-3t+4) \\ &= (6t^5-8t-16)-8t(t-1)^2(5t+4)<0, \end{aligned}$$

it follows that g is strictly decreasing on $[0,1]$. Since $g(0)=16$ and $g(1)=-7$, there exists a number $c\in(0,1)$ such that $g(c)=0$, $g(t)>0$ for $0<t<c$ and $g(t)<0$ for $c<t\le 1$. Therefore, f' is strictly increasing on $[0,c]$ and strictly decreasing on $[c,1]$. From $f'(0)=0$ and $f'(1)=\ln 2-1<0$, it follows that there exists a number $d\in(0,1)$ such that $f'(d)=0$, $f'(t)>0$ for $0<t<d$ and $f'(t)<0$ for $d<t\le 1$. As a consequence, f is strictly increasing on $[0,d]$ and strictly decreasing on $[d,1]$. Since $f(0)=0$ and $f(1)=0$, we have $f(t)\ge 0$ for $0\le t\le 1$.

Case 2: $-1<t\le 0$. From

$$g(t)=t^4(t-2)(t-6)+4(t+1)(2t^2-7t+3)+4>0,$$

it follows that f' is strictly increasing on $(-1,0]$. Since $f'(0)=0$, we have $f'(t)<0$ for $-1<t<0$, hence f is strictly decreasing on $(-1,0]$. From $f(0)=0$, it follows that $f(t)\ge 0$ for $-1<t\le 0$.

Conjecture. *If a,b are nonnegative real numbers such that $a+b=2$, then*

$$a^{3b}+b^{3a}+\left(\frac{a-b}{2}\right)^2\ge 2.$$

□

P 3.31. *If a,b are positive real numbers such that $a+b=2$, then*

$$a^{\frac{2}{a}}+b^{\frac{2}{b}}\le 2.$$

(Vasile Cîrtoaje, 2008)

Solution. Without loss of generality, assume that

$$0<a\le 1\le b<2,$$

and write the inequality as

$$\frac{1}{\left(\frac{1}{a^2}\right)^{1/a}}+\frac{1}{\left(\frac{1}{b}\right)^{2/b}}\le 2.$$

By Bernoulli's inequality, we have

$$\left(\frac{1}{a^2}\right)^{1/a}\ge 1+\frac{1}{a}\left(\frac{1}{a^2}-1\right)=\frac{a^3-a^2+1}{a^3},$$

$$\left(\frac{1}{b}\right)^{2/b}\ge 1+\frac{2}{b}\left(\frac{1}{b}-1\right)=\frac{b^2-2b+2}{b^2}.$$

Therefore, it suffices to show that

$$\frac{a^3}{a^3-a^2+1}+\frac{b^2}{b^2-2b+2}\le 2,$$

which is equivalent to

$$\frac{a^3}{a^3-a^2+1} \le \frac{(2-b)^2}{b^2-2b+2},$$

$$\frac{a^3}{a^3-a^2+1} \le \frac{a^2}{a^2-2a+2},$$

$$a^2(a-1)^2 \ge 0.$$

The equality happens for $a = b = 1$. □

P 3.32. *If a, b are positive real numbers such that $a + b = 2$, then*

$$a^{\frac{3}{a}} + b^{\frac{3}{b}} \ge 2.$$

(Vasile Cîrtoaje, 2008)

Solution. Assume that $a \le b$; that is,

$$0 < a \le 1 \le b < 2.$$

There are two cases to consider: $0 < a \le \frac{3}{5}$ and $\frac{3}{5} \le a \le 1$.

Case 1: $0 < a \le \frac{3}{5}$. From $a + b = 2$, we get $\frac{7}{5} \le b < 2$. Let

$$f(x) = x^{\frac{3}{x}}, \quad 0 < x < 2.$$

Since

$$f'(x) = 3x^{\frac{3}{x}-2}(1 - \ln x) > 0,$$

$f(x)$ is increasing on $(0, 2)$, hence $f(b) \ge f\left(\frac{7}{5}\right)$; that is,

$$b^{\frac{3}{b}} \ge \left(\frac{7}{5}\right)^{15/7}.$$

Using Bernoulli's inequality gives

$$\left(\frac{7}{5}\right)^{15/7} = \frac{7}{5}\left(1+\frac{2}{5}\right)^{8/7} > \frac{7}{5}\left(1+\frac{16}{35}\right) = \frac{51}{25} > 2,$$

therefore

$$a^{\frac{3}{a}} + b^{\frac{3}{b}} > 2.$$

Case 2: $\frac{3}{5} \le a \le 1$. From $a + b = 2$, we get $1 \le b \le \frac{7}{5}$. By Lemma below, we have

$$2a^{\frac{3}{a}} \ge 3 - 15a + 21a^2 - 7a^3$$

and

$$2b^{\frac{3}{b}} \geq 3 - 15b + 21b^2 - 7b^3.$$

Summing these inequalities, we get

$$2\left(a^{\frac{3}{a}} + b^{\frac{3}{b}}\right) \geq 6 - 15(a+b) + 21(a^2+b^2) - 7(a^3+b^3)$$
$$= 6 - 15(a+b) + 21(a+b)^2 - 7(a+b)^3 = 4.$$

This completes the proof. The equality holds for $a = b = 1$.

Lemma. *If* $\frac{3}{5} \leq x \leq 2$, *then*

$$2x^{\frac{3}{x}} \geq 3 - 15x + 21x^2 - 7x^3,$$

with equality for $x = 1$.

Proof. First, we show that $h(x) > 0$, where

$$h(x) = 3 - 15x + 21x^2 - 7x^3.$$

From

$$h'(x) = 3(-5 + 14x - 7x^2),$$

it follows that $h(x)$ is increasing on $\left[1 - \sqrt{\frac{2}{7}}, 1 + \sqrt{\frac{2}{7}}\right]$, and decreasing on $\left[1 + \sqrt{\frac{2}{7}}, \infty\right)$. Then, it suffices to show that $f\left(\frac{3}{5}\right) \geq 0$ and $f(2) \geq 0$. Indeed

$$f\left(\frac{3}{5}\right) = \frac{6}{125}, \quad f(2) = 1.$$

Write now the desired inequality as $f(x) \geq 0$, where

$$f(x) = \ln 2 + \frac{3}{x}\ln x - \ln(3 - 15x + 21x^2 - 7x^3), \quad \frac{3}{5} \leq x \leq 2.$$

We have

$$\frac{x^2}{3}f'(x) = g(x), \quad g(x) = 1 - \ln x + \frac{x^2(7x^2 - 14x + 5)}{3 - 15x + 21x^2 - 7x^3},$$

$$g'(x) = \frac{g_1(x)}{x(3 - 15x + 21x^2 - 7x^3)^2},$$

where

$$g_1(x) = -49x^7 + 245x^6 - 280x^5 - 147x^4 + 471x^3 - 321x^2 + 90x - 9.$$

In addition,

$$g_1(x = (x-1)^2 g_2(x), \quad g_2(x) = -49x^5 + 147x^4 + 63x^3 - 168x^2 + 72x - 9,$$

$$g_2(x) = 11x^5 + 3g_3(x), \quad g_3(x) = -20x^5 + 49x^4 + 21x^3 - 56x^2 + 24x - 3,$$
$$g_3(x) = (4x-1)g_4(x), \quad g_4(x) = -5x^4 + 11x^3 + 8x^2 - 12x + 3,$$
$$g_4(x) = x^5 + g_5(x), \quad g_5(x) = -6x^4 + 11x^3 + 8x^2 - 12x + 3,$$
$$g_5(x) = (2x-1)g_6(x), \quad g_6(x) = -3x^3 + 4x^2 + 6x - 3,$$
$$g_6(x) = 1 + (2-x)(3x^2 + 2x - 2).$$

