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Chapter 1

Half Convex Function Method

1.1 Theoretical Basis

Let $\mathbb{I}$ be a real interval, s an interior point of $\mathbb{I}$ and

$$\mathbb{I}_{\geq s} = \{u | u \in \mathbb{I}, u \geq s\}, \quad \mathbb{I}_{\leq s} = \{u | u \in \mathbb{I}, u \leq s\}.$$

The following statement is known as the Right Half Convex Function Theorem (**RHCF-Theorem**).

Right Half Convex Function Theorem (*Vasile Cîrtoaje, 2004*). *Let f be a real function defined on an interval $\mathbb{I}$ and convex on $\mathbb{I}_{\geq S}$, where $S \in \text{int}(\mathbb{I})$. The inequality*

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \quad (1)$$

holds for all $a_1, a_2, \dots, a_n \in \mathbb{I}$ satisfying $\frac{a_1 + a_2 + \cdots + a_n}{n} = S$ if and only if it holds for all $a_1, a_2, \dots, a_n \in \mathbb{I}$ such that

$$a_1 \leq a_2 = a_3 = \cdots = a_n.$$

Proof. Since the necessity is obvious, we will prove further the sufficiency. Assume that

$$a_1 \leq a_2 \leq \cdots \leq a_n, \quad a_1 < a_n.$$

If $a_1 \geq S$, then the required inequality is just Jensen's inequality for convex functions. Otherwise, if $a_1 < S$, then there exists

$$k \in \{1, 2, \dots, n-1\}$$

so that

$$a_1 \leq \cdots \leq a_k < S \leq a_{k+1} \leq \cdots \leq a_n.$$

Since f is convex on $\mathbb{I}_{\geq S}$, we may apply Jensen's inequality to get

$$f(a_{k+1}) + \cdots + f(a_n) \geq (n-k)f(z),$$

where

$$z = \frac{a_{k+1} + \cdots + a_n}{n-k}, \quad z \in \mathbb{I}.$$

Thus, it suffices to show that

$$f(a_1) + \cdots + f(a_k) + (n - k)f(z) \geq nf(S). \quad (2)$$

Let $b_1, \dots, b_k$ be defined by

$$a_i + (n - 1)b_i = nS, \quad i = 1, \dots, k.$$

We claim that

$$z \geq b_1 \geq \cdots \geq b_k > S,$$

which involves

$$b_1, \dots, b_k \in \mathbb{I}_{>S}.$$

Indeed, we have

$$\begin{aligned} b_1 &\geq \cdots \geq b_k, \\ b_k - S &= \frac{S - a_k}{n - 1} > 0, \end{aligned}$$

and

$$z \geq b_1$$

because

$$\begin{aligned} (n - 1)b_1 &= nS - a_1 = (a_2 + \cdots + a_k) + a_{k+1} + \cdots + a_n \\ &\leq (k - 1)S + a_{k+1} + \cdots + a_n \\ &= (k - 1)S + (n - k)z \leq (n - 1)z. \end{aligned}$$

Since $b_1, \dots, b_k \in \mathbb{I}$, $a_i - b_i = \frac{n(a_i - S)}{n - 1} < 0$ for $i \leq k$, and (1) is true for $a_1 \leq a_2 = a_3 = \cdots = a_n$, we have

$$\begin{aligned} f(a_1) + (n - 1)f(b_1) &\geq nf(S), \\ &\dots \\ f(a_k) + (n - 1)f(b_k) &\geq nf(S), \end{aligned}$$

hence

$$\begin{aligned} f(a_1) + \cdots + f(a_k) + (n - 1)[f(b_1) + \cdots + f(b_k)] &\geq knf(S), \\ f(a_1) + \cdots + f(a_k) &\geq knf(S) - (n - 1)[f(b_1) + \cdots + f(b_k)]. \end{aligned}$$

According to this result, the inequality (2) is true if

$$knf(S) - (n - 1)[f(b_1) + \cdots + f(b_k)] + (n - k)f(z) \geq nf(S),$$

which is equivalent to

$$pf(z) + (k - p)f(S) \geq f(b_1) + \cdots + f(b_k), \quad p = \frac{n - k}{n - 1} \leq 1.$$

For $k = 1$, this inequality is an equality. For $k > 1$, by Jensen's inequality, we have

$$pf(z) + (1 - p)f(S) \geq f(w), \quad w = pz + (1 - p)S.$$

Thus, we only need to show that

$$f(w) + (k-1)f(S) \geq f(b_1) + f(b_2) + \cdots + f(b_k).$$

From $w + (k-1)S = b_1 + b_2 + \cdots + b_k$, we obtain

$$w - b_1 = (b_2 - S) + \cdots + (b_k - S) > 0.$$

Since the decreasingly ordered vector $\vec{A}_k = (w, S, \dots, S)$ majorizes the decreasingly ordered vector $\vec{B}_k = (b_1, b_2, \dots, b_k)$, this inequality follows from Karamata's inequality for convex functions.

Similarly, we can prove the Left Half Convex Function Theorem (**LHCF-Theorem**).

Left Half Convex Function Theorem. *Let f be a real function defined on an interval $\mathbb{I}$ and convex on $\mathbb{I}_{\leq S}$, where $S \in \text{int}(\mathbb{I})$. The inequality*

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \quad (3)$$

holds for all $a_1, a_2, \dots, a_n \in \mathbb{I}$ satisfying $\frac{a_1 + a_2 + \cdots + a_n}{n} = S$ if and only if it holds for all $a_1, a_2, \dots, a_n \in \mathbb{I}$ such that

$$a_1 = a_2 = \cdots = a_{n-1} \leq a_n.$$

From the RHCF-Theorem and the LHCF-Theorem, we find the **HCF-Theorem** (Half Convex Function Theorem).

Half Convex Function Theorem. *Let f be a real function defined on an interval $\mathbb{I}$ and convex on $\mathbb{I}_{\geq S}$ or $\mathbb{I}_{\leq S}$, where $S \in \text{int}(\mathbb{I})$. The inequality*

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \dots, a_n \in \mathbb{I}$ satisfying $\frac{a_1 + a_2 + \cdots + a_n}{n} = S$ if and only if it holds for the case when $n-1$ of the numbers $a_1, a_2, \dots, a_n$ are equal.

Note 1. When $n-1$ of the numbers $a_1, a_2, \dots, a_n$ are equal, the inequality (1) in RHCF-Theorem and the inequality (3) in LHCF-Theorem can be written as

$$f(x) + (n-1)f(y) \geq nf(S),$$

where $x, y \in \text{int}(\mathbb{I})$ such that $x + (n-1)y = nS$. Let us denote

$$g(u) = \frac{f(u) - f(s)}{u - s}, \quad h(x, y) = \frac{g(x) - g(y)}{x - y}.$$

In many applications, it is useful to replace the hypothesis

$$f(x) + (n-1)f(y) \geq nf(S)$$

by the equivalent condition

$$h(x, y) \geq 0 \text{ for all } x, y \in \mathbb{I} \text{ so that } x + (n-1)y = nS.$$

This equivalence is true because

$$\begin{aligned} f(x) + (n-1)f(y) - nf(S) &= [f(x) - f(S)] + (n-1)[f(y) - f(S)] \\ &= (x-S)g(x) + (n-1)(y-S)g(y) \\ &= \frac{n-1}{n}(x-y)[g(x) - g(y)] \\ &= \frac{n-1}{n}(x-y)^2 h(x, y). \end{aligned}$$

Note 2. Assume that f is differentiable on $\mathbb{I}$, and let

$$H(x, y) = \frac{f'(x) - f'(y)}{x - y}.$$

In some applications with HCF-Theorem, it is useful to replace the hypothesis

$$f(x) + (n-1)f(y) \geq nf(S)$$

with the more restrictive condition

$$H(x, y) \geq 0 \text{ for all } x, y \in \mathbb{I} \text{ so that } x + (n-1)y = nS.$$

To prove this, we will show that the new condition $H(x, y) \geq 0$ implies

$$f(x) + (n-1)f(y) \geq nf(S)$$

for all $x, y \in \mathbb{I}$ so that $x + (n-1)y = nS$. Write this inequality as

$$f_1(x) \geq nf(S),$$

where

$$f_1(x) = f(x) + (n-1)f(y) = f(x) + (n-1)f\left(\frac{nS-x}{n-1}\right).$$

From

$$\begin{aligned} f'_1(x) &= f'(x) - f'\left(\frac{nS-x}{n-1}\right) \\ &= f'(x) - f'(y) \\ &= \frac{n}{n-1}(x-S)H(x, y), \end{aligned}$$

it follows that f_1 is decreasing on $\mathbb{I}_{\leq S}$ and increasing on $\mathbb{I}_{\geq S}$; therefore,

$$f_1(x) \geq f_1(S) = nf(S).$$

Note 3. From the proof of the RHCF-Theorem, it follows that the RHCF-Theorem, the LHCF-Theorem and the HCF-Theorem are also valid in the case when f is defined on $\mathbb{I} \setminus \{u_0\}$, where $u_0 \in \mathbb{I}_{< s}$ for the RHCF-Theorem, and $u_0 \in \mathbb{I}_{> s}$ for the LHCF-Theorem.

Note 4. The inequalities in the RHCF-Theorem, the LHCF-Theorem and the HCF-Theorem become equalities for

$$a_1 = a_2 = \cdots = a_n = s.$$

In addition, if there exist $x, y \in \mathbb{I}$ so that

$$x + (n-1)y = ns, \quad f(x) + (n-1)f(y) = nf(s), \quad x \neq y,$$

then the equality holds also for

$$a_1 = x, \quad a_2 = \cdots = a_n = y$$

(or any cyclic permutation). Notice that these equality conditions are equivalent to

$$x + (n-1)y = ns, \quad h(x, y) = 0$$

($x < y$ for the RHCF-Theorem, and $x > y$ for the LHCF-Theorem).

Note 5. Similarly, we can extend the *weighted* Jensen's inequality to right and left half convex functions establishing the WRHCF-Theorem, the WLHCF-Theorem and the WHCF-Theorem (Vasile Cîrtoaje, 2008).

WHCF-Theorem. Let $p_1, p_2, \dots, p_n$ be positive real numbers so that

$$p_1 + p_2 + \cdots + p_n = 1, \quad p = \min\{p_1, p_2, \dots, p_n\},$$

and let f be a real function defined on an interval $\mathbb{I}$ and convex on $\mathbb{I}_{\geq s}$ or $\mathbb{I}_{\leq s}$, where $s \in \text{int}(\mathbb{I})$. The inequality

$$p_1 f(a_1) + p_2 f(a_2) + \cdots + p_n f(a_n) \geq f(p_1 a_1 + p_2 a_2 + \cdots + p_n a_n)$$

holds for all $a_1, a_2, \dots, a_n \in \mathbb{I}$ so that

$$p_1 a_1 + p_2 a_2 + \cdots + p_n a_n = s,$$

if and only if

$$p f(x) + (1-p)f(y) \geq f(s)$$

for all $x, y \in \mathbb{I}$ satisfying

$$p x + (1-p)y = s.$$

The following **LCRCF-Theorem** is also useful to prove some symmetric inequalities.

Left Convex-Right Concave Function Theorem (Vasile Cîrtoaje, 2004). Let $a < c$ be real numbers, let $f(x)$ be a continuous function defined on $\mathbb{I} = [a, \infty)$, strictly convex for $x \leq c$ and strictly concave for $x \geq c$, and let

$$E(a_1, a_2, \dots, a_n) = f(a_1) + f(a_2) + \cdots + f(a_n).$$

If $a_1, a_2, \dots, a_n \in \mathbb{I}$ so that

$$a_1 + a_2 + \dots + a_n = S = \text{constant},$$

then

- (a) E is minimum for $a_1 = a_2 = \dots = a_{n-1} \leq a_n$;
- (b) E is maximum for either $a_1 = a$ or $a < a_1 \leq a_2 = \dots = a_n$.

Proof. Without loss of generality, assume that $a_1 \leq a_2 \leq \dots \leq a_n$. Since the sum $E(a_1, a_2, \dots, a_n)$ is a continuous function on the compact set

$$A = \{(a_1, a_2, \dots, a_n) : a_1 + a_2 + \dots + a_n = S, a_1, a_2, \dots, a_n \in \mathbb{I}\},$$

E attains its minimum and maximum.

- (a) For the sake of contradiction, suppose that E is minimum at $(b_1, b_2, \dots, b_n)$ with

$$b_1 \leq b_2 \leq \dots \leq b_n, \quad b_1 < b_{n-1}.$$

For $b_{n-1} \leq c$, by Jensen's inequality for strictly convex functions we have

$$f(b_1) + f(b_{n-1}) > 2f\left(\frac{b_1 + b_{n-1}}{2}\right),$$

while for $b_{n-1} > c$, by Karamata's inequality for strictly concave functions we have

$$f(b_{n-1}) + f(b_n) > f(c) + f(b_{n-1} + b_n - c).$$

The both results contradict the assumption that E is minimum at $(b_1, b_2, \dots, b_n)$.

- (b) For the sake of contradiction, suppose that E is maximum at $(b_1, b_2, \dots, b_n)$ with

$$a < b_1 \leq b_2 \leq \dots \leq b_n, \quad b_2 < b_n.$$

There are three cases to consider.

Case 1: $b_2 \geq c$. By Jensen's inequality for strictly concave functions, we have

$$f(b_2) + f(b_n) < 2f\left(\frac{b_2 + b_n}{2}\right).$$

Case 2: $b_2 < c$ and $b_1 + b_2 - a \leq c$. By Karamata's inequality for strictly convex functions, we have

$$f(b_1) + f(b_2) < f(a) + f(b_1 + b_2 - a).$$

Case 3: $b_2 < c$ and $b_1 + b_2 - c \geq a$. By Karamata's inequality for strictly convex functions, we have

$$f(b_1) + f(b_2) < f(b_1 + b_2 - c) + f(c).$$

Clearly, all these results contradict the assumption that E is maximum at $(b_1, b_2, \dots, b_n)$.

Note 6. The point (a) of LCRCF-Theorem is also true when $\mathbb{I} = (a, \infty)$ and $f(a_+) = \infty$.

Note 7. If $\mathbb{I} = [a, d]$ and $a < c < d$, then E is minimum for either $a_n = d$ or $a_1 = a_2 = \dots = a_{n-1} \leq a_n$.

Note 8. From the point (b) of LCRCF-Theorem, we can prove (by the induction way) that E is maximum for

$$a = a_1 = \dots = a_{j-1} \leq a_j \leq a_{j+1} = \dots = a_n,$$

where $j \in \{1, 2, \dots, n-1\}$.

1.2 Applications

1.1. If a, b, c are real numbers so that $a + b + c = 3$, then

$$3(a^4 + b^4 + c^4) + a^2 + b^2 + c^2 + 6 \geq 6(a^3 + b^3 + c^3).$$

1.2. If $a_1, a_2, \dots, a_n \geq \frac{1-2n}{n-2}$ so that $a_1 + a_2 + \dots + a_n = n$, then

$$a_1^3 + a_2^3 + \dots + a_n^3 \geq n.$$

1.3. If $a_1, a_2, \dots, a_n \geq \frac{-n}{n-2}$ so that $a_1 + a_2 + \dots + a_n = n$, then

$$a_1^3 + a_2^3 + \dots + a_n^3 \geq a_1^2 + a_2^2 + \dots + a_n^2.$$

1.4. If $a_1, a_2, \dots, a_n$ are real numbers so that $a_1 + a_2 + \dots + a_n = n$, then

$$(n^2 - 3n + 3)(a_1^4 + a_2^4 + \dots + a_n^4 - n) \geq 2(n^2 - n + 1)(a_1^2 + a_2^2 + \dots + a_n^2 - n).$$

1.5. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$, then

$$(n^2 + n + 1)(a_1^3 + a_2^3 + \dots + a_n^3 - n) \geq (n + 1)(a_1^4 + a_2^4 + \dots + a_n^4 - n).$$

1.6. If a, b, c are real numbers so that $a + b + c = 3$, then

$$\begin{aligned} \text{(a)} \quad & a^4 + b^4 + c^4 - 3 + 2(7 + 3\sqrt{7})(a^3 + b^3 + c^3 - 3) \geq 0; \\ \text{(b)} \quad & a^4 + b^4 + c^4 - 3 + 2(7 - 3\sqrt{7})(a^3 + b^3 + c^3 - 3) \geq 0. \end{aligned}$$

1.7. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$. If k is a positive integer satisfying $3 \leq k \leq n + 1$, then

$$\frac{a_1^k + a_2^k + \dots + a_n^k - n}{a_1^2 + a_2^2 + \dots + a_n^2 - n} \geq (n-1) \left[\left(\frac{n}{n-1} \right)^{k-1} - 1 \right].$$

1.8. Let $k \geq 3$ be an integer number. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$, then

$$\frac{a_1^k + a_2^k + \dots + a_n^k - n}{a_1^2 + a_2^2 + \dots + a_n^2 - n} \leq \frac{n^{k-1} - 1}{n - 1}.$$

1.9. If $a_1, a_2, \dots, a_n$ are positive real numbers so that $a_1 + a_2 + \dots + a_n = n$, then

$$n^2 \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} - n \right) \geq 4(n-1)(a_1^2 + a_2^2 + \dots + a_n^2 - n).$$

1.10. If $a_1, a_2, \dots, a_8$ are positive real numbers so that $a_1 + a_2 + \dots + a_8 = 8$, then

$$\frac{1}{a_1^2} + \frac{1}{a_2^2} + \dots + \frac{1}{a_8^2} \geq a_1^2 + a_2^2 + \dots + a_8^2.$$

1.11. If $a_1, a_2, \dots, a_n$ are positive real numbers so that $\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} = n$, then

$$a_1^2 + a_2^2 + \dots + a_n^2 - n \geq 2 \left(1 + \frac{\sqrt{n-1}}{n} \right) (a_1 + a_2 + \dots + a_n - n).$$

1.12. If a, b, c, d, e are positive real numbers so that $a^2 + b^2 + c^2 + d^2 + e^2 = 5$, then

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} - 5 + \frac{4(1+\sqrt{5})}{5} (a + b + c + d + e - 5) \geq 0.$$

1.13. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{1}{3a+b+c} + \frac{1}{3b+c+a} + \frac{1}{3c+a+b} \leq \frac{2}{5} \left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \right).$$

1.14. If $a, b, c, d \geq 3 - \sqrt{7}$ so that $a + b + c + d = 4$, then

$$\frac{1}{2+a^2} + \frac{1}{2+b^2} + \frac{1}{2+c^2} + \frac{1}{2+d^2} \geq \frac{4}{3}.$$

1.15. If $a_1, a_2, \dots, a_n \in [-\sqrt{n}, n-2]$ so that $a_1 + a_2 + \dots + a_n = n$, then

$$\frac{1}{n+a_1^2} + \frac{1}{n+a_2^2} + \dots + \frac{1}{n+a_n^2} \leq \frac{n}{n+1}.$$

1.16. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$\frac{3-a}{9+a^2} + \frac{3-b}{9+b^2} + \frac{3-c}{9+c^2} \geq \frac{3}{5}.$$

1.17. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$\frac{1}{1-a+2a^2} + \frac{1}{1-b+2b^2} + \frac{1}{1-c+2c^2} \geq \frac{3}{2}.$$

1.18. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$\frac{1}{5+a+a^2} + \frac{1}{5+b+b^2} + \frac{1}{5+c+c^2} \geq \frac{3}{7}.$$

1.19. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$. If

$$k \geq 1 - \frac{1}{n},$$

then

$$\frac{1}{1+ka_1^2} + \frac{1}{1+ka_2^2} + \dots + \frac{1}{1+ka_n^2} \geq \frac{n}{1+k}.$$

1.20. Let $a_1, a_2, \dots, a_n$ ($n \geq 3$) be nonnegative real numbers so that

$$a_1 + a_2 + \dots + a_n = n.$$

If $p \geq \frac{2}{n-2}$ and $q = (p+1)(p+n)$, then

$$\frac{1}{a_1^2 + pa_1 + q} + \frac{1}{a_2^2 + pa_2 + q} + \dots + \frac{1}{a_n^2 + pa_n + q} \leq \frac{n}{1+p+q}.$$

1.21. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$. If

$$k \geq \frac{n^2 - n + 1}{n - 1},$$

then

$$\frac{1}{a_1^2 + k} + \frac{1}{a_2^2 + k} + \dots + \frac{1}{a_n^2 + k} \leq \frac{n}{1+k}.$$

1.22. Let $a_1, a_2, \dots, a_n$ be nonnegative numbers so that $a_1 + a_2 + \dots + a_n = n$. If

$$k \geq \frac{n^2}{4(n-1)},$$

then

$$\frac{a_1(a_1-1)}{a_1^2+k} + \frac{a_2(a_2-1)}{a_2^2+k} + \dots + \frac{a_n(a_n-1)}{a_n^2+k} \geq 0.$$

1.23. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$, then

$$\frac{a_1 - 1}{(n - 2a_1)^2} + \frac{a_2 - 1}{(n - 2a_2)^2} + \dots + \frac{a_n - 1}{(n - 2a_n)^2} \geq 0.$$

1.24. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that

$$a_1 + a_2 + \dots + a_n = n, \quad a_1, a_2, \dots, a_n > -k, \quad k \geq 1 + \frac{n}{\sqrt{n-1}},$$

then

$$\frac{a_1^2 - 1}{(a_1 + k)^2} + \frac{a_2^2 - 1}{(a_2 + k)^2} + \dots + \frac{a_n^2 - 1}{(a_n + k)^2} \geq 0.$$

1.25. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$. If $0 < k \leq 1 + \sqrt{\frac{2n-1}{n-1}}$, then

$$\frac{a_1^2 - 1}{(a_1 + k)^2} + \frac{a_2^2 - 1}{(a_2 + k)^2} + \dots + \frac{a_n^2 - 1}{(a_n + k)^2} \leq 0.$$

1.26. If $a_1, a_2, \dots, a_n \geq n - 1 - \sqrt{n^2 - n + 1}$ so that $a_1 + a_2 + \dots + a_n = n$, then

$$\frac{a_1^2 - 1}{(a_1 + 2)^2} + \frac{a_2^2 - 1}{(a_2 + 2)^2} + \dots + \frac{a_n^2 - 1}{(a_n + 2)^2} \leq 0.$$

1.27. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$. If $k \geq \frac{(n-1)(2n-1)}{n^2}$, then

$$\frac{1}{1 + ka_1^3} + \frac{1}{1 + ka_2^3} + \dots + \frac{1}{1 + ka_n^3} \geq \frac{n}{1 + k}.$$

1.28. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$. If $0 < k \leq \frac{n-1}{n^2 - 2n + 2}$, then

$$\frac{1}{1 + ka_1^3} + \frac{1}{1 + ka_2^3} + \dots + \frac{1}{1 + ka_n^3} \leq \frac{n}{1 + k}.$$

1.29. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$. If $k \geq \frac{n^2}{n-1}$, then

$$\sqrt{\frac{a_1}{k - a_1}} + \sqrt{\frac{a_2}{k - a_2}} + \dots + \sqrt{\frac{a_n}{k - a_n}} \leq \frac{n}{\sqrt{k-1}}.$$

1.30. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$, then

$$n^{-a_1^2} + n^{-a_2^2} + \dots + n^{-a_n^2} \geq 1.$$

1.31. If a, b, c, d are nonnegative real numbers so that $a + b + c + d = 4$, then

$$(3a^2 + 1)(3b^2 + 1)(3c^2 + 1)(3d^2 + 1) \leq 256.$$

1.32. If $a, b, c, d, e \geq -1$ so that $a + b + c + d + e = 5$, then

$$(a^2 + 1)(b^2 + 1)(c^2 + 1)(d^2 + 1)(e^2 + 1) \geq (a + 1)(b + 1)(c + 1)(d + 1)(e + 1).$$

1.33. Let $a_1, a_2, \dots, a_n$ be positive numbers so that $a_1 + a_2 + \dots + a_n = n$. If

$$k \leq \frac{2\sqrt{n-1}}{n} + 2\sqrt{\frac{2\sqrt{n-1}}{n}}, \quad k \leq 3,$$

then

$$k(\sqrt{a_1} + \sqrt{a_2} + \dots + \sqrt{a_n}) + \frac{1}{\sqrt{a_1}} + \frac{1}{\sqrt{a_2}} + \dots + \frac{1}{\sqrt{a_n}} \geq (k + 1)n.$$

1.34. If $a_1, a_2, \dots, a_n$ ($n \geq 3$) are positive numbers so that $a_1 + a_2 + \dots + a_n = 1$, then

$$\left(\frac{1}{\sqrt{a_1}} - \sqrt{a_1}\right) \left(\frac{1}{\sqrt{a_2}} - \sqrt{a_2}\right) \dots \left(\frac{1}{\sqrt{a_n}} - \sqrt{a_n}\right) \geq \left(\sqrt{n} - \frac{1}{\sqrt{n}}\right)^n.$$

1.35. Let $a_1, a_2, \dots, a_n$ be positive real numbers so that $a_1 + a_2 + \dots + a_n = n$. If

$$k \leq \left(1 + \frac{2\sqrt{n-1}}{n}\right)^2,$$

then

$$\left(ka_1 + \frac{1}{a_1}\right) \left(ka_2 + \frac{1}{a_2}\right) \dots \left(ka_n + \frac{1}{a_n}\right) \geq (k + 1)^n.$$

1.36. If a, b, c, d are nonzero real numbers so that

$$a, b, c, d \geq \frac{-1}{2}, \quad a + b + c + d = 4,$$

then

$$3\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2}\right) + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \geq 16.$$

1.37. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that $a_1^2 + a_2^2 + \dots + a_n^2 = n$, then

$$a_1^3 + a_2^3 + \dots + a_n^3 - n + \sqrt{\frac{n}{n-1}} (a_1 + a_2 + \dots + a_n - n) \geq 0.$$

1.38. If a, b, c, d, e are nonnegative real numbers so that $a^2 + b^2 + c^2 + d^2 + e^2 = 5$, then

$$\frac{1}{7-2a} + \frac{1}{7-2b} + \frac{1}{7-2c} + \frac{1}{7-2d} + \frac{1}{7-2e} \leq 1.$$

1.39. Let $0 \leq a_1, a_2, \dots, a_n < k$ so that $a_1^2 + a_2^2 + \dots + a_n^2 = n$. If

$$1 < k \leq 1 + \sqrt{\frac{n}{n-1}},$$

then

$$\frac{1}{k-a_1} + \frac{1}{k-a_2} + \dots + \frac{1}{k-a_n} \geq \frac{n}{k-1}.$$

1.40. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\sqrt{1 + \frac{48a}{b+c}} + \sqrt{1 + \frac{48b}{c+a}} + \sqrt{1 + \frac{48c}{a+b}} \geq 15.$$

1.41. If a, b, c are nonnegative real numbers, then

$$\sqrt{\frac{3a^2}{7a^2 + 5(b+c)^2}} + \sqrt{\frac{3b^2}{7b^2 + 5(c+a)^2}} + \sqrt{\frac{3c^2}{7c^2 + 5(a+b)^2}} \leq 1.$$

1.42. If a, b, c are nonnegative real numbers, then

$$\sqrt{\frac{a^2}{a^2 + 2(b+c)^2}} + \sqrt{\frac{b^2}{b^2 + 2(c+a)^2}} + \sqrt{\frac{c^2}{c^2 + 2(a+b)^2}} \geq 1.$$

1.43. Let a, b, c be nonnegative real numbers, no two of which are zero. If

$$k \geq k_0, \quad k_0 = \frac{\ln 3}{\ln 2} - 1 \approx 0.585,$$

then

$$\left(\frac{2a}{b+c}\right)^k + \left(\frac{2b}{c+a}\right)^k + \left(\frac{2c}{a+b}\right)^k \geq 3.$$

1.44. If $a, b, c \in [1, 7 + 4\sqrt{3}]$, then

$$\sqrt{\frac{2a}{b+c}} + \sqrt{\frac{2b}{c+a}} + \sqrt{\frac{2c}{a+b}} \geq 3.$$

1.45. Let a, b, c be nonnegative real numbers so that $a + b + c = 3$. If

$$0 < k \leq k_0, \quad k_0 = \frac{\ln 2}{\ln 3 - \ln 2} \approx 1.71,$$

then

$$a^k(b+c) + b^k(c+a) + c^k(a+b) \leq 6.$$

1.46. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$\sqrt{a} + \sqrt{b} + \sqrt{c} - 3 \geq 13 \left(\sqrt{\frac{a+b}{2}} + \sqrt{\frac{b+c}{2}} + \sqrt{\frac{c+a}{2}} - 3 \right).$$

1.47. Let a, b, c be nonnegative real numbers so that $a + b + c = 3$. If $k > 2$, then

$$a^k + b^k + c^k + 3 \geq 2 \left(\frac{a+b}{2} \right)^k + 2 \left(\frac{b+c}{2} \right)^k + 2 \left(\frac{c+a}{2} \right)^k.$$

1.48. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$, then

$$\sqrt{a_1} + \sqrt{a_2} + \dots + \sqrt{a_n} + n(k-1) \leq k \left(\sqrt{\frac{n-a_1}{n-1}} + \sqrt{\frac{n-a_2}{n-1}} + \dots + \sqrt{\frac{n-a_n}{n-1}} \right),$$

where

$$k = (\sqrt{n} - 1)(\sqrt{n} + \sqrt{n-1}).$$

1.49. If a, b, c are the lengths of the sides of a triangle so that $a + b + c = 3$, then

$$\frac{1}{a+b-c} + \frac{1}{b+c-a} + \frac{1}{c+a-b} - 3 \geq 4(2 + \sqrt{3}) \left(\frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{c+a} - 3 \right).$$