Therefore, we get in succession $g_6(x) > 0$, $g_5(x) > 0$, $g_4(x) > 0$, $g_3(x) > 0$, $g_2(x) > 0$, $g_1(x) \geq 0$, $g'(x) \geq 0$, $g(x)$ is increasing. Since $g(1) = 0$, we have $g(x) < 0$ on $\left[\frac{3}{5}, 1\right)$ and $g(x) > 0$ on $(1, 2]$. Then, $f(x)$ is decreasing on $\left[\frac{3}{5}, 1\right]$ and increasing on $[1, 2]$, hence $f(x) \geq f(1) = 0$.

□

P 3.33. *If a, b are positive real numbers such that $a + b = 2$, then*

$$a^{5b^2} + b^{5a^2} \leq 2.$$

(Vasile Cîrtoaje, 2010)

Solution. Assume that $a \geq b$. For $a = 2$ and $b = 0$, the inequality is obvious. Otherwise, using the substitution $a = 1+x$ and $b = 1-x$, $0 \leq x < 1$, we can write the desired inequality as

$$e^{5(1-x)^2 \ln(1+x)} + e^{5(1+x)^2 \ln(1-x)} \leq 2.$$

According to Lemma below, it suffices to show that $f(x) \leq 2$, where

$$f(x) = e^{5(u-v)} + e^{-5(u+v)},$$

$$u = x + \frac{7}{3}x^3 + \frac{31}{30}x^5, \quad v = \frac{5}{2}x^2 + \frac{17}{12}x^4 + \frac{9}{20}x^6.$$

If $f'(x) \leq 0$, then $f(x)$ is decreasing, hence

$$f(x) \leq f(0) = 2.$$

Since

$$f'(x) = 5(u' - v')e^{5(u-v)} - 5(u' + v')e^{-5(u+v)},$$

$$u' = 1 + 7x^2 + \frac{31}{6}x^4, \quad v' = 5x + \frac{17}{3}x^3 + \frac{27}{10}x^5,$$

the inequality $f'(x) \leq 0$ becomes

$$e^{-10u}(u' + v') \geq u' - v'$$

For the nontrivial case $u'-v'>0$, we rewrite this inequality as $g(x)\ge 0$, where

$$g(x)=-10u+\ln(u'+v')-\ln(u'-v').$$

If $g'(x)\ge 0$, then $g(x)$ is increasing, hence

$$g(x)\ge f(0)=0.$$

We have

$$g'(x)=-10u'+\frac{u''+v''}{u'+v'}-\frac{u''-v''}{u'-v'},$$

where

$$u''=14x+\frac{62}{3}x^3,\quad v''=5+17x^2+\frac{27}{2}x^4.$$

Write the inequality $g'(x)\ge 0$ as

$$u'v''-v'u''\ge 5u'(u'+v')(u'-v'),$$

$$a_1t+a_2t^2+a_3t^3+a_4t^4+a_5t^5+a_6t^6+a_7t^7\ge 0,$$

where $t=x^2$, $0\le t<1$, and

$$a_1=2,\quad a_2=321.5,\quad a_3\approx 152.1,\quad a_4\approx -498.2,$$

$$a_5\approx -168.5,\quad a_6\approx 356.0,\quad a_7\approx 188.3.$$

This inequality is true if

$$300t^2+150t^3-500t^4-200t^5+250t^6\ge 0.$$

Since the last inequality is equivalent to the obvious inequality

$$50t^2(1-t)(6+9t-t^2-5t^3)\ge 0,$$

the proof is completed. The equality holds for $a=b=1$.

Lemma. *If* $-1<t<1$, *then*

$$(1-t)^2\ln(1+t)\le t-\frac{5}{2}t^2+\frac{7}{3}t^3-\frac{17}{12}t^4+\frac{31}{30}t^5-\frac{9}{20}t^6.$$

Proof. We show that

$$(1-t)^2\ln(1+t)\le(1-t)^2\left(t-\frac{1}{2}t^2+\frac{1}{3}t^3-\frac{1}{4}t^4+\frac{1}{5}t^5\right)$$
$$\le t-\frac{5}{2}t^2+\frac{7}{3}t^3-\frac{17}{12}t^4+\frac{31}{30}t^5-\frac{9}{20}t^6.$$

The left inequality is equivalent to $f(t)\ge 0$, where

$$f(t)=t-\frac{1}{2}t^2+\frac{1}{3}t^3-\frac{1}{4}t^4+\frac{1}{5}t^5-\ln(1+t).$$

Since
$$f'(t) = \frac{t^5}{1+t},$$
$f(t)$ is decreasing on $(-1, 0]$ and increasing on $[0, 1)$; therefore, $f(t) \ge f(0) = 0$.
The right inequality is equivalent to $t^6(t-1) \le 0$, which is clearly true.

□

P 3.34. *If a, b are positive real numbers such that $a + b = 2$, then*
$$a^{2\sqrt{b}} + b^{2\sqrt{a}} \le 2.$$

(Vasile Cîrtoaje, 2010)

Solution. Assume that $a \ge b$. For $a = 2$ and $b = 0$, the inequality is obvious. Otherwise, using the substitution $a = 1+x$ and $b = 1-x$, $0 \le x < 1$, we can write the desired inequality as $f(x) \le 2$, where
$$f(x) = (1+x)^{2\sqrt{1-x}} + (1-x)^{2\sqrt{1+x}} = e^{2\sqrt{1-x}\ln(1+x)} + e^{2\sqrt{1+x}\ln(1-x)}.$$

There are two cases to consider.