1.50. Let $a_1, a_2, \dots, a_5$ be nonnegative numbers so that $a_1 + a_2 + a_3 + a_4 + a_5 \leq 5$. If

$$k \geq k_0, \quad k_0 = \frac{29 + \sqrt{761}}{10} \approx 5.66,$$

then

$$\sum \frac{1}{ka_1^2 + a_2 + a_3 + a_4 + a_5} \geq \frac{5}{k+4}.$$

1.51. Let $a_1, a_2, \dots, a_5$ be nonnegative numbers so that $a_1 + a_2 + a_3 + a_4 + a_5 \leq 5$. If

$$0 < k \leq k_0, \quad k_0 = \frac{11 - \sqrt{101}}{10} \approx 0.095,$$

then

$$\sum \frac{1}{ka_1^2 + a_2 + a_3 + a_4 + a_5} \geq \frac{5}{k+4}.$$

1.52. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n \leq n$. If

$$0 < k \leq \frac{1}{n+1},$$

then

$$\frac{a_1}{ka_1^2 + a_2 + \dots + a_n} + \frac{a_2}{a_1 + ka_2^2 + \dots + a_n} + \dots + \frac{a_n}{a_1 + a_2 + \dots + ka_n^2} \geq \frac{n}{k+n-1}.$$

1.53. If $a_1, a_2, a_3, a_4, a_5 \leq \frac{7}{2}$ so that $a_1 + a_2 + a_3 + a_4 + a_5 = 5$, then

$$\frac{a_1}{a_1^2 - a_1 + 5} + \frac{a_2}{a_2^2 - a_2 + 5} + \frac{a_3}{a_3^2 - a_3 + 5} + \frac{a_4}{a_4^2 - a_4 + 5} + \frac{a_5}{a_5^2 - a_5 + 5} \leq 1.$$

1.54. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n \geq n$. If

$$0 < k \leq \frac{1}{1 + \frac{1}{4(n-1)^2}},$$

then

$$\frac{a_1^2}{ka_1^2 + a_2 + \dots + a_n} + \frac{a_2^2}{a_1 + ka_2^2 + \dots + a_n} + \dots + \frac{a_n^2}{a_1 + a_2 + \dots + ka_n^2} \geq \frac{n}{k+n-1}.$$

1.55. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n \leq n$. If $k \geq n-1$, then

$$\frac{a_1^2}{ka_1^2 + a_2 + \dots + a_n} + \frac{a_2^2}{a_1 + ka_2^2 + \dots + a_n} + \dots + \frac{a_n^2}{a_1 + a_2 + \dots + ka_n^2} \leq \frac{n}{k+n-1}.$$

1.56. Let $a_1, a_2, \dots, a_n \in [0, n]$ so that $a_1 + a_2 + \dots + a_n \geq n$. If $0 < k \leq \frac{1}{n}$, then

$$\frac{a_1 - 1}{ka_1^2 + a_2 + \dots + a_n} + \frac{a_2 - 1}{a_1 + ka_2^2 + \dots + a_n} + \dots + \frac{a_n - 1}{a_1 + a_2 + \dots + ka_n^2} \geq 0.$$

1.57. If a, b, c are positive real numbers so that $abc = 1$, then

$$\sqrt{a^2 - a + 1} + \sqrt{b^2 - b + 1} + \sqrt{c^2 - c + 1} \geq a + b + c.$$

1.58. If $a, b, c, d \geq \frac{1}{1 + \sqrt{6}}$ so that $abcd = 1$, then

$$\frac{1}{a+2} + \frac{1}{b+2} + \frac{1}{c+2} + \frac{1}{d+2} \leq \frac{4}{3}.$$

1.59. If a, b, c are positive real numbers so that $abc = 1$, then

$$a^2 + b^2 + c^2 - 3 \geq 2(ab + bc + ca - a - b - c).$$

1.60. If a, b, c are positive real numbers so that $abc = 1$, then

$$a^2 + b^2 + c^2 - 3 \geq 18(a + b + c - ab - bc - ca).$$

1.61. If $a_1, a_2, \dots, a_n$ are positive real numbers so that $a_1 a_2 \cdots a_n = 1$, then

$$a_1^2 + a_2^2 + \cdots + a_n^2 - n \geq 6\sqrt{3} \left(a_1 + a_2 + \cdots + a_n - \frac{1}{a_1} - \frac{1}{a_2} - \cdots - \frac{1}{a_n} \right).$$

1.62. If $a_1, a_2, \dots, a_n$ ($n \geq 4$) are positive real numbers so that $a_1 a_2 \cdots a_n = 1$, then

$$(n-1)(a_1^2 + a_2^2 + \cdots + a_n^2) + n(n+3) \geq (2n+2)(a_1 + a_2 + \cdots + a_n).$$

1.63. Let $a_1, a_2, \dots, a_n$ ($n \geq 3$) be positive real numbers so that $a_1 a_2 \cdots a_n = 1$. If p and q are nonnegative real numbers so that $p + q \geq n - 1$, then

$$\frac{1}{1 + pa_1 + qa_1^2} + \frac{1}{1 + pa_2 + qa_2^2} + \cdots + \frac{1}{1 + pa_n + qa_n^2} \geq \frac{n}{1 + p + q}.$$

1.64. Let a, b, c, d be positive real numbers so that $abcd = 1$. If p and q are nonnegative real numbers so that $p + q = 3$, then

$$\frac{1}{1 + pa + qa^3} + \frac{1}{1 + pb + qb^3} + \frac{1}{1 + pc + qc^3} + \frac{1}{1 + pd + qd^3} \geq 1.$$

1.65. If $a_1, a_2, \dots, a_n$ are positive real numbers so that $a_1 a_2 \cdots a_n = 1$, then

$$\frac{1}{1 + a_1 + \cdots + a_1^{n-1}} + \frac{1}{1 + a_2 + \cdots + a_2^{n-1}} + \cdots + \frac{1}{1 + a_n + \cdots + a_n^{n-1}} \geq 1.$$

1.66. Let $a_1, a_2, \dots, a_n$ be positive real numbers so that $a_1 a_2 \cdots a_n = 1$. If

$$k \geq n^2 - 1,$$

then

$$\frac{1}{\sqrt{1 + ka_1}} + \frac{1}{\sqrt{1 + ka_2}} + \cdots + \frac{1}{\sqrt{1 + ka_n}} \geq \frac{n}{\sqrt{1 + k}}.$$

1.67. Let $a_1, a_2, \dots, a_n$ be positive real numbers so that $a_1 a_2 \cdots a_n = 1$. If $p, q \geq 0$ so that $0 < p + q \leq \frac{1}{n-1}$, then

$$\frac{1}{1 + pa_1 + qa_1^2} + \frac{1}{1 + pa_2 + qa_2^2} + \cdots + \frac{1}{1 + pa_n + qa_n^2} \leq \frac{n}{1 + p + q}.$$

1.68. Let $a_1, a_2, \dots, a_n$ ($n \geq 3$) be positive real numbers so that $a_1 a_2 \cdots a_n = 1$. If

$$0 < k \leq \frac{2n-1}{(n-1)^2},$$

then

$$\frac{1}{\sqrt{1 + ka_1}} + \frac{1}{\sqrt{1 + ka_2}} + \cdots + \frac{1}{\sqrt{1 + ka_n}} \leq \frac{n}{\sqrt{1 + k}}.$$

1.69. If $a_1, a_2, \dots, a_n$ are positive real numbers so that $a_1 a_2 \cdots a_n = 1$, then

$$\sqrt{a_1^4 + \frac{2n-1}{(n-1)^2}} + \sqrt{a_2^4 + \frac{2n-1}{(n-1)^2}} + \cdots + \sqrt{a_n^4 + \frac{2n-1}{(n-1)^2}} \geq \frac{1}{n-1} (a_1 + a_2 + \cdots + a_n)^2.$$

1.70. If $a_1, a_2, \dots, a_n$ are positive real numbers so that $a_1 a_2 \cdots a_n = 1$, then

$$a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1} + n(n-2) \geq (n-1) \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right).$$

1.71. Let $a_1, a_2, \dots, a_n$ be positive real numbers so that $a_1 a_2 \cdots a_n = 1$. If $k \geq n$, then

$$a_1^k + a_2^k + \cdots + a_n^k + kn \geq (k+1) \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right).$$

1.72. If $a_1, a_2, \dots, a_n$ are positive real numbers so that $a_1 a_2 \cdots a_n = 1$, then

$$\left(1 - \frac{1}{n}\right)^{a_1} + \left(1 - \frac{1}{n}\right)^{a_2} + \cdots + \left(1 - \frac{1}{n}\right)^{a_n} \leq n - 1.$$

1.73. If a, b, c are positive real numbers so that $abc = 1$, then

$$\frac{1}{1 + \sqrt{1 + 3a}} + \frac{1}{1 + \sqrt{1 + 3b}} + \frac{1}{1 + \sqrt{1 + 3c}} \leq 1.$$

1.74. If $a_1, a_2, \dots, a_n$ are positive real numbers so that $a_1 a_2 \cdots a_n = 1$, then

$$\frac{1}{1 + \sqrt{1 + 4n(n-1)a_1}} + \frac{1}{1 + \sqrt{1 + 4n(n-1)a_2}} + \cdots + \frac{1}{1 + \sqrt{1 + 4n(n-1)a_n}} \geq \frac{1}{2}.$$

1.75. If a, b, c are positive real numbers so that $abc = 1$, then

$$\frac{a^6}{1 + 2a^5} + \frac{b^6}{1 + 2b^5} + \frac{c^6}{1 + 2c^5} \geq 1.$$

1.76. If a, b, c are positive real numbers so that $abc = 1$, then

$$\sqrt{25a^2 + 144} + \sqrt{25b^2 + 144} + \sqrt{25c^2 + 144} \leq 5(a + b + c) + 24.$$

1.77. If a, b, c are positive real numbers so that $abc = 1$, then

$$\sqrt{16a^2 + 9} + \sqrt{16b^2 + 9} + \sqrt{16c^2 + 9} \geq 4(a + b + c) + 3.$$

1.78. If ABC is a triangle, then

$$\sin A \left(2 \sin \frac{A}{2} - 1\right) + \sin B \left(2 \sin \frac{B}{2} - 1\right) + \sin C \left(2 \sin \frac{C}{2} - 1\right) \geq 0.$$

1.79. If ABC is an acute or right triangle, then

$$\sin 2A \left(1 - 2 \sin \frac{A}{2}\right) + \sin 2B \left(1 - 2 \sin \frac{B}{2}\right) + \sin 2C \left(1 - 2 \sin \frac{C}{2}\right) \geq 0.$$

1.80. If a, b, c, d are real numbers so that $a + b + c + d = 4$, then

$$\frac{a}{a^2 - a + 4} + \frac{b}{b^2 - b + 4} + \frac{c}{c^2 - c + 4} + \frac{d}{d^2 - d + 4} \leq 1.$$

1.81. Let a, b, c be nonnegative real numbers so that $a + b + c = 2$. If

$$k_0 \leq k \leq 3, \quad k_0 = \frac{\ln 2}{\ln 3 - \ln 2} \approx 1.71,$$

then

$$a^k(b + c) + b^k(c + a) + c^k(a + b) \leq 2.$$

1.82. If $a_1, a_2, \dots, a_n$ are positive real numbers so that $a_1 + a_2 + \dots + a_n = n$, then

$$(n + 1)^2 \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right) \geq 4(n + 2)(a_1^2 + a_2^2 + \dots + a_n^2) + n(n^2 - 3n - 6).$$

1.83. If a, b, c, d, e are positive real numbers such that $a + b + c + d + e = 5$, then

$$27\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e}\right) \geq 4(a^3 + b^3 + c^3 + d^3 + e^3) + 115.$$

1.84. If a, b, c are nonnegative real numbers so that $a + b + c = 12$, then

$$(a^2 + 10)(b^2 + 10)(c^2 + 10) \geq 13310.$$

1.85. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$, then

$$(a_1^2 + 1)(a_2^2 + 1) \cdots (a_n^2 + 1) \geq \frac{(n^2 - 2n + 2)^n}{(n - 1)^{2n-2}}.$$

1.86. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$(a^2 + 2)(b^2 + 2)(c^2 + 2) \leq 44.$$

1.87. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$(a^2 + 1)(b^2 + 1)(c^2 + 1) \leq \frac{169}{16}.$$

1.88. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$(2a^2 + 1)(2b^2 + 1)(2c^2 + 1) \leq \frac{121}{4}.$$

1.89. If a, b, c are nonnegative real numbers so that $a + b + c \geq k_0$, where

$$k_0 = \frac{3}{8}\sqrt{66 + 10\sqrt{105}} \approx 4.867,$$

then

$$\sqrt[3]{(a^2 + 1)(b^2 + 1)(c^2 + 1)} \leq \left(\frac{a + b + c}{3}\right)^2 + 1.$$

1.90. If a, b, c, d are nonnegative real numbers so that $a + b + c + d = 4$, then

$$(a^2 + 3)(b^2 + 3)(c^2 + 3)(d^2 + 3) \leq 513.$$

1.91. If a, b, c, d are nonnegative real numbers so that $a + b + c + d = 4$, then

$$(a^2 + 2)(b^2 + 2)(c^2 + 2)(d^2 + 2) \leq 144.$$

1.92. If a, b, c, d are nonnegative real numbers such that

$$a + b + c + d = 4,$$

then

$$\frac{a}{3a^3 + 2} + \frac{b}{3b^3 + 2} + \frac{c}{3c^3 + 2} + \frac{d}{3d^3 + 2} \leq \frac{4}{5}.$$

1.93. If a, b, c, d are positive real numbers such that

$$\frac{1}{9a + 2} + \frac{1}{9b + 2} + \frac{1}{9c + 2} + \frac{1}{9d + 2} = \frac{4}{11},$$

then

$$3(a + b + c + d) + 20 \geq 8 \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right).$$

1.94. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers such that $a_1 + a_2 + \dots + a_n = 1$, then

$$a_1^3 + a_2^3 + \dots + a_n^3 \leq \frac{1}{8} + a_1^4 + a_2^4 + \dots + a_n^4.$$

1.95. If $a_1, a_2, \dots, a_n$ ($n \geq 4$) are nonnegative real numbers such that

$$a_1 + a_2 + \dots + a_n = n,$$

then

$$n(n-1)(2n-1) \sum_{i=1}^n \frac{1}{a_i + n-1} \geq a_1^2 + a_2^2 + \dots + a_n^2 + n^2(2n-3).$$

1.96. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers such that

$$a_1 + a_2 + \dots + a_n = n,$$

then

$$n(n^2 - n + 1) \sum_{i=1}^n \frac{1}{a_i + n-1} \leq a_1^2 + a_2^2 + \dots + a_n^2 + n^2(n-1).$$

1.97. If $a_1, a_2, \dots, a_n$ ($n \geq 4$) are nonnegative real numbers such that

$$\frac{1}{a_1 + n-1} + \frac{1}{a_2 + n-1} + \dots + \frac{1}{a_n + n-1} \geq 1,$$

then

$$a_1^2 + a_2^2 + \dots + a_n^2 - n \geq \frac{n^2 - n - 1}{n-2} (a_1 + a_2 + \dots + a_n - n).$$

1.98. Let a, b, c be nonnegative real numbers, no two of which are zero, and

$$x = \frac{2a}{b+c}, \quad y = \frac{2b}{c+a}, \quad z = \frac{2c}{a+b}.$$

Prove that

$$\frac{1}{\sqrt{5x+4}} + \frac{1}{\sqrt{5y+4}} + \frac{1}{\sqrt{5z+4}} \geq 1.$$

1.99. If a, b, c, d are positive real numbers such that $abcd = 1$, then

$$a^3 + b^3 + c^3 + d^3 + 4 \geq 2(a^2 + b^2 + c^2 + d^2).$$

1.100. If $a, b, c, d \in [-1, 1]$ such that $a + b + c + d = 1$, then

$$\frac{-1}{9} \leq a^3 + b^3 + c^3 + d^3 \leq \frac{7}{4}.$$

1.101. Prove that $\frac{19}{4}$ is the minimum positive value of the constant k such that

$$\sqrt[3]{\frac{b+c}{ka+b+c}} + \sqrt[3]{\frac{c+a}{kb+c+a}} + \sqrt[3]{\frac{a+b}{kc+a+b}} \geq 3\sqrt[3]{\frac{2}{k+2}}$$

holds for any nonnegative real numbers a, b, c with $a + b + c > 0$.

1.3 Solutions

P 1.1. If a, b, c are real numbers so that $a + b + c = 3$, then

$$3(a^4 + b^4 + c^4) + a^2 + b^2 + c^2 + 6 \geq 6(a^3 + b^3 + c^3).$$

(Vasile C., 2006)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) \geq 3f(s), \quad s = \frac{a + b + c}{3} = 1,$$

where

$$f(u) = 3u^4 - 6u^3 + u^2, \quad u \in \mathbb{R}.$$

From

$$f''(u) = 2(18u^2 - 18u + 1),$$

it follows that $f''(u) > 0$ for $u \geq 1$, hence f is convex on $[s, \infty)$. By the RHCF-Theorem, it suffices to show that $f(x) + 2f(y) \geq 3f(1)$ for all real x, y so that $x + 2y = 3$. Let

$$E = f(x) + 2f(y) - 3f(1).$$

We have

$$\begin{aligned} E &= [f(x) - f(1)] + 2[f(y) - f(1)] \\ &= (3x^4 - 6x^3 + x^2 + 2) + 2(3y^4 - 6y^3 + y^2 + 2) \\ &= (x - 1)(3x^3 - 3x^2 - 2x - 2) + 2(y - 1)(3y^3 - 3y^2 - 2y - 2) \\ &= (x - 1)[(3x^3 - 3x^2 - 2x - 2) - (3y^3 - 3y^2 - 2y - 2)] \\ &= (x - 1)[3(x^3 - y^3) - 3(x^2 - y^2) - 2(x - y)] \\ &= (x - 1)(x - y)[3(x^2 + xy + y^2) - 3(x + y) - 2] \\ &= \frac{(x - 1)^2[27(x^2 + xy + y^2) - 9(x + y)(x + 2y) - 2(x + 2y)^2]}{6} \\ &= \frac{(x - 1)^2(4x - y)^2}{6} \geq 0. \end{aligned}$$

The equality holds for $a = b = c = 1$, and also for $a = \frac{1}{3}$ and $b = c = \frac{4}{3}$ (or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization:

- If $a_1, a_2, \dots, a_n$ are real numbers so that $a_1 + a_2 + \dots + a_n = n$, then

$$(a_1^2 - a_1)^2 + (a_2^2 - a_2)^2 + \dots + (a_n^2 - a_n)^2 \geq \frac{n-1}{n^2 - 3n + 3}(a_1^2 + a_2^2 + \dots + a_n^2 - n),$$

with equality for $a_1 = a_2 = \dots = a_n = 1$, and also for

$$a_1 = \frac{1}{n^2 - 3n + 3}, \quad a_2 = a_3 = \dots = a_n = 1 + \frac{n-2}{n^2 - 3n + 3}$$

(or any cyclic permutation).

□

P 1.2. If $a_1, a_2, \dots, a_n \geq \frac{1-2n}{n-2}$ so that $a_1 + a_2 + \dots + a_n = n$, then

$$a_1^3 + a_2^3 + \dots + a_n^3 \geq n.$$

(Vasile C., 2000)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \dots + a_n}{n} = 1,$$

where

$$f(u) = u^3, \quad u \geq \frac{1-2n}{n-2}.$$

From $f''(u) = 6u$, it follows that f is convex on $[s, \infty)$. By the RHCF-Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for all $x, y \geq \frac{1-2n}{n-2}$ so that $x + (n-1)y = n$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = u^2 + u + 1,$$

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = x + y + 1 = \frac{(n-2)x + 2n - 1}{n-1} \geq 0.$$

From $x + (n-1)y = n$ and $h(x, y) = 0$, we get

$$x = \frac{1-2n}{n-2}, \quad y = \frac{n+1}{n-2}.$$

Therefore, according to Note 4, the equality holds for $a_1 = a_2 = \dots = a_n = 1$, and also for

$$a_1 = \frac{1-2n}{n-2}, \quad a_2 = a_3 = \dots = a_n = \frac{n+1}{n-2}$$

(or any cyclic permutation).

□

P 1.3. If $a_1, a_2, \dots, a_n \geq \frac{-n}{n-2}$ so that $a_1 + a_2 + \dots + a_n = n$, then

$$a_1^3 + a_2^3 + \dots + a_n^3 \geq a_1^2 + a_2^2 + \dots + a_n^2.$$

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \dots + a_n}{n} = 1,$$

where

$$f(u) = u^3 - u^2, \quad u \geq \frac{-n}{n-2}.$$

From $f''(u) = 6u - 2$, it follows that f is convex on $[s, \infty)$. According to the RHCF-Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for $x, y \geq \frac{-n}{n-2}$ so that $x + (n-1)y = n$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = u^2,$$

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = x + y = \frac{(n-2)x + n}{n-1} \geq 0.$$

From $x + (n-1)y = n$ and $h(x, y) = 0$, we get

$$x = \frac{-n}{n-2}, \quad y = \frac{n}{n-2}.$$

Therefore, in accordance with Note 4, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = \frac{-n}{n-2}, \quad a_2 = a_3 = \cdots = a_n = \frac{n}{n-2}$$

(or any cyclic permutation).

□

P 1.4. If $a_1, a_2, \dots, a_n$ are real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then

$$(n^2 - 3n + 3)(a_1^4 + a_2^4 + \cdots + a_n^4 - n) \geq 2(n^2 - n + 1)(a_1^2 + a_2^2 + \cdots + a_n^2 - n).$$

(Vasile C., 2009)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = (n^2 - 3n + 3)u^4 - 2(n^2 - n + 1)u^2, \quad u \in \mathbb{I} = \mathbb{R}.$$

For $u \geq s = 1$, we have

$$\begin{aligned} \frac{1}{4}f''(u) &= 3(n^2 - 3n + 3)u^2 - (n^2 - n + 1) \\ &\geq 3(n^2 - 3n + 3) - (n^2 - n + 1) = 2(n-2)^2 \geq 0; \end{aligned}$$

therefore, f is convex on $\mathbb{I}_{\geq s}$. By the RHCF-Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for $x, y \in \mathbb{R}$ so that $x + (n-1)y = n$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

We have

$$g(u) = (n^2 - 3n + 3)(u^3 + u^2 + u + 1) - 2(n^2 - n + 1)(u + 1)$$

and

$$\begin{aligned} h(x, y) &= (n^2 - 3n + 3)(x^2 + xy + y^2 + x + y + 1) - 2(n^2 - n + 1) \\ &= [(n^2 - 3n + 3)y - n^2 + n + 1]^2 \geq 0. \end{aligned}$$

The equality holds for $a_1 = a_2 = \dots = a_n = 1$, and also for

$$a_1 = -1 + \frac{2}{n^2 - 3n + 3}, \quad a_2 = a_3 = \dots = a_n = 1 + \frac{2n - 4}{n^2 - 3n + 3}$$

(or any cyclic permutation).

□

P 1.5. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$, then

$$(n^2 + n + 1)(a_1^3 + a_2^3 + \dots + a_n^3 - n) \geq (n + 1)(a_1^4 + a_2^4 + \dots + a_n^4 - n).$$

(Vasile C., 2009)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \dots + a_n}{n} = 1,$$

where

$$f(u) = (n^2 + n + 1)u^3 - (n + 1)u^4, \quad u \in \mathbb{I} = [0, n].$$

The function f is convex on $\mathbb{I}_{\leq s}$ because

$$\begin{aligned} f''(u) &= 6u[n^2 + n + 1 - 2(n + 1)u] \geq 6u[n^2 + n + 1 - 2(n + 1)] \\ &= 6(n^2 - n - 1)u \geq 0. \end{aligned}$$

By the LHCF-Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for $x, y \geq 0$ so that $x + (n - 1)y = n$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

We have

$$\begin{aligned} g(u) &= (n^2 + n + 1)(u^2 + u + 1) - (n + 1)(u^3 + u^2 + u + 1) \\ &= -(n + 1)u^3 + n^2(u^2 + u + 1) \end{aligned}$$

and

$$\begin{aligned} h(x, y) &= -(n + 1)(x^2 + xy + y^2) + n^2(x + y + 1) \\ &= -(n + 1)(x^2 + xy + y^2) + n(x + y)[x + (n - 1)y] + [x + (n - 1)y]^2 \\ &= (n^2 + n - 3)xy + 2n(n - 2)y^2 \geq 0. \end{aligned}$$

The equality holds for $a_1 = a_2 = \dots = a_n = 1$, and also for

$$a_1 = n, \quad a_2 = a_3 = \dots = a_n = 0$$

(or any cyclic permutation).

□

P 1.6. Let a, b, c be real numbers so that $a + b + c = 3$. If

$$-14 - 6\sqrt{7} \leq k \leq -14 + 6\sqrt{7},$$

then

$$a^4 + b^4 + c^4 - 3 \geq k(a^3 + b^3 + c^3 - 3).$$

(Vasile C., 2009)

Solution. Write the desired inequalities as

$$f(a) + f(b) + f(c) \geq 3f(s), \quad s = \frac{a + b + c}{3} = 1,$$

where

$$f(u) = u^4 - ku^3, \quad u \in \mathbb{R}.$$

From

$$f''(u) = 6u(2u^2 - k),$$

it follows that $f''(u) > 0$ for $u \geq 1$, hence f is convex on $[s, \infty)$. By the RHCF-Theorem, it suffices to show that $f(x) + 2f(y) \geq 3f(1)$ for all real x, y so that $x + 2y = 3$. Using Note 1, we only need to show that $h(x, y) \geq 0$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

We have

$$g(u) = u^3 + u^2 + u + 1 - k(u^2 + u + 1) + u + 1 = u^3 + (1 - k)(u^2 + u + 1),$$

$$\begin{aligned} h(x, y) &= x^2 + xy + y^2 + (1 - k)(x + y + 1) = 3y^2 - (10 - k)y + 13 - 4k \\ &= 3 \left(y - \frac{10 - k}{6} \right)^2 + \frac{(6\sqrt{7} + 14 + k)(6\sqrt{7} - 14 - k)}{12} \geq 0. \end{aligned}$$

The equality holds for $a = b = c = 1$. If $k = -14 - 6\sqrt{7}$, then the equality holds also for

$$a = -5 - 2\sqrt{7}, \quad b = c = 4 + \sqrt{7}$$

(or any cyclic permutation). If $k = -14 + 6\sqrt{7}$, then the equality holds also for

$$a = -5 + 2\sqrt{7}, \quad b = c = 4 - \sqrt{7}$$

(or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- Let $a_1, a_2, \dots, a_n$ be real numbers so that $a_1 + a_2 + \dots + a_n = n$. If $k_1 \leq k \leq k_2$, where

$$k_1 = \frac{-2(n^2 - n + 1) - 2\sqrt{3(n^2 - n + 1)(n^2 - 3n + 3)}}{(n - 2)^2},$$

$$k_2 = \frac{-2(n^2 - n + 1) + 2\sqrt{3(n^2 - n + 1)(n^2 - 3n + 3)}}{(n - 2)^2},$$

then

$$a_1^4 + a_2^4 + \cdots + a_n^4 - n \geq k(a_1^3 + a_2^3 + \cdots + a_n^3 - n).$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $k \in \{k_1, k_2\}$, then the equality holds also for

$$a_1 = \frac{-2(n^2 - 3n + 1) + (n - 1)(n - 2)k}{2(n^2 - 3n + 3)},$$

$$a_2 = a_3 = \cdots = a_n = \frac{2(n^2 - n - 1) - (n - 2)k}{2(n^2 - 3n + 3)}$$

(or any cyclic permutation).