Case 1: $13/20 \le x < 1$. If f is decreasing on $[13/20, 1)$, then
$$f(x) \le f\left(\frac{13}{20}\right) = \left(\frac{33}{20}\right)^{\sqrt{7/5}} + \left(\frac{7}{20}\right)^{\sqrt{33/5}} < \left(\frac{5}{3}\right)^{5/4} + \left(\frac{1}{4}\right)^{2} < 2.$$

Since the function $(1-x)^{2\sqrt{1+x}}$ is decreasing, it suffices to show that
$$g(x) = (1+x)^{2\sqrt{1-x}}$$
is decreasing. This is true if $g'(x) \le 0$ for $x \in [13/20, 1)$, that is equivalent to $h(x) \le 0$, where
$$h(x) = \frac{2(1-x)}{1+x} - \ln(1+x).$$

Clearly, h is decreasing, hence
$$h(x) \le h\left(\frac{13}{20}\right) = \frac{14}{33} - \ln\frac{33}{20} < 0.$$

Case 2: $0 \le x \le 13/20$. By Lemma below, it suffices to show that $g(x) \le 2$, where
$$g(x) = e^{2x-2x^2+\frac{11}{12}x^3-\frac{1}{2}x^4} + e^{-(2x+2x^2+\frac{11}{12}x^3+\frac{1}{2}x^4)}.$$

If $g'(x) \le 0$ for $x \in [0, 13/20]$, then g is decreasing, hence $g(x) \le g(0) = 2$. Since
$$\begin{aligned} g'(x) =&(2 - 4x + \frac{11}{4}x^2 - 2x^3)e^{2x-2x^2+\frac{11}{12}x^3-\frac{1}{2}x^4} \\ &- (2 + 4x + \frac{11}{4}x^2 + 2x^3)e^{-(2x+2x^2+\frac{11}{12}x^3+\frac{1}{2}x^4)}, \end{aligned}$$

the inequality $g'(x) \le 0$ is equivalent to

$$e^{-4x-\frac{11}{6}x^3} \ge \frac{8-16x+11x^2-8x^3}{8+16x+11x^2+8x^3}.$$

For the nontrivial case $8-16x+11x^2-8x^3>0$, rewrite this inequality as $h(x) \ge 0$, where

$$h(x) = -4x - \frac{11}{6}x^3 - \ln(8-16x+11x^2-8x^3) + \ln(8+16x+11x^2+8x^3).$$

If $h' \ge 0$, then h is increasing, hence $h(x) \ge h(0) = 0$. From

$$h'(x) = -4 - \frac{11}{2}x^2 + \frac{(16+24x^2)-22x}{8+11x^2-(16x+8x^3)} + \frac{(16+24x^2)+22x}{8+11x^2+(16x+8x^3)},$$

it follows that $h'(x) \ge 0$ is equivalent to

$$(16+24x^2)(8+11x^2) - 22x(16x+8x^3) \ge \frac{1}{4}(8+11x^2)[(8+11x^2)^2-(16x+8x^3)^2].$$

Since

$$(8+11x^2)^2-(16x+8x^3)^2 \le (8+11x^2)^2 - 256x^2 - 256x^4 \le 16(4-5x^2),$$

it suffices to show that

$$(4+6x^2)(8+11x^2) - 11x(8x+4x^3) \ge (8+11x^2)(4-5x^2).$$

This inequality reduces to $77x^4 \ge 0$. The proof is completed. The equality holds for $a=b=1$.

Lemma. *If* $-1 < t \le \frac{13}{20}$, *then*

$$\sqrt{1-t}\,\ln(1+t) \le t - t^2 + \frac{11}{24}t^3 - \frac{1}{4}t^4.$$

Proof. Consider two cases.

Case 1: $0 \le t \le \frac{13}{20}$. We can prove the desired inequality by multiplying the following inequalities

$$\sqrt{1-t} \le 1 - \frac{1}{2}t - \frac{1}{8}t^2 - \frac{1}{16}t^3,$$

$$\ln(1+t) \le t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \frac{1}{4}t^4 + \frac{1}{5}t^5,$$

$$\left(1 - \frac{1}{2}t - \frac{1}{8}t^2 - \frac{1}{16}t^3\right)\left(t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \frac{1}{4}t^4 + \frac{1}{5}t^5\right) \le t - t^2 + \frac{11}{24}t^3 - \frac{1}{4}t^4.$$

The first inequality is equivalent to $f(t) \ge 0$, where

$$f(t) = \ln\left(1 - \frac{1}{2}t - \frac{1}{8}t^2 - \frac{1}{16}t^3\right) - \frac{1}{2}\ln(1-t).$$

Since

$$f'(t) = \frac{1}{2(1-t)} - \frac{8+4t+3t^2}{16-8t-2t^2-t^3} = \frac{5t^3}{2(1-t)(16-8t-2t^2-t^3)} \ge 0,$$

f is increasing, hence $f(t) \ge f(0) = 0$.
The second inequality is equivalent to $f(t) \ge 0$, where

$$f(t) = t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \frac{1}{4}t^4 + \frac{1}{5}t^5 - \ln(1+t).$$

Since

$$f'(t) = 1 - t + t^2 - t^3 + t^4 - \frac{1}{1+t} = \frac{t^5}{1+t} \ge 0,$$

$f(t)$ is increasing, hence $f(t) \ge f(0) = 0$.
The third inequality is equivalent to

$$t^4(160 - 302t + 86t^2 + 9t^3 + 12t^4) \ge 0.$$

This is true since

$$160 - 302t + 86t^2 + 9t^3 + 12t^4 \ge 2(80 - 151t + 43t^2) > 0.$$

Case 2: $-1 < t \le 0$. Write the desired inequality as

$$-\sqrt{1-t}\ln(1+t) \ge -t + t^2 - \frac{11}{24}t^3 + \frac{1}{4}t^4.$$

This is true if

$$\sqrt{1-t} \ge 1 - \frac{1}{2}t - \frac{1}{8}t^2,$$

$$-\ln(1+t) \ge -t + t^2 - \frac{1}{3}t^3 + \frac{1}{4}t^4,$$

$$\left(1 - \frac{1}{2}t - \frac{1}{8}t^2\right)\left(-t + t^2 - \frac{1}{3}t^3 + \frac{1}{4}t^4\right) \ge -t + t^2 - \frac{11}{24}t^3 + \frac{1}{4}t^4.$$

The first inequality is equivalent to $f(t) \ge 0$, where

$$f(t) = \frac{1}{2}\ln(1-t) - \ln\left(1 - \frac{1}{2}t - \frac{1}{8}t^2\right).$$

Since

$$f'(t) = \frac{-1}{2(1-t)} + \frac{2(2+t)}{8-4t-t^2} = \frac{-3t^2}{2(1-t)(8-4t-t^2)} \le 0,$$

f is decreasing, hence $f(t) \ge f(0) = 0$.

The second inequality is equivalent to $f(t) \ge 0$, where

$$f(t) = t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \frac{1}{4}t^4 - \ln(1+t).$$

Since

$$f'(t) = 1 - t + t^2 - t^3 - \frac{1}{1+t} = \frac{-t^4}{1+t} \le 0,$$

f is decreasing, hence $f(t) \ge f(0) = 0$.

The third inequality reduces to the obvious inequality

$$t^4(10 - 8t - 3t^2) \ge 0.$$

□

P 3.35. *If a, b are nonnegative real numbers such that $a + b = 2$, then*

$$\frac{ab(1-ab)^2}{2} \le a^{b+1} + b^{a+1} - 2 \le \frac{ab(1-ab)^2}{3}.$$

(Vasile Cîrtoaje, 2010)

Solution. Assume that $a \ge b$, which yields $1 \le a \le 2$ and $0 \le b \le 1$.

(a) To prove the left inequality we apply Lemma 1 below. For $x = a$ and $k = b$, we have

$$a^{b+1} \ge 1 + (1+b)(a-1) + \frac{b(1+b)}{2}(a-1)^2 - \frac{b(1+b)(1-b)}{6}(a-1)^3,$$

$$a^{b+1} \ge a - b + ab + \frac{b(1+b)}{2}(a-1)^2 - \frac{b(1+b)}{6}(a-1)^4. \qquad (*)$$

Also, for $x = b$ and $k = a - 1$, we have

$$b^a \ge 1 + a(b-1) + \frac{a(a-1)}{2}(b-1)^2 - \frac{a(a-1)(2-a)}{6}(b-1)^3,$$

$$b^a \ge 1 - a + ab + \frac{a}{2}(a-1)^3 + \frac{ab}{6}(a-1)^4,$$

$$b^{a+1} \ge b - ab + ab^2 + \frac{ab}{2}(a-1)^3 + \frac{ab^2}{6}(a-1)^4. \qquad (**)$$

Summing up (*) and (**) gives

$$\begin{aligned} a^{b+1}+b^{a+1}-2 &\geq -b(a-1)^2+\frac{b(3-ab)}{2}(a-1)^2-\frac{b(1+b-ab)}{6}(a-1)^4 \\ &= \frac{b}{2}(a-1)^4-\frac{b(1+b-ab)}{6}(a-1)^4 \\ &= \frac{ab(1+b)}{6}(a-1)^4 \geq \frac{ab}{6}(a-1)^4=\frac{ab(1-ab)^2}{6}. \end{aligned}$$

The equality holds for $a=b=1$, for $a=2$ and $b=0$, and for $a=0$ and $b=2$.

(b) To prove the right inequality we apply Lemma 2 below. For $x=a$ and $k=b$, we have

$$\begin{aligned} a^{b+1} \leq 1+(b+1)(a-1)+\frac{(b+1)b}{2}(a-1)^2+\frac{(b+1)b(b-1)}{6}(a-1)^3 \\ +\frac{(b+1)b(b-1)(b-2)}{24}(a-1)^4, \end{aligned}$$

$$a^{b+1} \leq 1+(b+1)(a-1)+\frac{b(b+1)}{2}(a-1)^2-\frac{b(b+1)}{6}(a-1)^4+\frac{ab(b+1)}{24}(a-1)^5.$$

Also, for $x=b$ and $k=a$, we have

$$b^{a+1} \leq 1+(a+1)(b-1)+\frac{a(a+1)}{2}(b-1)^2-\frac{a(a+1)}{6}(b-1)^4+\frac{ab(a+1)}{24}(b-1)^5.$$

Summing up these inequalities and having in view that

$$(b+1)(a-1)^5+(a+1)(b-1)^5=-2(a-1)^5 \leq 0$$

give

$$\begin{aligned} a^{b+1}+b^{a+1}-2 &\leq -2(a-1)^2+\frac{a^2+b^2+2}{2}(a-1)^2-\frac{a^2+b^2+2}{6}(a-1)^4 \\ &\leq \frac{a^2+b^2-2}{2}(a-1)^2-\frac{a^2+b^2+2}{6}(a-1)^4 \\ &= (a-1)^4-\frac{a^2+b^2+2}{6}(a-1)^4 \\ &= \frac{ab}{3}(a-1)^4=\frac{ab(1-ab)^2}{3}. \end{aligned}$$

The equality holds for $a=b=1$, for $a=2$ and $b=0$, and for $a=0$ and $b=2$.

Lemma 1. *If $x \geq 0$ and $0 \leq k \leq 1$, then*

$$x^{k+1} \geq 1+(1+k)(x-1)+\frac{k(1+k)}{2}(x-1)^2-\frac{k(1+k)(1-k)}{6}(x-1)^3,$$

with equality for $x=1$, for $k=0$ and for $k=1$.

Proof. For $k = 0$ and $k = 1$, the inequality is an identity. For fixed k, $0 < k < 1$, let us define

$$f(x) = x^{k+1} - 1 - (1+k)(x-1) - \frac{k(1+k)}{2}(x-1)^2 + \frac{k(1+k)(1-k)}{6}(x-1)^3.$$

We need to show that $f(x) \ge 0$. We have

$$\frac{1}{1+k}f'(x) = x^k - 1 - k(x-1) + \frac{k(1-k)}{2}(x-1)^2,$$

$$\frac{1}{k(1+k)}f''(x) = x^{k-1} - 1 + (1-k)(x-1),$$

$$\frac{1}{k(1+k)(1-k)}f'''(x) = -x^{k-2} + 1.$$

Case 1: $0 \le x \le 1$. Since $f''' \le 0$, f'' is decreasing, $f''(x) \ge f''(1) = 0$, f' is increasing, $f'(x) \le f'(1) = 0$, f is decreasing, hence $f(x) \ge f(1) = 0$.

Case 2: $x \ge 1$. Since $f''' \ge 0$, f'' is increasing, $f''(x) \ge f''(1) = 0$, f' is increasing, $f'(x) \ge f'(1) = 0$, f is increasing, hence $f(x) \ge f(1) = 0$.

Lemma 2. *If either $x \ge 1$ and $0 \le k \le 1$, or $0 \le x \le 1$ and $1 \le k \le 2$, then*

$$x^{k+1} \le 1 + (k+1)(x-1) + \frac{(k+1)k}{2}(x-1)^2 + \frac{(k+1)k(k-1)}{6}(x-1)^3$$
$$+\frac{(k+1)k(k-1)(k-2)}{24}(x-1)^4,$$

with equality for $x = 1$, for $k = 0$, for $k = 1$ and for $k = 2$.

Proof. For $k = 0$, $k = 1$ and $k = 2$, the inequality is an identity. For fixed k, $k \in (0,1)\cup(1,2)$, let us define

$$f(x) = x^{k+1} - 1 - (k+1)(x-1) - \frac{(k+1)k}{2}(x-1)^2 - \frac{(k+1)k(k-1)}{6}(x-1)^3$$
$$-\frac{(k+1)k(k-1)(k-2)}{24}(x-1)^4.$$

We need to show that $f(x) \le 0$. We have

$$\frac{1}{k+1}f'(x) = x^k - 1 - k(x-1) - \frac{k(k-1)}{2}(x-1)^2 - \frac{k(k-1)(k-2)}{6}(x-1)^3,$$

$$\frac{1}{k(k+1)}f''(x) = x^{k-1} - 1 - (k-1)(x-1) - \frac{(k-1)(k-2)}{2}(x-1)^2,$$

$$\frac{1}{k(k+1)(k-1)}f'''(x) = x^{k-2} - 1 - (k-2)(x-1),$$

$$\frac{1}{k(k+1)(k-1)(k-2)}f^{(4)}(x) = x^{k-3} - 1.$$

Case 1: $x \ge 1,\ 0 < k < 1$. Since $f^{(4)}(x) \le 0$, $f'''(x)$ is decreasing, $f'''(x) \le f'''(1) = 0$, f'' is decreasing, $f''(x) \le f''(1) = 0$, f' is decreasing, $f'(x) \le f'(1) = 0$, f is decreasing, hence $f(x) \le f(1) = 0$.