□

P 1.7. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$. If k is a positive integer satisfying $3 \leq k \leq n + 1$, then

$$\frac{a_1^k + a_2^k + \cdots + a_n^k - n}{a_1^2 + a_2^2 + \cdots + a_n^2 - n} \geq (n - 1) \left[\left(\frac{n}{n - 1} \right)^{k-1} - 1 \right].$$

(Vasile C., 2012)

Solution. Denote

$$m = (n - 1) \left[\left(\frac{n}{n - 1} \right)^{k-1} - 1 \right] = \left(\frac{n}{n - 1} \right)^{k-2} + \left(\frac{n}{n - 1} \right)^{k-3} + \cdots + 1,$$

and write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = u^k - mu^2, \quad u \in [0, n].$$

We will show that f is convex on $[1, n]$. Since

$$f''(u) = k(k - 1)u^{k-2} - 2m \geq k(k - 1) - 2m,$$

we need to show that

$$\frac{k(k - 1)}{2} \geq \left(\frac{n}{n - 1} \right)^{k-2} + \left(\frac{n}{n - 1} \right)^{k-3} + \cdots + 1.$$

Since $n \geq k - 1$, this inequality is true if

$$\frac{k(k - 1)}{2} \geq \left(\frac{k - 1}{k - 2} \right)^{k-2} + \left(\frac{k - 1}{k - 2} \right)^{k-3} + \cdots + 1.$$

By Bernoulli's inequality, we have

$$\left(\frac{k-1}{k-2}\right)^j = \frac{1}{\left(1 - \frac{1}{k-1}\right)^j} \leq \frac{1}{1 - \frac{j}{k-1}} = \frac{k-1}{k-j-1}, \quad j = 0, 1, \dots, k-2.$$

Therefore, it suffices to show that

$$\frac{k(k-1)}{2} \geq (k-1) \left(1 + \frac{1}{2} + \dots + \frac{1}{k-1}\right).$$

This is true if

$$\frac{k}{2} \geq 1 + \frac{1}{2} + \dots + \frac{1}{k-1},$$

which can be easily proved by induction. According to the RHCF-Theorem and Note 1, we only need to show that $h(x, y) \geq 0$ for $x, y \geq 0$ so that $x + (n-1)y = n$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

We have

$$g(u) = \frac{(u^k - 1) - m(u^2 - 1)}{u - 1} = (u^{k-1} + u^{k-2} + \dots + 1) - m(u + 1),$$

$$\begin{aligned} h(x, y) &= \left(\frac{x^{k-1} - y^{k-1}}{x - y} + \frac{x^{k-2} - y^{k-2}}{x - y} + \dots + 1 \right) - m \\ &= \sum_{j=1}^{k-2} \left[\frac{x^{j+1} - y^{j+1}}{x - y} - \left(\frac{n}{n-1} \right)^j \right]. \end{aligned}$$

It suffices to show that $f_j(y) \geq 0$ for $y \in \left[0, \frac{n}{n-1}\right]$ and $j = 1, 2, \dots, k-2$, where

$$f_j(y) = x^j + x^{j-1}y + \dots + xy^{j-1} + y^j - \left(\frac{n}{n-1} \right)^j, \quad x = n - (n-1)y.$$

For $j = 1$, we have

$$f_1(y) = x + y - \frac{n}{n-1} = \frac{(n-2)x}{n-1} \geq 0.$$

For $j \geq 2$, from $x' = -(n-1)$ and $n-1 \geq k-2 \geq j$, we get

$$\begin{aligned} f'_j(y) &= -(n-1)[jx^{j-1} + (j-1)x^{j-2}y + \dots + y^{j-1}] + x^{j-1} + 2x^{j-2}y + \dots + jy^{j-1} \\ &\leq -j[jx^{j-1} + (j-1)x^{j-2}y + \dots + y^{j-1}] + x^{j-1} + 2x^{j-2}y + \dots + jy^{j-1} \\ &= -(j \cdot j - 1)x^{j-1} - [j \cdot (j-1) - 2]x^{j-2}y - \dots - (j \cdot 2 - j + 1)xy^{j-2} \leq 0. \end{aligned}$$

As a consequence, f_j is decreasing, hence it is minimum for $y = \frac{n}{n-1}$ (when $x = 0$):

$$f_j(y) \geq f_j\left(\frac{n}{n-1}\right) = 0.$$

From $x + (n - 1)y = n$ and $h(x, y) = 0$, we get

$$x = 0, \quad y = \frac{n}{n - 1}.$$

Therefore, the equality holds for

$$a_1 = 0, \quad a_2 = a_3 = \cdots = a_n = \frac{n}{n - 1}$$

(or any cyclic permutation).

Remark. For $k = 3$ and $k = 4$, we get the following statements (Vasile C., 2002):

- If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then

$$(n - 1)(a_1^3 + a_2^3 + \cdots + a_n^3 - n) \geq (2n - 1)(a_1^2 + a_2^2 + \cdots + a_n^2 - n),$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = 0, \quad a_2 = a_3 = \cdots = a_n = \frac{n}{n - 1}$$

(or any cyclic permutation).

- If $a_1, a_2, \dots, a_n$ ($n \geq 3$) are nonnegative real numbers so that

$$a_1 + a_2 + \cdots + a_n = n,$$

then

$$(n - 1)^2(a_1^4 + a_2^4 + \cdots + a_n^4 - n) \geq (3n^2 - 3n + 1)(a_1^2 + a_2^2 + \cdots + a_n^2 - n),$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = 0, \quad a_2 = a_3 = \cdots = a_n = \frac{n}{n - 1}$$

(or any cyclic permutation).

□

P 1.8. Let $k \geq 3$ be an integer number. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then

$$\frac{a_1^k + a_2^k + \cdots + a_n^k - n}{a_1^2 + a_2^2 + \cdots + a_n^2 - n} \leq \frac{n^{k-1} - 1}{n - 1}.$$

(Vasile C., 2012)

Solution. Denote

$$m = \frac{n^{k-1} - 1}{n - 1} = n^{k-2} + n^{k-3} + \cdots + 1,$$

and write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = mu^2 - u^k, \quad u \in [0, n].$$

We will show that f is convex on $[0, 1]$. Since

$$f''(u) = 2m - k(k-1)u^{k-2} \geq 2m - k(k-1),$$

we need to show that

$$n^{k-2} + n^{k-3} + \dots + 1 \geq \frac{k(k-1)}{2}.$$

This is true if

$$2^{k-2} + 2^{k-3} + \dots + 1 \geq \frac{k(k-1)}{2},$$

which is equivalent to

$$\begin{aligned} 2^{k-1} - 1 &\geq \frac{k(k-1)}{2}, \\ 2^k &\geq k^2 - k + 2. \end{aligned}$$

Since

$$\begin{aligned} 2^k &= (1+1)^k \geq 1 + \binom{k}{1} + \binom{k}{2} + \binom{k}{3} \\ &= 1 + k + \frac{k(k-1)}{2} + \frac{k(k-1)(k-2)}{6}, \end{aligned}$$

it suffices to show that

$$1 + k + \frac{k(k-1)}{2} + \frac{k(k-1)(k-2)}{6} \geq k^2 - k + 2,$$

which reduces to

$$(k-1)(k-2)(k-3) \geq 0.$$

According to the LHCF-Theorem and Note 1, we only need to show that $h(x, y) \geq 0$ for $x, y \geq 0$ so that $x + (n-1)y = n$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

We have

$$g(u) = \frac{m(u^2 - 1) - (u^k - 1)}{u - 1} = m(u + 1) - (u^{k-1} + u^{k-2} + \dots + 1)$$

and

$$\begin{aligned} h(x, y) &= m - \frac{x^{k-1} - y^{k-1}}{x - y} - \frac{x^{k-2} - y^{k-2}}{x - y} - \dots - 1 \\ &= \left(n^{k-2} - \frac{x^{k-1} - y^{k-1}}{x - y} \right) + \left(n^{k-3} - \frac{x^{k-2} - y^{k-2}}{x - y} \right) + \dots + \left(n - \frac{x^2 - y^2}{x - y} \right). \end{aligned}$$

It suffices to show that

$$n^j \geq \frac{x^{j+1} - y^{j+1}}{x - y}, \quad j = 1, 2, \dots, k-2.$$

We will show that

$$n^j \geq (x+y)^j \geq \frac{x^{j+1} - y^{j+1}}{x-y}.$$

The left inequality is true since

$$n - (x+y) = x + (n-1)y - (x+y) = (n-2)y \geq 0.$$

The right inequality is also true since

$$(x+y)^j = x^j + \binom{j}{1}x^{j-1}y + \cdots + \binom{j}{j-1}xy^{j-1} + y^j$$

and

$$\frac{x^{j+1} - y^{j+1}}{x-y} = x^j + x^{j-1}y + \cdots + xy^{j-1} + y^j.$$

The equality holds for $a_1 = n$ and $a_2 = a_3 = \cdots = a_n = 0$ (or any cyclic permutation).

Remark. For $k = 3$ and $k = 4$, we get the following statements (Vasile C., 2002):

- If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then

$$a_1^3 + a_2^3 + \cdots + a_n^3 - n \leq (n+1)(a_1^2 + a_2^2 + \cdots + a_n^2 - n),$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = n, \quad a_2 = a_3 = \cdots = a_n = 0$$

(or any cyclic permutation).

- If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then

$$a_1^4 + a_2^4 + \cdots + a_n^4 - n \leq (n^2 + n + 1)(a_1^2 + a_2^2 + \cdots + a_n^2 - n),$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = n, \quad a_2 = a_3 = \cdots = a_n = 0$$

(or any cyclic permutation).

□

P 1.9. If $a_1, a_2, \dots, a_n$ are positive real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then

$$n^2 \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n \right) \geq 4(n-1)(a_1^2 + a_2^2 + \cdots + a_n^2 - n).$$

(Vasile C., 2004)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{n^2}{u} - 4(n-1)u^2, \quad u \in \mathbb{I} = (0, n).$$

For $u \in (0, 1]$, we have

$$f''(u) = \frac{2n^2}{u^3} - 8(n-1) \geq 2n^2 - 8(n-1) = 2(n-2)^2 \geq 0.$$

Thus, f is convex on $\mathbb{I}_{\leq s}$. By the LHCF-Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for $x, y > 0$ so that $x + (n-1)y = n$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

We have

$$g(u) = \frac{-n^2}{u} - 4(n-1)(u+1)$$

and

$$h(x, y) = \frac{n^2}{xy} - 4(n-1) = \frac{[x + (n-1)y]^2}{xy} - 4(n-1) = \frac{[x - (n-1)y]^2}{xy}.$$

In accordance with Note 4, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = \frac{n}{2}, \quad a_2 = a_3 = \cdots = a_n = \frac{n}{2(n-1)}$$

(or any cyclic permutation).

Remark. The inequality is equivalent to

$$n \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right) + 8 \left(1 - \frac{1}{n} \right) \sum_{1 \leq i < j \leq n} a_i a_j \geq 5n^2 - 8n + 4.$$

□

P 1.10. If $a_1, a_2, \dots, a_8$ are positive real numbers so that $a_1 + a_2 + \cdots + a_8 = 8$, then

$$\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_8^2} \geq a_1^2 + a_2^2 + \cdots + a_8^2.$$

(Vasile C., 2007)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_8) \geq 8f(s), \quad s = \frac{a_1 + a_2 + \cdots + a_8}{8} = 1,$$

where

$$f(u) = \frac{1}{u^2} - u^2, \quad u \in (0, 8).$$

For $u \in (0, 1]$, we have

$$f''(u) = \frac{6}{u^4} - 2 \geq 6 - 2 > 0.$$

Thus, f is convex on $(0, s]$. By the LHCF-Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for $x, y > 0$ so that $x + 7y = 8$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

We have

$$g(u) = -u - 1 - \frac{1}{u} - \frac{1}{u^2}$$

and

$$h(x, y) = -1 + \frac{1}{xy} + \frac{x + y}{x^2 y^2}.$$

From $8 = x + 7y \geq 2\sqrt{7xy}$, we get $xy \leq 16/7$. Therefore,

$$\begin{aligned} h(x, y) &\geq -1 + \frac{1}{xy} + \frac{7(x + y)}{16xy} = \frac{112y^2 - 170y + 72}{16xy} \\ &> \frac{112y^2 - 176y + 72}{16xy} = \frac{14y^2 - 22y + 9}{2xy} > 0. \end{aligned}$$

The equality holds for $a_1 = a_2 = \cdots = a_8 = 1$.

Remark. In the same manner, we can prove the following generalization:

- If $a_1, a_2, \dots, a_n$ ($n \geq 4$) are positive real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then

$$\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2} + 8 - n \geq \frac{8}{n} (a_1^2 + a_2^2 + \cdots + a_n^2).$$

□

P 1.11. If $a_1, a_2, \dots, a_n$ are positive real numbers so that $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = n$, then

$$a_1^2 + a_2^2 + \cdots + a_n^2 - n \geq 2 \left(1 + \frac{\sqrt{n-1}}{n} \right) (a_1 + a_2 + \cdots + a_n - n).$$

(Vasile C., 2006)

Solution. Replacing each a_i by $1/a_i$, we need to prove that

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{1}{u^2} - \frac{2k}{u}, \quad k = 1 + \frac{\sqrt{n-1}}{n}, \quad u \in (0, n).$$

For $u \in (0, 1]$, we have

$$f''(u) = \frac{6-4ku}{u^4} \geq \frac{6-4k}{u^4} = \frac{2(\sqrt{n-1}-1)^2}{nu^4} \geq 0.$$

Thus, f is convex on $(0, s]$. By the LHCF-Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for $x, y > 0$ so that $x + (n-1)y = n$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

We have

$$g(u) = \frac{-1}{u^2} + \frac{2k-1}{u}$$

and

$$h(x, y) = \frac{1}{xy} \left(\frac{1}{x} + \frac{1}{y} + 1 - 2k \right).$$

We only need to show that

$$\frac{1}{x} + \frac{1}{y} \geq 2k - 1.$$

Indeed, using the Cauchy-Schwarz inequality, we get

$$\frac{1}{x} + \frac{1}{y} \geq \frac{(1 + \sqrt{n-1})^2}{x + (n-1)y} = \frac{(1 + \sqrt{n-1})^2}{n} = 2k - 1,$$

with equality for $x = \sqrt{n-1}y$. From $x + (n-1)y = n$ and $h(x, y) = 0$, we get

$$x = \frac{n}{1 + \sqrt{n-1}}, \quad y = \frac{n}{n-1 + \sqrt{n-1}}.$$

In accordance with Note 4, the original equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = \frac{1 + \sqrt{n-1}}{n}, \quad a_2 = a_3 = \cdots = a_n = \frac{n-1 + \sqrt{n-1}}{n}$$

(or any cyclic permutation).

□

P 1.12. If a, b, c, d, e are positive real numbers so that $a^2 + b^2 + c^2 + d^2 + e^2 = 5$, then

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} - 5 + \frac{4(1 + \sqrt{5})}{5} (a + b + c + d + e - 5) \geq 0.$$

(Vasile C., 2006)

Solution. Replacing a, b, c, d, e by $\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}, \sqrt{e}$, respectively, we need to prove that

$$f(a) + f(b) + f(c) + f(d) + f(e) \geq 5f(s), \quad s = \frac{a + b + c + d + e}{5} = 1,$$

where

$$f(u) = \frac{1}{\sqrt{u}} + k\sqrt{u}, \quad k = \frac{4(1 + \sqrt{5})}{5} \approx 2.59, \quad u \in (0, 5).$$

For $u \in (0, 1]$, we have

$$f''(u) = \frac{3 - ku}{4u^2\sqrt{u}} > 0;$$

therefore, f is convex on $(0, s]$. By the LHCF-Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for $x, y > 0$ so that $x + 4y = 5$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{k\sqrt{u} - 1}{u + \sqrt{u}}$$

and

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{\sqrt{x} + \sqrt{y} + 1 - k\sqrt{xy}}{\sqrt{xy}(\sqrt{x} + \sqrt{y})(\sqrt{x} + 1)(\sqrt{y} + 1)}.$$

Thus, we only need to show that

$$\sqrt{x} + \sqrt{y} + 1 - k\sqrt{xy} \geq 0,$$

which is true if

$$2\sqrt[4]{xy} + 1 - k\sqrt{xy} \geq 0.$$

Let

$$t = \sqrt[4]{xy}.$$

From

$$5 = x + 4y \geq 4\sqrt{xy} = 4t^2,$$

we get

$$t \leq \frac{\sqrt{5}}{2}.$$

Thus,

$$\begin{aligned} 2\sqrt[4]{xy} + 1 - k\sqrt{xy} &= 2t + 1 - kt^2 \\ &= \left(1 - \frac{2}{\sqrt{5}}t\right) \left[1 + 2\left(1 + \frac{1}{\sqrt{5}}\right)t\right] \geq 0. \end{aligned}$$

The equality holds for $a = b = c = d = e = 1$.