Case 2: $0 \le x \le 1,\ 1 < k < 2$. Since $f^{(4)} \le 0$, f''' is decreasing, $f'''(x) \ge f'''(1) = 0$, f'' is increasing, $f''(x) \le f''(1) = 0$, f' is decreasing, $f'(x) \ge f'(1) = 0$, f is increasing, hence $f(x) \le f(1) = 0$.

□

P 3.36. *If a, b are nonnegative real numbers such that $a + b = 1$, then*

$$a^{2b} + b^{2a} \le 1.$$

(Vasile Cîrtoaje, 2007)

Solution. Without loss of generality, assume that

$$0 \le b \le \frac{1}{2} \le a \le 1.$$

Applying Lemma 1 below for $c = 2b$, $0 \le c \le 1$, we get

$$a^{2b} \le (1-2b)^2 + 4ab(1-b) - 2ab(1-2b)\ln a,$$

which is equivalent to

$$a^{2b} \le 1 - 4ab^2 - 2ab(a-b)\ln a.$$

Similarly, applying Lemma 2 below for $d = 2a - 1$, $d \ge 0$, we get

$$b^{2a-1} \le 4a(1-a) + 2a(2a-1)\ln(2a+b-1),$$

which is equivalent to

$$b^{2a} \le 4ab^2 + 2ab(a-b)\ln a.$$

Adding up these inequalities, the desired inequality follows. The equality holds for $a = b = 1/2$, for $a = 0$ and $b = 1$, and for $a = 1$ and $b = 0$.

Lemma 1. *If $0 < a \le 1$ and $c \ge 0$, then*

$$a^c \le (1-c)^2 + ac(2-c) - ac(1-c)\ln a,$$

with equality for $a = 1$, for $c = 0$ and for $c = 1$.

Proof. Making the substitution

$$a = e^{-x}, \qquad x \ge 0,$$

we need to prove that $f(x) \ge 0$, where

$$f(x) = (1-c)^2 e^x + c(2-c) + c(1-c)x - e^{(1-c)x},$$

$$f'(x) = (1-c)[(1-c)e^x + c - e^{(1-c)x}].$$

If $f' \ge 0$ on $[0, \infty)$, then f is increasing, and hence $f(x) \ge f(0) = 0$. In order to prove that $f' \ge 0$, we consider two cases.

Case 1: $0 \le c \le 1$. By the weighted AM-GM inequality, we have

$$(1-c)e^x + c \ge e^{(1-c)x},$$

hence $f'(x) \ge 0$.

Case 2: $c \ge 1$. By the weighted AM-GM inequality, we have

$$(c-1)e^x + e^{(1-c)x} \ge c,$$

which yields

$$f'(x) = (c-1)[(c-1)e^x + e^{(1-c)x} - c] \ge 0.$$

Lemma 2. *If* $0 \le b \le 1$ *and* $d \ge 0$*, then*

$$b^d \le 1 - d^2 + d(1+d)\ln(b+d),$$

with equality for $b = 0$ *and for* $d = 0$*.*

Proof. Consider $0 < b \le 1$ and $d > 0$, and write the inequality as

$$(1+d)[1 - d + d\ln(b+d)] \ge b^d.$$

Since

$$1 - d + d\ln(b+d) > 1 - d + d\ln d \ge 0,$$

we can rewrite the inequality in the form

$$\ln(1+d) + \ln[1 - d + d\ln(b+d)] \ge d\ln b.$$

Using the substitution

$$b = e^{-x} - d, \qquad -\ln(1+d) \le x < -\ln d,$$

we need to prove that $f(x) \ge 0$, where

$$f(x) = \ln(1+d) + \ln(1 - d - dx) + dx - d\ln(1 - de^x).$$

Since

$$f'(x) = \frac{d^2(e^x - 1 - x)}{(1 - d - dx)(1 - de^x)} \ge 0,$$

f is increasing, hence

$$f(x) \geq f(-\ln(1+d)) = \ln[1 - d^2 + d(1+d)\ln(1+d)].$$

To complete the proof, we only need to show that $-d^2 + d(1+d)\ln(1+d) \geq 0$; that is,

$$(1+d)\ln(1+d) \geq d.$$

This inequality follows from $e^x \geq 1 + x$, where $x = \dfrac{-d}{1+d}$.

Conjecture. *If a, b are nonnegative real numbers such that $1 \leq a + b \leq 15$, then*

$$a^{2b} + b^{2a} \leq a^{a+b} + b^{a+b}.$$

□

P 3.37. *If a, b are positive real numbers such that $a + b = 1$, then*

$$2a^a b^b \geq a^{2b} + b^{2a}.$$

Solution. Taking into account the inequality $a^{2b} + b^{2a} \leq 1$ from the preceding P 3.36, it suffices to show that

$$2a^a b^b \geq 1.$$

Write this inequality as

$$2a^a b^b \geq a^{a+b} + b^{a+b},$$

$$2 \geq \left(\frac{a}{b}\right)^b + \left(\frac{b}{a}\right)^a.$$

Since $a < 1$ and $b < 1$, we apply Bernoulli's inequality as follows:

$$\left(\frac{a}{b}\right)^b + \left(\frac{b}{a}\right)^a \leq 1 + b\left(\frac{a}{b} - 1\right) + 1 + a\left(\frac{b}{a} - 1\right) = 2.$$

Thus, the proof is completed. The equality holds for $a = b = 1/2$.

□

P 3.38. *If a, b are positive real numbers such that $a + b = 1$, then*

$$a^{-2a} + b^{-2b} \leq 4.$$

Solution. Applying Lemma below, we have

$$a^{-2a} \le 4 - 2\ln 2 - 4(1-\ln 2)a,$$

$$b^{-2b} \le 4 - 2\ln 2 - 4(1-\ln 2)b.$$

Adding these inequalities, the desired inequality follows. The equality holds for $a = b = 1/2$.

Lemma. *If* $x \in (0,1]$, *then*

$$x^{-2x} \le 4 - 2\ln 2 - 4(1-\ln 2)x,$$

with equality for $x = 1/2$.

Proof. Write the inequality as

$$\frac{1}{4}x^{-2x} \le 1 - c - (1-2c)x, \quad c = \frac{1}{2}\ln 2 \approx 0.346.$$

This is true if $f(x) \le 0$, where

$$f(x) = -2\ln 2 - 2x\ln x - \ln[1 - c - (1-2c)x].$$

We have

$$f'(x) = -2 - 2\ln x + \frac{1-2c}{1-c-(1-2c)x}\ ,$$

$$f''(x) = -\frac{2}{x} + \frac{(1-2c)^2}{[1-c-(1-2c)x]^2} = \frac{g(x)}{x[1-c-(1-2c)x]^2}\ ,$$

where

$$g(x) = 2(1-2c)^2x^2 - (1-2c)(5-6c)x + 2(1-c)^2.$$

Since

$$\begin{aligned} g'(x) &= (1-2c)[4(1-2c)x - 5 + 6c] \le (1-2c)[4(1-2c) - 5 + 6c] \\ &= (1-2c)(-1-2c) < 0, \end{aligned}$$

g is decreasing on $(0,1]$, hence $g(x) \ge g(1) = -2c^2 + 4c - 1 > 0$, $f''(x) > 0$ for $x \in (0,1]$, f' is increasing. Since $f'(1/2) = 0$, we have $f'(x) \le 0$ for $x \in (0,1/2]$ and $f'(x) \ge 0$ for $x \in [1/2,1]$. Therefore, f is decreasing on $(0,1/2]$ and increasing on $[1/2,1]$, hence $f(x) \ge f(1/2) = 0$.

Remark. According to the inequalities in P 3.36 and P 3.38, the following inequality holds for all positive numbers a, b such that $a + b = 1$:

$$\left(a^{2b} + b^{2a}\right)\left(\frac{1}{a^{2a}} + \frac{1}{b^{2b}}\right) \le 4.$$

Actually, this inequality holds for all $a, b \in (0,1]$. In this case, it is sharper than the inequality in P 3.19.

□

P 3.39. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1a_2\cdots a_n = 1$, then*

$$\left(1-\frac{1}{n}\right)^{a_1} + \left(1-\frac{1}{n}\right)^{a_2} + \cdots + \left(1-\frac{1}{n}\right)^{a_n} \le n-1.$$

(Vasile Cîrtoaje, 2004)

Solution. We will prove the more general inequality

$$\left(1-\frac{1}{n}\right)^{a_1} + \left(1-\frac{1}{n}\right)^{a_2} + \cdots + \left(1-\frac{1}{n}\right)^{a_n} \le n\left(1-\frac{1}{n}\right)^{a}, \tag{*}$$

where $a = \sqrt[n]{a_1a_2\cdots a_n} \le 1$. Using the substitution

$$x_i = a_i \ln\frac{n}{n-1}, \qquad i = 1, 2, \ldots, n,$$

the inequality becomes as follows:

$$e^{-x_1} + e^{-x_2} + \cdots + e^{-x_n} \le ne^{-r}, \tag{**}$$

where

$$r = \sqrt[n]{x_1x_2\cdots x_n} \le \ln\frac{n}{n-1}.$$

To prove this inequality, we use the induction technique. For $n = 1$, (**) is an equality. Consider now that (**) holds for $n-1$ numbers, $n \ge 2$, and show that it also holds for n numbers. Assume that

$$x_1 \le x_2 \le \cdots \le x_n,$$

and denote

$$x = \sqrt[n-1]{x_1x_2\cdots x_{n-1}}.$$

Because

$$x \le r \le \ln\frac{n}{n-1} < \ln\frac{n-1}{(n-1)-1},$$

the induction hypothesis yields

$$e^{-x_1} + e^{-x_2} + \cdots + e^{-x_{n-1}} \le (n-1)e^{-x}.$$

Thus, we only need to show that

$$e^{-x_n} + (n-1)e^{-x} \le ne^{-r},$$

which is equivalent to

$$f(x) \le ne^{-r}$$

for

$$0 < x \le r \le \ln\frac{n}{n-1} < 1,$$

where

$$f(x) = e^{-r^n/x^{n-1}} + (n-1)e^{-x}.$$

We have

$$\frac{x^n e^{r^n/x^{n-1}}}{n-1} f'(x) = g(x), \qquad g(x) = r^n - x^n e^{r^n/x^{n-1}-x},$$

$$e^{x-r^n/x^{n-1}} g'(x) = h(x), \qquad h(x) = x^n - nx^{n-1} + (n-1)r^n,$$

$$h'(x) = nx^{n-2}(x-n+1).$$

Since $h'(x) < 0$, h is strictly decreasing, and from

$$h(0) = (n-1)r^n > 0, \qquad h(r) = nr^{n-1}(r-1) < 0,$$

it follows that there exists $x_1 \in (0,r)$ such that $h(x_1) = 0$, $h(x) > 0$ for $x \in (0,x_1)$, $h(x) < 0$ for $x \in (x_1, r]$. Therefore, g is strictly increasing on $(0, x_1]$ and strictly decreasing on $[x_1, r]$. Since $g(0_+) = -\infty$ and $g(r) = 0$, there exists $x_2 \in (0, x_1)$ such that $g(x_2) = 0$, $g(x) < 0$ for $x \in (0, x_2)$, $g(x) > 0$ for $x \in (x_2, r]$. Consequently, f is strictly decreasing on $(0, x_2]$ and strictly increasing on $[x_2, r]$, hence

$$f(x) \le \max\{f(0_+), f(r)\} = \max\{n-1, ne^{-r}\} = ne^{-r}.$$

Thus, the proof is completed. The inequality (**) is an equality for

$$x_1 = x_2 = \cdots = x_n \le \ln \frac{n}{n-1},$$

the inequality (*) for

$$a_1 = a_2 = \cdots = a_n \le 1,$$

and the original inequality for

$$a_1 = a_2 = \cdots = a_n = 1.$$

□

Appendix A

Glosar

1. AM-GM (ARITHMETIC MEAN-GEOMETRIC MEAN) INEQUALITY

If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers, then

$$a_1 + a_2 + \cdots + a_n \geq n\sqrt[n]{a_1 a_2 \cdots a_n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

2. WEIGHTED AM-GM INEQUALITY

Let $p_1, p_2, \ldots, p_n$ be positive real numbers satisfying

$$p_1 + p_2 + \cdots + p_n = 1.$$

If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers, then

$$p_1 a_1 + p_2 a_2 + \cdots + p_n a_n \geq a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

3. AM-HM (ARITHMETIC MEAN-HARMONIC MEAN) INEQUALITY

If $a_1, a_2, \ldots, a_n$ are positive real numbers, then

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \geq n^2,$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

4. POWER MEAN INEQUALITY

The power mean of order k of positive real numbers $a_1, a_2, \ldots, a_n$, that is

$$M_k = \begin{cases} \left(\frac{a_1^k + a_2^k + \cdots + a_n^k}{n}\right)^{\frac{1}{k}}, & k \neq 0 \\ \sqrt[n]{a_1 a_2 \cdots a_n}, & k = 0 \end{cases},$$

is an increasing function with respect to $k \in \mathbb{R}$. For instant, $M_2 \geq M_1 \geq M_0 \geq M_{-1}$ is equivalent to

$$\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \geq \frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}.$$

5. BERNOULLI'S INEQUALITY

For any real number $x \geq -1$, we have

a) $(1+x)^r \geq 1 + rx$ for $r \geq 1$ and $r \leq 0$;

b) $(1+x)^r \leq 1 + rx$ for $0 \leq r \leq 1$.