□

P 1.13. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{1}{3a + b + c} + \frac{1}{3b + c + a} + \frac{1}{3c + a + b} \leq \frac{2}{5} \left(\frac{1}{b + c} + \frac{1}{c + a} + \frac{1}{a + b} \right).$$

(Vasile C., 2006)

Solution. Due to homogeneity, we may assume that $a + b + c = 3$. So, we need to show that

$$f(a) + f(b) + f(c) \geq 3f(s), \quad s = \frac{a + b + c}{3} = 1,$$

where

$$f(u) = \frac{2}{3-u} - \frac{5}{2u+3}, \quad u \in [0, 3).$$

For $u \in [1, 3)$, we have

$$f''(u) = \frac{4}{(3-u)^3} - \frac{40}{(2u+3)^3} = \frac{36[2u^3 + 3u^2 + 9(u-1)(3-u)]}{(3-u)^3(2u+3)^3} > 0;$$

therefore, f is convex on $[s, 3)$. By the RHCF-Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for $x, y \geq 0$ so that $x + 2y = 3$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

We have

$$g(u) = \frac{1}{3-u} + \frac{2}{2u+3}$$

and

$$\begin{aligned} h(x, y) &= \frac{1}{(3-x)(3-y)} - \frac{4}{(2x+3)(2y+3)} \\ &= \frac{9(2x+2y-3)}{(3-x)(3-y)(2x+3)(2y+3)} \\ &= \frac{9x}{(3-x)(3-y)(2x+3)(2y+3)} \geq 0. \end{aligned}$$

The equality holds for $a = b = c$, and also for $a = 0$ and $b = c$ (or any cyclic permutation). □

P 1.14. If $a, b, c, d \geq 3 - \sqrt{7}$ so that $a + b + c + d = 4$, then

$$\frac{1}{2+a^2} + \frac{1}{2+b^2} + \frac{1}{2+c^2} + \frac{1}{2+d^2} \geq \frac{4}{3}.$$

(Vasile C., 2008)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) + f(d) \geq 4f(s), \quad s = \frac{a + b + c + d}{4} = 1,$$

where

$$f(u) = \frac{1}{2+u^2}, \quad u \geq 3 - \sqrt{7}.$$

For $u \geq s = 1$, $f(u)$ is convex because

$$f''(u) = \frac{3(3u^2 - 2)}{(2 + u^2)^3} > 0.$$

By the RHCF-Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for $x, y \geq 3 - \sqrt{7}$ so that $x + 3y = 4$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{-1 - u}{3(2 + u^2)}$$

and

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{xy + x + y - 2}{3(2 + x^2)(2 + y^2)},$$

where

$$\begin{aligned} xy + x + y - 2 &= \frac{-x^2 + 6x - 2}{3} = \frac{(3 + \sqrt{7} - x)(x - 3 + \sqrt{7})}{3} \\ &= \frac{(-1 + \sqrt{7} + 3y)(x - 3 + \sqrt{7})}{3} \geq 0. \end{aligned}$$

In accordance with Note 4, the equality holds for $a = b = c = d = 1$, and also for

$$a = 3 - \sqrt{7}, \quad b = c = d = \frac{1 + \sqrt{7}}{3}$$

(or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- If $a_1, a_2, \dots, a_n \geq n - 1 - \sqrt{n^2 - 3n + 3}$ so that $a_1 + a_2 + \dots + a_n = n$, then

$$\frac{1}{2 + a_1^2} + \frac{1}{2 + a_2^2} + \dots + \frac{1}{2 + a_n^2} \geq \frac{n}{3},$$

with equality for $a_1 = a_2 = \dots = a_n = 1$, and also for

$$a_1 = n - 1 - \sqrt{n^2 - 3n + 3}, \quad a_2 = a_3 = \dots = a_n = \frac{1 + \sqrt{n^2 - 3n + 3}}{n - 1}$$

(or any cyclic permutation).

□

P 1.15. If $a_1, a_2, \dots, a_n \in [-\sqrt{n}, n - 2]$ so that $a_1 + a_2 + \dots + a_n = n$, then

$$\frac{1}{n + a_1^2} + \frac{1}{n + a_2^2} + \dots + \frac{1}{n + a_n^2} \leq \frac{n}{n + 1}.$$

(Vasile C., 2008)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{-1}{n + u^2}, \quad n \geq 3, \quad u \in [-\sqrt{n}, n - 2].$$

For $u \in [-\sqrt{n}, 1]$, we have

$$f''(u) = \frac{2(n - u^2)}{(n + u^2)^3} \geq 0,$$

hence f is convex on $[-\sqrt{n}, s]$. By the LHCF-Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for $x, y \in [-\sqrt{n}, n - 2]$ so that $x + (n - 1)y = n$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{u + 1}{(n + 1)(n + u^2)}$$

and

$$\begin{aligned} h(x, y) &= \frac{g(x) - g(y)}{x - y} = \frac{n - x - y - xy}{(n + 1)(n + x^2)(n + y^2)} \\ &= \frac{(n - x)(n - 2 - x)}{(n^2 - 1)(n + x^2)(n + y^2)} \geq 0. \end{aligned}$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = n - 2, \quad a_2 = a_3 = \cdots = a_n = \frac{2}{n - 1}$$

(or any cyclic permutation).

□

P 1.16. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$\frac{3 - a}{9 + a^2} + \frac{3 - b}{9 + b^2} + \frac{3 - c}{9 + c^2} \geq \frac{3}{5}.$$

(Vasile C., 2013)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) \geq 3f(s), \quad s = \frac{a + b + c}{3} = 1,$$

where

$$f(u) = \frac{3 - u}{9 + u^2}, \quad u \in [0, 3].$$

For $u \in [1, 3]$, we have

$$\frac{1}{2}f''(u) = \frac{u^2(9 - u) + 27(u - 1)}{(9 + u^2)^3} > 0.$$

Thus, f is convex on $[s, 3]$. By the RHCF-Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for $x, y \geq 0$ so that $x + 2y = 3$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

We have

$$g(u) = \frac{-(6 + u)}{5(9 + u^2)}$$

and

$$h(x, y) = \frac{xy + 6x + 6y - 9}{5(9 + x^2)(9 + y^2)} = \frac{x(9 - x)}{10(9 + x^2)(9 + y^2)} \geq 0.$$

The equality holds for $a = b = c = 1$, and also for $a = 0$ and $b = c = \frac{3}{2}$ (or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization:

- If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$, then

$$\frac{n - a_1}{n^2 + (n^2 - 3n + 1)a_1^2} + \frac{n - a_2}{n^2 + (n^2 - 3n + 1)a_2^2} + \dots + \frac{n - a_n}{n^2 + (n^2 - 3n + 1)a_n^2} \geq \frac{n}{2n - 1},$$

with equality for $a_1 = a_2 = \dots = a_n = 1$, and also for

$$a_1 = 0, \quad a_2 = a_3 = \dots = a_n = \frac{n}{n - 1}$$

(or any cyclic permutation).

□

P 1.17. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$\frac{1}{1 - a + 2a^2} + \frac{1}{1 - b + 2b^2} + \frac{1}{1 - c + 2c^2} \geq \frac{3}{2}.$$

(Vasile C., 2012)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) \geq 3f(s), \quad s = \frac{a + b + c}{3} = 1,$$

where

$$f(u) = \frac{1}{1 - u + 2u^2}, \quad u \in [0, 3].$$

For $u \in [1, 3]$, we have

$$\frac{1}{2}f''(u) = \frac{12u^2 - 6u - 1}{(1 - u + 2u^2)^3} > 0.$$

Thus, f is convex on $[s, 3]$. By the RHCF-Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for $x, y \geq 0$ so that $x + 2y = 3$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

We have

$$g(u) = \frac{-(1 + 2u)}{2(1 - u + 2u^2)}$$

and

$$h(x, y) = \frac{4xy + 2x + 2y - 3}{2(1 - x + 2x^2)(1 - y + 2y^2)} = \frac{x(1 + 4y)}{2(1 - x + 2x^2)(1 - y + 2y^2)} \geq 0.$$

The equality holds for $a = b = c = 1$, and also for $a = 0$ and $b = c = \frac{3}{2}$ (or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization:

- Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$. If

$$k \geq k_1, \quad k_1 = \frac{3n - 2 + \sqrt{5n^2 - 8n + 4}}{2n},$$

then

$$\frac{1}{1 - a_1 + ka_1^2} + \frac{1}{1 - a_2 + ka_2^2} + \dots + \frac{1}{1 - a_n + ka_n^2} \geq \frac{n}{k},$$

with equality for $a_1 = a_2 = \dots = a_n = 1$. If $k = k_1$, then the equality holds also for

$$a_1 = 0, \quad a_2 = a_3 = \dots = a_n = \frac{n}{n - 1}$$

(or any cyclic permutation).

□

P 1.18. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$\frac{1}{5 + a + a^2} + \frac{1}{5 + b + b^2} + \frac{1}{5 + c + c^2} \geq \frac{3}{7}.$$

(Vasile C., 2008)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) \geq 3f(s), \quad s = \frac{a + b + c}{3} = 1,$$

where

$$f(u) = \frac{1}{5 + u + u^2}, \quad u \in [0, 3].$$

For $u \geq 1$, from

$$f''(u) = \frac{2(3u^2 + 3u - 4)}{(5 + u + u^2)^3} > 0,$$

it follows that f is convex on $[s, 3]$. By the RHCF-Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for $x, y \geq 0$ so that $x + 2y = 3$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{-2 - u}{7(5 + u + u^2)}$$

and

$$\begin{aligned} h(x, y) &= \frac{g(x) - g(y)}{x - y} = \frac{xy + 2(x + y) - 3}{7(5 + x + x^2)(5 + y + y^2)} \\ &= \frac{x(5 - x)}{14(5 + x + x^2)(5 + y + y^2)} \geq 0. \end{aligned}$$

According to Note 4, the equality holds for $a = b = c = 1$, and also for $a = 0$ and $b = c = \frac{3}{2}$ (or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$. If

$$0 < k \leq k_1, \quad k_1 = \frac{2(2n - 1)}{n - 1},$$

then

$$\frac{1}{k + a_1 + a_1^2} + \frac{1}{k + a_2 + a_2^2} + \dots + \frac{1}{k + a_n + a_n^2} \geq \frac{n}{k + 2},$$

with equality for $a_1 = a_2 = \dots = a_n = 1$. If $k = k_1$, then the equality holds also for

$$a_1 = 0, \quad a_2 = a_3 = \dots = a_n = \frac{n}{n - 1}$$

(or any cyclic permutation).

□

P 1.19. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$. If

$$k \geq 1 - \frac{1}{n},$$

then

$$\frac{1}{1 + ka_1^2} + \frac{1}{1 + ka_2^2} + \dots + \frac{1}{1 + ka_n^2} \geq \frac{n}{1 + k}.$$

(Vasile C., 2005)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{1}{1 + ku^2}, \quad u \in [0, n].$$

For $u \in [1, n]$, we have

$$f''(u) = \frac{2k(3ku^2 - 1)}{(1 + ku^2)^3} \geq \frac{2k(3k - 1)}{(1 + ku^2)^3} > 0.$$

Thus, f is convex on $[s, n]$. By the RHCF-Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for $x, y \geq 0$ so that $x + (n - 1)y = n$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{-k(u + 1)}{(1 + k)(1 + ku^2)}$$

and

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{k^2(x + y + xy) - k}{(1 + k)(1 + kx^2)(1 + ky^2)}.$$

We need to show that

$$k(x + y + xy) - 1 \geq 0.$$

Indeed, we have

$$k(x + y + xy) - 1 \geq \left(1 - \frac{1}{n}\right)(x + y + xy) - 1 = \frac{x(2n - 2 - x)}{n} \geq 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $k = 1 - \frac{1}{n}$, then the equality holds also for

$$a_1 = 0, \quad a_2 = a_3 = \cdots = a_n = \frac{n}{n - 1}$$

(or any cyclic permutation).