If $a_1, a_2, \ldots, a_n$ are real numbers such that either $a_1, a_2, \ldots, a_n \geq 0$ or

$$-1 \leq a_1, a_2, \ldots, a_n \leq 0,$$

then

$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) \geq 1 + a_1 + a_2 + \cdots + a_n.$$

6. SCHUR'S INEQUALITY

For any nonnegative real numbers a, b, c and any positive number k, the inequality holds

$$a^k(a-b)(a-c) + b^k(b-c)(b-a) + c^k(c-a)(c-b) \geq 0,$$

with equality for $a = b = c$, and for $a = 0$ and $b = c$ (or any cyclic permutation).
For $k = 1$, we get the third degree Schur's inequality, which can be rewritten as follows

$$a^3 + b^3 + c^3 + 3abc \geq ab(a+b) + bc(b+c) + ca(c+a),$$

$$(a+b+c)^3 + 9abc \geq 4(a+b+c)(ab+bc+ca),$$

$$a^2 + b^2 + c^2 + \frac{9abc}{a+b+c} \geq 2(ab+bc+ca),$$

$$(b-c)^2(b+c-a) + (c-a)^2(c+a-b) + (a-b)^2(a+b-c) \geq 0.$$

For $k = 2$, we get the fourth degree Schur's inequality, which holds for any real numbers a, b, c, and can be rewritten as follows

$$a^4 + b^4 + c^4 + abc(a + b + c) \geq ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2),$$

$$a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2 \geq (ab + bc + ca)(a^2 + b^2 + c^2 - ab - bc - ca),$$

$$(b - c)^2(b + c - a)^2 + (c - a)^2(c + a - b)^2 + (a - b)^2(a + b - c)^2 \geq 0,$$

$$6abcp \geq (p^2 - q)(4q - p^2), \qquad p = a + b + c, \quad q = ab + bc + ca.$$

A generalization of the fourth degree Schur's inequality for any real numbers a, b, c and any real number m, is the following (*V. Cîrtoaje*, 2004):

$$\sum (a - mb)(a - mc)(a - b)(a - c) \geq 0,$$

where the equality holds for $a = b = c$, and for $a/m = b = c$ (or any cyclic permutation). This inequality is equivalent to

$$\sum a^4 + m(m + 2)\sum a^2b^2 + (1 - m^2)abc\sum a \geq (m + 1)\sum ab(a^2 + b^2),$$

$$\sum (b - c)^2(b + c - a - ma)^2 \geq 0.$$

A more general result is given by the following theorem (*V. Cîrtoaje*, 2008).

Theorem. *Let*

$$f_4(a, b, c) = \sum a^4 + \alpha \sum a^2b^2 + \beta abc \sum a - \gamma \sum ab(a^2 + b^2),$$

where α, β, γ *are real constants such that* $1 + \alpha + \beta = 2\gamma$. *Then,*

(a) $f_4(a, b, c) \geq 0$ *for all* $a, b, c \in \mathbb{R}$ *if and only if*

$$1 + \alpha \geq \gamma^2;$$

(b) $f_4(a, b, c) \geq 0$ *for all* $a, b, c \geq 0$ *if and only if*

$$\alpha \geq (\gamma - 1)\max\{2, \gamma + 1\}.$$

7. CAUCHY-SCHWARZ INEQUALITY

If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers, then

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2,$$

with equality for

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}.$$

Notice that the equality conditions are also valid for $a_i = b_i = 0$, where $1 \leq i \leq n$.

8. HÖLDER'S INEQUALITY

If x_{ij} $(i = 1, 2, \cdots, m; j = 1, 2, \cdots n)$ are nonnegative real numbers, then

$$\prod_{i=1}^{m}\left(\sum_{j=1}^{n} x_{ij}\right) \geq \left(\sum_{j=1}^{n} \sqrt[m]{\prod_{i=1}^{m} x_{ij}}\right)^{m}.$$

9. REARRANGEMENT INEQUALITY

(1) If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are two increasing (or decreasing) real sequences, and $(i_1, i_2, \cdots, i_n)$ is an arbitrary permutation of $(1, 2, \cdots, n)$, then

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \geq a_1b_{i_1} + a_2b_{i_2} + \cdots + a_nb_{i_n}.$$

(2) If $a_1, a_2, \ldots, a_n$ is decreasing and $b_1, b_2, \ldots, b_n$ is increasing, then

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \leq a_1b_{i_1} + a_2b_{i_2} + \cdots + a_nb_{i_n}.$$

(3) Let $b_1, b_2, \ldots, b_n$ and $c_1, c_2, \ldots, c_n$ be two real sequences such that

$$b_1 + \cdots + b_k \geq c_1 + \cdots + c_k, \quad k = 1, 2, \cdots, n.$$

If $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$, then

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \geq a_1c_1 + a_2c_2 + \cdots + a_nc_n.$$

10. CHEBYSHEV'S INEQUALITY

Let $a_1 \geq a_2 \geq \cdots \geq a_n$ be real numbers.

a) If $b_1 \geq b_2 \geq \cdots b_n$, then

$$n\sum_{i=1}^{n} a_ib_i \geq \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right);$$

b) If $b_1 \leq b_2 \leq \cdots \leq b_n$, then

$$n\sum_{i=1}^{n} a_ib_i \leq \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right).$$

11. MINKOWSKI'S INEQUALITY

For any real number $k \geq 1$ and any positive real numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$, the following inequalities hold:

$$\sum_{i=1}^{n} \left(a_i^k + b_i^k\right)^{\frac{1}{k}} \geq \left[\left(\sum_{i=1}^{n} a_i\right)^k + \left(\sum_{i=1}^{n} b_i\right)^k\right]^{\frac{1}{k}};$$

$$\sum_{i=1}^{n} \left(a_i^k + b_i^k + c_i^k\right)^{\frac{1}{k}} \geq \left[\left(\sum_{i=1}^{n} a_i\right)^k + \left(\sum_{i=1}^{n} b_i\right)^k + \left(\sum_{i=1}^{n} c_i\right)^k\right]^{\frac{1}{k}}.$$

12. CONVEX FUNCTIONS

A function f defined on a real interval $\mathbb{I}$ is said to be *convex* if

$$f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y)$$

for all x, $y \in \mathbb{I}$ and any α, $\beta \geq 0$ with $\alpha + \beta = 1$. If the inequality is reversed, then f is said to be concave.
If f is differentiable on $\mathbb{I}$, then f is (strictly) convex if and only if the derivative f' is (strictly) increasing. If $f'' \geq 0$ on $\mathbb{I}$, then f is convex on $\mathbb{I}$.

Jensen's inequality. *Let $p_1, p_2, \ldots, p_n$ be positive real numbers. If f is a convex function on a real interval $\mathbb{I}$, then for any a_1, a_2, $\ldots$, $a_n \in \mathbb{I}$, the inequality holds*

$$\frac{p_1 f(a_1) + p_2 f(a_2) + \cdots + p_n f(a_n)}{p_1 + p_2 + \cdots + p_n} \geq f\left(\frac{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n}{p_1 + p_2 + \cdots + p_n}\right).$$

For $p_1 = p_2 = \cdots = p_n$, Jensen's inequality becomes

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right).$$

13. KARAMATA'S MAJORIZATION INEQUALITY

Let f be a convex function on a real interval $\mathbb{I}$. If a decreasingly ordered sequence

$$A = (a_1, a_2, \ldots, a_n), \quad a_i \in \mathbb{I},$$

majorizes a decreasingly ordered sequence

$$B = (b_1, b_2, \ldots, b_n), \quad b_i \in \mathbb{I},$$

then

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq f(b_1) + f(b_2) + \cdots + f(b_n).$$