□

P 1.20. Let $a_1, a_2, \dots, a_n$ ($n \geq 3$) be nonnegative real numbers so that

$$a_1 + a_2 + \cdots + a_n = n.$$

If $p \geq \frac{2}{n - 2}$ and $q \geq q_0 = (p + 1)(p + n)$, then

$$\frac{1}{a_1^2 + pa_1 + q} + \frac{1}{a_2^2 + pa_2 + q} + \cdots + \frac{1}{a_n^2 + pa_n + q} \leq \frac{n}{1 + p + q}.$$

(Vasile C., 2023)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{-1}{u^2 + u + 2n + 2}, \quad u \in [0, n].$$

We have

$$f''(u) = \frac{2A(u)}{(u^2 + pu + q)^3},$$

where

$$A(u) = q - p^2 - 3pu - 3u^2.$$

For $u \in [0, 1]$, we have

$$A(u) = q - p^2 - 3pu - 3u^2 \geq A(1) = q - p^2 - 3p - 3 \geq q_0 - p^2 - 3p - 3 = (n - 2)p + n - 3 > 0.$$

Thus, f is convex on $[0, s]$. By the LHCF-Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for $x \geq 1 \geq y \geq 0$ so that $x + (n - 1)y = n$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{u + 1 + p}{(1 + p + q)(u^2 + pu + q)},$$

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{B(x, y)}{(1 + p + q)(x^2 + px + q)(y^2 + py + q)},$$

where

$$B(x, y) = q - p(p + 1) - (p + 1)(x + y) - xy.$$

For $x + (n - 1)y = n$, we get

$$B \geq q_0 - p^2 - 3p - 3 = (n - 1)y^2 + [(n - 2)p - 2]y \geq 0,$$

hence $h(x, y) \geq 0$. The equality occurs for $a_1 = a_2 = \cdots = a_n = 1$. If $q = q_0$, then the equality also occurs for $a_1 = n$ and $a_2 = a_3 = \cdots = a_n = 0$ (or any cyclic permutation). □

P 1.21. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$. If

$$k \geq k_0 = \frac{n^2 - n + 1}{n - 1},$$

then

$$\frac{1}{a_1^2 + k} + \frac{1}{a_2^2 + k} + \cdots + \frac{1}{a_n^2 + k} \leq \frac{n}{1 + k}.$$

(Vasile C., 2005)

Solution. Write this inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{-1}{u^2 + k}, \quad u \in [0, n].$$

For $u \in [0, 1]$, we have

$$f''(u) = \frac{2(k - 3u^2)}{(u^2 + k)^3} \geq 0,$$

since

$$k - 3u^2 \geq k_0 - 3 = \frac{n^2 - n + 1}{n - 1} - 3 = \frac{(n - 2)^2}{n - 1} \geq 0.$$

Thus, f is convex on $[0, s]$. By the LHCF-Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for $x, y \geq 0$ so that $x + (n - 1)y = n$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{u + 1}{(1 + k)(u^2 + k)}$$

and

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{k - x - y - xy}{(1 + k)(x^2 + k)(y^2 + k)}.$$

It suffices to show that

$$k - x - y - xy \geq 0.$$

Indeed, we have

$$k - x - y - xy \geq k_0 - x - y - xy = \frac{[(n - 1)y - 1]^2}{n - 1} \geq 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $k = k_0$, then the equality holds also for

$$a_1 = n - 1, \quad a_2 = a_3 = \cdots = a_n = \frac{1}{n - 1}$$

(or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- Let $a_1, a_2, \dots, a_n$ ($n \geq 3$) be nonnegative real numbers so that

$$a_1 + a_2 + \cdots + a_n = n.$$

If $-1 \leq p \leq \frac{2}{n - 2}$ and $q \geq q_0 = \frac{n^2 p^2 + 4(n^2 - n + 1)(p + 1)}{4(n - 1)}$, then

$$\frac{1}{a_1^2 + pa_1 + q} + \frac{1}{a_2^2 + pa_2 + q} + \cdots + \frac{1}{a_n^2 + pa_n + q} \leq \frac{n}{1 + p + q}.$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$. In addition, if $q = q_0$, the equality also occurs for $a_1 = n - 1 + \frac{(n - 2)p}{2}$ and $a_2 = a_3 = \cdots = a_n = \frac{2 - (n - 2)p}{2(n - 1)}$ (or any cyclic permutation).

The proof is similar to the one of the preceding P 1.20. So, it suffices to show that $A(u) \geq 0$ for $u \in [0, 1]$, and $B(x, y) \geq 0$ for $x \geq 1 \geq y \geq 0$ such that $x + (n-1)y = n$, where

$$A(u) = q - p^2 - 3pu - 3u^2,$$

$$B(x, y) = q - p(p+1) - (p+1)(x+y) - xy.$$

Since $A(u)$ is concave, we have $A(u) \geq 0$ for $u \in [0, 1]$ if $A(0) \geq 0$ and $A(1) \geq 0$. Indeed,

$$A(0) = q - p^2 \geq q_0 - p^2 = \frac{(n-2)^2 p^2 + 4(n^2 - n + 1)(p+1)}{4(n-1)} > 0,$$

$$A(1) = q - p^2 - 3p - 3 \geq q_0 - p^2 - 3p - 3 = \frac{(n-2)^2(p+2)^2}{4(n-1)} > 0.$$

Also, we have

$$\begin{aligned} B &= (n-1)y^2 + [(n-2)p-2]y + q - (p+1)(p+n) \geq (n-1)y^2 + [(n-2)p-2]y + q_0 - (p+1)(p+n) \\ &= (n-1)y^2 + [(n-2)p-2]y + \frac{[(n-2)p-2]^2}{4(n-1)} = \frac{[2(n-1)y + (n-2)p-2]^2}{4(n-1)} \geq 0. \end{aligned}$$

□

P 1.22. Let $a_1, a_2, \dots, a_n$ be nonnegative numbers so that $a_1 + a_2 + \dots + a_n = n$. If $k \geq \frac{n^2}{4(n-1)}$, then

$$\frac{a_1(a_1-1)}{a_1^2+k} + \frac{a_2(a_2-1)}{a_2^2+k} + \dots + \frac{a_n(a_n-1)}{a_n^2+k} \geq 0.$$

(Vasile C., 2012)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \dots + a_n}{n} = 1,$$

where

$$f(u) = \frac{u(u-1)}{u^2+k}, \quad u \in [0, n].$$

From

$$f'(u) = \frac{u^2 + 2ku - k}{(u^2+k)^2}, \quad f''(u) = \frac{2(k^2 - u^3) + 6ku(1-u)}{(u^2+k)^3},$$

it follows that f is convex on $[0, 1]$. By the LHCF-Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for $x, y \geq 0$ so that $x + (n-1)y = n$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

We have

$$g(u) = \frac{u}{u^2+k}$$

and

$$\begin{aligned} h(x, y) &= \frac{k - xy}{(x^2 + k)(y^2 + k)} \geq \frac{n^2 - 4(n-1)xy}{4(n-1)(x^2 + k)(y^2 + k)} \\ &= \frac{[x + (n-1)y]^2 - 4(n-1)xy}{4(n-1)(x^2 + k)(y^2 + k)} = \frac{[x - (n-1)y]^2}{4(n-1)(x^2 + k)(y^2 + k)} \geq 0. \end{aligned}$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = n/2, \quad a_2 = a_3 = \cdots = a_n = n/(2n-2)$$

(or any cyclic permutation).

□

P 1.23. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then

$$\frac{a_1 - 1}{(n - 2a_1)^2} + \frac{a_2 - 1}{(n - 2a_2)^2} + \cdots + \frac{a_n - 1}{(n - 2a_n)^2} \geq 0.$$

(Vasile C., 2012)

Solution. For $n = 2$, the inequality is an identity. Consider further $n \geq 3$ and write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{u - 1}{(n - 2u)^2}, \quad u \in \mathbb{I} = [0, n] \setminus \{n/2\}.$$

From

$$f'(u) = \frac{2u + n - 4}{(n - 2u)^3}, \quad f''(u) = \frac{8(u + n - 3)}{(n - 2u)^4},$$

it follows that f is convex on $\mathbb{I}_{\leq s}$. By the LHCF-Theorem, Note 1 and Note 3, it suffices to show that $h(x, y) \geq 0$ for $x, y \in \mathbb{I}$ so that $x + (n-1)y = n$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{1}{(n - 2u)^2}$$

and

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{4(n - x - y)}{(n - 2x)^2(n - 2y)^2} = \frac{4(n - 2)y}{(n - 2x)^2(n - 2y)^2} \geq 0.$$

In accordance with Note 4, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = n, \quad a_2 = a_3 = \cdots = a_n = 0$$

(or any cyclic permutation).

□

P 1.24. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that

$$a_1 + a_2 + \dots + a_n = n, \quad a_1, a_2, \dots, a_n > -k, \quad k \geq 1 + \frac{n}{\sqrt{n-1}},$$

then

$$\frac{a_1^2 - 1}{(a_1 + k)^2} + \frac{a_2^2 - 1}{(a_2 + k)^2} + \dots + \frac{a_n^2 - 1}{(a_n + k)^2} \geq 0.$$

(Vasile C., 2008)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \dots + a_n}{n} = 1,$$

where

$$f(u) = \frac{u^2 - 1}{(u + k)^2}, \quad u > -k.$$

For $u \in (-k, 1]$, we have

$$f''(u) = \frac{2(k^2 - 3 - 2ku)}{(u + k)^4} \geq \frac{2(k^2 - 2k - 3)}{(u + k)^4} = \frac{2(k + 1)(k - 3)}{(u + k)^4} \geq 0.$$

Thus, f is convex on $(-k, s]$. By the LHCF-Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for $x, y > -k$ so that $x + (n - 1)y = n$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{u + 1}{(u + k)^2}$$

and

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{(k - 1)^2 - (1 + x)(1 + y)}{(x + k)^2(y + k)^2}.$$

Since

$$(k - 1)^2 \geq \frac{n^2}{n - 1},$$

we need to show that

$$n^2 \geq (n - 1)(1 + x)(1 + y).$$

Indeed,

$$n^2 - (n - 1)(1 + x)(1 + y) = n^2 - (1 + x)(2n - 1 - x) = (x - n + 1)^2 \geq 0.$$

The equality holds for $a_1 = a_2 = \dots = a_n = 1$. If $k = 1 + \frac{n}{\sqrt{n-1}}$, then the equality holds also for

$$a_1 = n - 1, \quad a_2 = a_3 = \dots = a_n = \frac{1}{n - 1}$$

(or any cyclic permutation).

□

P 1.25. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$. If $0 < k \leq 1 + \sqrt{\frac{2n-1}{n-1}}$, then

$$\frac{a_1^2 - 1}{(a_1 + k)^2} + \frac{a_2^2 - 1}{(a_2 + k)^2} + \dots + \frac{a_n^2 - 1}{(a_n + k)^2} \leq 0.$$

(Vasile C., 2008)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \dots + a_n}{n} = 1,$$

where

$$f(u) = \frac{1 - u^2}{(u + k)^2}, \quad u \in [0, n].$$

For $u \geq 1$, we have

$$f''(u) = \frac{2(2ku - k^2 + 3)}{(u + k)^4} \geq \frac{2(2k - k^2 + 3)}{(u + k)^4} = \frac{2(1 + k)(3 - k)}{(u + k)^4} > 0.$$

Thus, f is convex on $[s, n]$. By the RHCF-Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for $x, y \geq 0$ so that $x + (n - 1)y = n$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{-u - 1}{(u + k)^2}$$

and

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{2k - k^2 + x + y + xy}{(x + k)^2(y + k)^2} \geq \frac{2k - k^2 + x + y}{(x + k)^2(y + k)^2}.$$

Since

$$x + y \geq \frac{x + (n - 1)y}{n - 1} = \frac{n}{n - 1},$$

we get

$$2k - k^2 + x + y \geq 2k - k^2 + \frac{n}{n - 1} = -(k - 1)^2 + \frac{2n - 1}{n - 1} \geq 0,$$

hence $h(x, y) \geq 0$.

The equality holds for $a_1 = a_2 = \dots = a_n = 1$. If $k = 1 + \sqrt{\frac{2n-1}{n-1}}$, then the equality holds also for

$$a_1 = 0, \quad a_2 = a_3 = \dots = a_n = \frac{n}{n - 1}$$

(or any cyclic permutation).

□

P 1.26. If $a_1, a_2, \dots, a_n \geq n - 1 - \sqrt{n^2 - n + 1}$ so that $a_1 + a_2 + \dots + a_n = n$, then

$$\frac{a_1^2 - 1}{(a_1 + 2)^2} + \frac{a_2^2 - 1}{(a_2 + 2)^2} + \dots + \frac{a_n^2 - 1}{(a_n + 2)^2} \leq 0.$$

(Vasile C., 2008)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \dots + a_n}{n} = 1,$$

where

$$f(u) = \frac{1 - u^2}{(u + 2)^2}, \quad u \geq n - 1 - \sqrt{n^2 - n + 1}.$$

For $u \geq 1$, we have

$$f''(u) = \frac{2(4u - 1)}{(u + 2)^4} > 0.$$

Thus, $f(u)$ is convex for $u \geq s$. By the RHCF-Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for

$$n - 1 - \sqrt{n^2 - n + 1} \leq x \leq 1 \leq y, \quad x + (n - 1)y = n.$$

Since

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{-u - 1}{(u + 2)^2},$$

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{x + y + xy}{(x + 2)^2(y + 2)^2} = \frac{-x^2 + 2(n - 1)x + n}{(n - 1)(x + 2)^2(y + 2)^2},$$

we need to show that

$$n - 1 - \sqrt{n^2 - n + 1} \leq x \leq n - 1 + \sqrt{n^2 - n + 1}.$$

This is true because

$$n - 1 - \sqrt{n^2 - n + 1} \leq x \leq 1 < n - 1 + \sqrt{n^2 - n + 1}.$$

The equality holds for $a_1 = a_2 = \dots = a_n = 1$, and also for

$$a_1 = n - 1 - \sqrt{n^2 - n + 1}, \quad a_2 = a_3 = \dots = a_n = \frac{1 + \sqrt{n^2 - n + 1}}{n - 1}$$

(or any cyclic permutation).

□

P 1.27. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$. If $k \geq \frac{(n - 1)(2n - 1)}{n^2}$, then

$$\frac{1}{1 + ka_1^3} + \frac{1}{1 + ka_2^3} + \dots + \frac{1}{1 + ka_n^3} \geq \frac{n}{1 + k}.$$

(Vasile C., 2008)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{1}{1 + ku^3}, \quad u \in [0, n].$$

For $u \in [1, n]$, we have

$$f''(u) = \frac{6ku(2ku^3 - 1)}{(1 + ku^3)^3} \geq \frac{6ku(2k - 1)}{(1 + ku^3)^3} > 0.$$

Thus, f is convex on $[s, n]$. By the RHCF-Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for $x, y \geq 0$ so that $x + (n - 1)y = n$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

We have

$$g(u) = \frac{-k(u^2 + u + 1)}{(1 + k)(1 + ku^3)}$$

and

$$\frac{h(x, y)}{k^2} = \frac{x^2y^2 + xy(x + y - 1) + (x + y)^2 - (x + y + 1)/k}{(1 + k)(1 + kx^3)(1 + ky^3)}.$$

Since

$$x + y \geq \frac{x + (n - 1)y}{n - 1} = \frac{n}{n - 1} > 1,$$

it suffices to show that

$$(x + y)^2 \geq \frac{x + y + 1}{k}.$$

From $x + y \geq \frac{n}{n - 1}$, we get

$$k(x + y) \geq \frac{2n - 1}{n},$$

hence

$$k(x + y)^2 - x - y = (x + y)[k(x + y) - 1] \geq \frac{n}{n - 1} \left(\frac{2n - 1}{n} - 1 \right) = 1.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $k = \frac{(n - 1)(2n - 1)}{n^2}$, then the equality holds also for

$$a_1 = 0, \quad a_2 = a_3 = \cdots = a_n = \frac{n}{n - 1}$$

(or any cyclic permutation).