We say that a sequence $A = (a_1, a_2, \ldots, a_n)$ with $a_1 \geq a_2 \geq \cdots \geq a_n$ majorizes a sequence $B = (b_1, b_2, \ldots, b_n)$ with $b_1 \geq b_2 \geq \cdots \geq b_n$, and write it as

$$A \succ B,$$

if

$$\begin{gathered} a_1 \geq b_1, \\ a_1 + a_2 \geq b_1 + b_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_1 + a_2 + \cdots + a_{n-1} \geq b_1 + b_2 + \cdots + b_{n-1}, \\ a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n. \end{gathered}$$

14. SYMMETRIC INEQUALITIES OF DEGREE THREE, FOUR OR FIVE

Theorem (*Vasile Cirtoaje*, 2010) *Let $f_n(a,b,c)$ be a symmetric homogeneous polynomial of degree n.*

(a) *The inequality $f_4(a,b,c) \geq 0$ holds for all real numbers a, b, c if and only if $f_4(a,1,1) \geq 0$ for all real a;*

(b) *For $n \in \{3,4,5\}$, the inequality $f_n(a,b,c) \geq 0$ holds for all $a,b,c \geq 0$ if and only if $f_n(a,1,1) \geq 0$ and $f_n(0,b,c) \geq 0$ for all $a,b,c \geq 0$.*

15. SYMMETRIC HOMOGENEOUS INEQUALITIES OF DEGREE SIX

Any sixth degree symmetric homogeneous polynomial $f_6(a,b,c)$ can be written in the form

$$f_6(a,b,c) = Ar^2 + B(p,q)r + C(p,q),$$

where A is called *the highest coefficient* of f_6, and

$$p = a + b + c, \qquad q = ab + bc + ca, \qquad r = abc.$$

Theorem (*Vasile Cirtoaje*, 2010). *Let $f_6(a,b,c)$ be a sixth degree symmetric homogeneous polynomial having the highest coefficient $A \leq 0$.*

(a) *The inequality $f_6(a,b,c) \geq 0$ holds for all real numbers a,b,c if and only if $f_6(a,1,1) \geq 0$ for all real a;*

(b) *The inequality $f_6(a,b,c) \geq 0$ holds for all $a,b,c \geq 0$ if and only if $f_6(a,1,1) \geq 0$ and $f_6(0,b,c) \geq 0$ for all $a,b,c \geq 0$.*

This theorem is also valid for the case where $B(p,q)$ and $C(p,q)$ are homogeneous rational functions.

For $A > 0$, we can use the *highest coefficient cancellation method* (*Vasile Cirtoaje*, 2010). This method consists in finding some suitable real numbers B, C and D such that the following sharper inequality holds

$$f_6(a,b,c) \ge A\left(r + Bp^3 + Cpq + D\frac{q^2}{p}\right)^2.$$

Because the function g_6 defined by

$$g_6(a,b,c) = f_6(a,b,c) - A\left(r + Bp^3 + Cpq + D\frac{q^2}{p}\right)^2$$

has the highest coefficient $A_1 = 0$, we can prove the inequality $g_6(a,b,c) \ge 0$ using Theorem above.

Notice that sometimes it is useful to break the problem into two parts, $p^2 \le \xi q$ and $p^2 > \xi q$, where ξ is a suitable real number.

A symmetric homogeneous polynomial of degree six in three variables has the form

$$f_6(a,b,c) = A_1\sum a^6 + A_2\sum ab(a^4+b^4) + A_3\sum a^2b^2(a^2+b^2)$$

$$+A_4\sum a^3b^3 + A_5abc\sum a^3 + A_6abc\sum ab(a+b) + 3A_7a^2b^2c^2,$$

where $A_1,\dots,A_7$ are real constants. In order to write this polynomial as a function of p, q and r, the following relations are useful:

$$\sum a^3 = 3r + p^3 - 3pq,$$

$$\sum ab(a+b) = -3r + pq,$$

$$\sum a^3b^3 = 3r^2 - 3pqr + q^3,$$

$$\sum a^2b^2(a^2+b^2) = -3r^2 - 2(p^3 - 2pq)r + p^2q^2 - 2q^3,$$

$$\sum ab(a^4+b^4) = -3r^2 - 2(p^3 - 7pq)r + p^4q - 4p^2q^2 + 2q^3,$$

$$\sum a^6 = 3r^2 + 6(p^3 - 2pq)r + p^6 - 6p^4q + 9p^2q^2 - 2q^3,$$

$$(a-b)^2(b-c)^2(c-a)^2 = -27r^2 + 2(9pq - 2p^3)r + p^2q^2 - 4q^3.$$

According to these relations, the highest coefficient A of the polynomial $f_6(a,b,c)$ is

$$A = 3(A_1 - A_2 - A_3 + A_4 + A_5 - A_6 + A_7).$$

The polynomials

$$P_1(a,b,c) = \sum(A_1a^2 + A_2bc)(B_1a^2 + B_2bc)(C_1a^2 + C_2bc),$$

$$P_2(a,b,c) = \sum(A_1a^2 + A_2bc)(B_1b^2 + B_2ca)(C_1c^2 + C_2ab)$$

and

$$P_3(a,b,c) = (A_1a^2 + A_2bc)(A_1b^2 + A_2ca)(A_1c^2 + A_2ab)$$

has the highest coefficients

$$P_1(1,1,1), \quad P_2(1,1,1), \quad P_3(1,1,1),$$

respectively. The polynomial

$$P_4(a,b,c) = (a^2 + mab + b^2)(b^2 + mbc + c^2)(c^2 + mca + a^2)$$

has the highest coefficient

$$A = (m-1)^3.$$

16. SQUARE PRODUCT INEQUALITY

Let a, b, c be real numbers, and let

$$p = a+b+c, \quad q = ab+bc+ca, \quad r = abc,$$
$$s = \sqrt{p^2-3q} = \sqrt{a^2+b^2+c^2-ab-bc-ca}.$$

From the identity

$$(a-b)^2(b-c)^2(c-a)^2 = -27r^2 + 2(9pq-2p^3)r + p^2q^2 - 4q^3$$
$$= \frac{4(p^2-3q)^3 - (2p^3-9pq+27r)^2}{27},$$

it follows that

$$\frac{-2p^3+9pq-2(p^2-3q)\sqrt{p^2-3q}}{27} \le r \le \frac{-2p^3+9pq+2(p^2-3q)\sqrt{p^2-3q}}{27},$$

which is equivalent to

$$\frac{p^3-3ps^2-2s^3}{27} \le r \le \frac{p^3-3ps^2+2s^3}{27}.$$

Therefore, for constant p and q, the product r is minimum and maximum when two of a, b, c are equal.

17. VASC'S POWER EXPONENTIAL INEQUALITIES

Theorem. *Let* $0 < k \le e$.

(a) *If* $a, b > 0$, *then (Vasile Cîrtoaje, 2006)*

$$a^{ka} + b^{kb} \ge a^{kb} + b^{ka};$$

(b) *If* $a, b \in (0,1]$, *then (Vasile Cîrtoaje, 2010)*

$$2\sqrt{a^{ka}b^{kb}} \ge a^{kb} + b^{ka}.$$

Appendix B

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