□

P 1.28. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$. If $0 < k \leq \frac{n-1}{n^2-2n+2}$, then

$$\frac{1}{1+ka_1^3} + \frac{1}{1+ka_2^3} + \dots + \frac{1}{1+ka_n^3} \leq \frac{n}{1+k}.$$

(Vasile C., 2008)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \dots + a_n}{n} = 1,$$

where

$$f(u) = \frac{-1}{1+ku^3}, \quad u \in [0, n].$$

For $u \in [0, 1]$, we have

$$f''(u) = \frac{6ku(1-2ku^3)}{(1+ku^3)^3} \geq \frac{6ku(1-2k)}{(1+ku^3)^3} \geq 0.$$

Thus, f is convex on $[0, s]$. By the LHCF-Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for $x, y \geq 0$ so that $x + (n-1)y = n$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

We have

$$g(u) = \frac{k(u^2 + u + 1)}{(1+k)(1+ku^3)}$$

and

$$\frac{h(x, y)}{k^2} = \frac{(x+y+1)/k - x^2y^2 - xy(x+y-1) - (x+y)^2}{(1+k)(1+kx^3)(1+ky^3)}.$$

It suffices to show that

$$\frac{(n^2 - 2n + 2)(x + y + 1)}{n - 1} - x^2y^2 - xy(x + y - 1) - (x + y)^2 \geq 0,$$

which is equivalent to

$$[2 + ny - (n-1)y^2][1 - (n-1)y]^2 \geq 0.$$

This is true because

$$2 + ny - (n-1)y^2 = 2 + y[n - (n-1)y] = 2 + xy > 0.$$

The equality holds for $a_1 = a_2 = \dots = a_n = 1$. If $k = \frac{n-1}{n^2-2n+2}$, then the equality holds also for

$$a_1 = n-1, \quad a_2 = a_3 = \dots = a_n = \frac{1}{n-1}$$

(or any cyclic permutation).

□

P 1.29. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$. If $k \geq \frac{n^2}{n-1}$, then

$$\sqrt{\frac{a_1}{k-a_1}} + \sqrt{\frac{a_2}{k-a_2}} + \dots + \sqrt{\frac{a_n}{k-a_n}} \leq \frac{n}{\sqrt{k-1}}.$$

(Vasile C., 2008)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \dots + a_n}{n} = 1,$$

where

$$f(u) = -\sqrt{\frac{u}{k-u}}, \quad u \in [0, n].$$

For $u \in [0, 1]$, we have

$$f''(u) = \frac{k(k-4u)}{4u^{3/2}(k-u)^{5/2}} \geq \frac{k(k-4)}{4u^{3/2}(k-u)^{5/2}} \geq 0.$$

Thus, f is convex on $[0, s]$. By the LHCF-Theorem, it suffices to prove that

$$f(x) + (n-1)f(y) \geq nf(1)$$

for $x \geq 1 \geq y \geq 0$ so that $x + (n-1)y = n$. We write the inequality as

$$\sqrt{\frac{(k-1)x}{k-x}} + (n-1)\sqrt{\frac{(k-1)y}{k-y}} \leq n,$$

$$\sqrt{1 + \frac{(n-1)k(1-y)}{(n-1)y + k - n}} \leq 1 + (n-1) \left[1 - \sqrt{\frac{(k-1)y}{k-y}} \right].$$

Let

$$z = \sqrt{\frac{(k-1)y}{k-y}}, \quad z \leq 1,$$

which yields

$$y = \frac{kz^2}{z^2 + k - 1},$$

$$1 - y = \frac{(k-1)(1-z^2)}{z^2 + k - 1}, \quad (n-1)y + k - n = \frac{(k-1)(nz^2 + k - n)}{z^2 + k - 1}.$$

Since

$$\begin{aligned} \frac{k(1-y)}{(n-1)y + k - n} &= \frac{k(1-z^2)}{k - n(1-z^2)} = \frac{1-z^2}{1 - n(1-z^2)/k} \\ &\leq \frac{1-z^2}{1 - (1-z^2)(n-1)/n} = \frac{n(1-z^2)}{(n-1)z^2 + 1}, \end{aligned}$$

it suffices to show that

$$\sqrt{1 + \frac{n(n-1)(1-z^2)}{(n-1)z^2 + 1}} \leq 1 + (n-1)(1-z).$$

By squaring, we get the obvious inequality

$$(z-1)^2[(n-1)z-1]^2 \geq 0.$$

The equality holds for $a_1 = a_2 = \dots = a_n = 1$. If $k = \frac{n^2}{n-1}$, then the equality holds also for

$$a_1 = \frac{n(n-1)^2}{n^2 - 2n + 2}, \quad a_2 = a_3 = \dots = a_n = \frac{n}{(n-1)(n^2 - 2n + 2)}$$

(or any cyclic permutation).

□

P 1.30. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$, then

$$n^{-a_1^2} + n^{-a_2^2} + \dots + n^{-a_n^2} \geq 1.$$

(Vasile C., 2006)

Solution. Let $k = \ln n$. Write the inequality as

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \dots + a_n}{n} = 1,$$

where

$$f(u) = n^{-u^2}, \quad u \in [0, n].$$

For $u \geq 1$, we have

$$f''(u) = 2kn^{-u^2}(2ku^2 - 1) \geq 2kn^{-u^2}(2k - 1) \geq 2kn^{-u^2}(2 \ln 2 - 1) > 0;$$

therefore, f is convex on $[s, n]$. By the RHCF-Theorem, it suffices to show that

$$f(x) + (n-1)f(y) \geq nf(1)$$

for $0 \leq x \leq 1 \leq y$ and $x + (n-1)y = n$. The desired inequality is equivalent to $g(x) \geq 0$, where

$$g(x) = n^{-x^2} + (n-1)n^{-y^2} - 1, \quad y = \frac{n-x}{n-1}, \quad 0 \leq x \leq 1.$$

Since $y' = -1/(n-1)$, we get

$$g'(x) = -2xkn^{-x^2} - 2(n-1)ky y' n^{-y^2} = 2k(yn^{-y^2} - xn^{-x^2}).$$

The derivative $g'(x)$ has the same sign as $g_1(x)$, where

$$g_1(x) = \ln(yn^{-y^2}) - \ln(xn^{-x^2}) = \ln y - \ln x + k(x^2 - y^2),$$

$$g_1'(x) = \frac{y'}{y} - \frac{1}{x} + 2k(x - yy') = n \left[\frac{-1}{x(n-x)} + \frac{2k(1 + nx - 2x)}{(n-1)^2} \right].$$

For $0 < x \leq 1$, $g_1'(x)$ has the same sign as

$$h(x) = \frac{-(n-1)^2}{2k} + x(n-x)(1 + nx - 2x).$$

Since

$$\begin{aligned} h'(x) &= n + 2(n^2 - 2n - 1)x - 3(n-2)x^2 \\ &\geq nx + 2(n^2 - 2n - 1)x - 3(n-2)x \\ &= 2(n-1)(n-2)x \geq 0, \end{aligned}$$

h is strictly increasing on $[0, 1]$. From

$$h(0) = \frac{-(n-1)^2}{2k} < 0, \quad h(1) = (n-1)^2 \left(1 - \frac{1}{2k} \right) > 0,$$

it follows that there is $x_1 \in (0, 1)$ so that $h(x_1) = 0$, $h(x) < 0$ for $x \in [0, x_1)$ and $h(x) > 0$ for $x \in (x_1, 1]$. Therefore, g_1 is strictly decreasing on $(0, x_1]$ and strictly increasing on $[x_1, 1]$. Since $g_1(0_+) = \infty$ and $g_1(1) = 0$, there is $x_2 \in (0, x_1)$ so that $g_1(x_2) = 0$, $g_1(x) > 0$ for $x \in (0, x_2)$ and $g_1(x) < 0$ for $x \in (x_2, 1)$. Consequently, g is strictly increasing on $[0, x_2]$ and strictly decreasing on $[x_2, 1]$. Because $g(0) > 0$ and $g(1) = 0$, it follows that $g(x) \geq 0$ for $x \in [0, 1]$. The proof is completed.

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 1.31. If a, b, c, d are nonnegative real numbers so that $a + b + c + d = 4$, then

$$(3a^2 + 1)(3b^2 + 1)(3c^2 + 1)(3d^2 + 1) \leq 256.$$

(Vasile C., 2006)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) + f(d) \geq 4f(s), \quad s = \frac{a + b + c + d}{4} = 1,$$

where

$$f(u) = -\ln(3u^2 + 1), \quad u \in [0, 4].$$

For $u \in [1, 4]$, we have

$$f''(u) = \frac{6(3u^2 - 1)}{(3u^2 + 1)^2} > 0.$$

Therefore, f is convex on $[s, 4]$. By the RHCF-Theorem, we only need to show that

$$f(x) + 3f(y) \geq 4f(1)$$

for $0 \leq x \leq 1 \leq y$ so that $x + 3y = 4$; that is, to show that $g(x) \geq 0$ for $x \in [0, 1]$, where

$$g(x) = f(x) + 3f(y) - 4f(1), \quad y = \frac{4-x}{3}.$$

Since $y'(x) = -1/3$, we have

$$\begin{aligned} g'(x) &= f'(x) + 3y'f'(y) = \frac{-6x}{3x^2+1} + \frac{6y}{3y^2+1} \\ &= \frac{6(x-y)(3xy-1)}{(3x^2+1)(3y^2+1)} = \frac{8(1-x)(x^2-4x+1)}{(3x^2+1)(3y^2+1)} \geq 0. \end{aligned}$$

Since g is increasing on $[0, 2 - \sqrt{3}]$ and decreasing on $[2 - \sqrt{3}, 1]$, it suffices to show that $g(0) \geq 0$ and $g(1) \geq 0$. The inequality $g(0) \geq 0$ is true if the original inequality holds for $a = 0$ and $b = c = d = 4/3$. This reduces to $19^3 \leq 27 \cdot 256$, which is true because $27 \cdot 256 - 19^3 = 53 > 0$. The inequality $g(1) \geq 0$ is also true because $g(1) = 0$.

The equality holds for $a = b = c = d = 1$.

□

P 1.32. If $a, b, c, d, e \geq -1$ so that $a + b + c + d + e = 5$, then

$$(a^2 + 1)(b^2 + 1)(c^2 + 1)(d^2 + 1)(e^2 + 1) \geq (a + 1)(b + 1)(c + 1)(d + 1)(e + 1).$$

(Vasile C., 2007)

Solution. Consider the nontrivial case $a, b, c, d, e > -1$, and write the inequality as

$$f(a) + f(b) + f(c) + f(d) + f(e) \geq nf(s), \quad s = \frac{a + b + c + d + e}{5} = 1,$$

where

$$f(u) = \ln(u^2 + 1) - \ln(u + 1), \quad u > -1.$$

For $u \in (-1, 1]$, we have

$$f''(u) = \frac{2(1-u^2)}{(u^2+1)^2} + \frac{1}{(u+1)^2} > 0.$$

Therefore, f is convex on $(-1, s]$. By the LHCF-Theorem and Note 2, it suffices to show that $H(x, y) \geq 0$ for $x, y > -1$ so that $x + 4y = 5$, where

$$H(x, y) = \frac{f'(x) - f'(y)}{x - y} = \frac{2(1 - xy)}{(x^2 + 1)(y^2 + 1)} + \frac{1}{(x + 1)(y + 1)};$$

thus, we need to show that

$$2(1 - xy) + \frac{(x^2 + 1)(y^2 + 1)}{(x + 1)(y + 1)} \geq 0.$$

Since

$$\frac{x^2 + 1}{x + 1} \geq \frac{x + 1}{2}, \quad \frac{y^2 + 1}{y + 1} \geq \frac{y + 1}{2},$$

it suffices to prove that

$$2(1 - xy) + \frac{(x+1)(y+1)}{4} \geq 0,$$

which is equivalent to

$$x + y + 9 - 7xy \geq 0,$$

$$28x^2 - 38x + 14 \geq 0,$$

$$(28x - 19)^2 + 31 \geq 0.$$

The equality holds for $a = b = c = d = e = 1$.

□

P 1.33. Let $a_1, a_2, \dots, a_n$ be positive numbers so that $a_1 + a_2 + \dots + a_n = n$. If

$$k \leq \frac{2\sqrt{n-1}}{n} + 2\sqrt{\frac{2\sqrt{n-1}}{n}}, \quad k \leq 3,$$

then

$$k(\sqrt{a_1} + \sqrt{a_2} + \dots + \sqrt{a_n}) + \frac{1}{\sqrt{a_1}} + \frac{1}{\sqrt{a_2}} + \dots + \frac{1}{\sqrt{a_n}} \geq (k+1)n.$$

(Vasile C., 2006)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \dots + a_n}{n} = 1,$$

where

$$f(u) = \frac{k}{\sqrt{u}} + \sqrt{u}, \quad u \in (0, n).$$

From

$$f''(u) = \frac{3 - ku}{4u^{5/2}},$$

it follows that f is convex on $(0, 1]$. Thus, according to the LHCF-Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for $x \geq 1 \geq y > 0$ such that $x + (n-1)y = n$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

We have

$$g(u) = \frac{k}{\sqrt{u} + 1} - \frac{1}{u + \sqrt{u}}$$

and

$$(\sqrt{x} + \sqrt{y})(\sqrt{x} + 1)(\sqrt{y} + 1)h(x, y) = -k + \frac{\sqrt{x} + \sqrt{y} + 1}{\sqrt{xy}}.$$

So, we need to show that

$$\frac{\sqrt{x} + \sqrt{y} + 1}{\sqrt{xy}} \geq k.$$

Since

$$\sqrt{x} + \sqrt{y} \geq 2\sqrt[4]{xy},$$

it suffices to show that

$$\frac{2\sqrt[4]{xy} + 1}{\sqrt{xy}} \geq k,$$

which is equivalent to

$$\frac{1}{\sqrt{xy}} + \frac{2}{\sqrt[4]{xy}} \geq k.$$

From

$$n = x + (n-1)y \geq 2\sqrt{(n-1)xy},$$

we get

$$\frac{1}{\sqrt{xy}} \geq \frac{2\sqrt{n-1}}{n},$$

hence

$$\frac{1}{\sqrt{xy}} + \frac{2}{\sqrt[4]{xy}} \geq \frac{2\sqrt{n-1}}{n} + 2\sqrt{\frac{2\sqrt{n-1}}{n}} \geq k.$$

The proof is completed. The equality holds for $a_1 = a_2 = \dots = a_n = 1$.

Remark. Since

$$1 < \frac{2\sqrt{n-1}}{n} + 2\sqrt{\frac{2\sqrt{n-1}}{n}}$$

for $n \leq 134$, the following inequality holds for $a_1, a_2, \dots, a_{134} > 0$ such that $a_1 + a_2 + \dots + a_{134} = 134$:

$$\sqrt{a_1} + \sqrt{a_2} + \dots + \sqrt{a_{134}} + \frac{1}{\sqrt{a_1}} + \frac{1}{\sqrt{a_2}} + \dots + \frac{1}{\sqrt{a_{134}}} \geq 268.$$

Since

$$2 < \frac{2\sqrt{n-1}}{n} + 2\sqrt{\frac{2\sqrt{n-1}}{n}}$$

for $n \leq 12$, the following inequality holds for $a_1, a_2, \dots, a_{12} > 0$ such that $a_1 + a_2 + \dots + a_{12} = 12$:

$$2(\sqrt{a_1} + \sqrt{a_2} + \dots + \sqrt{a_{12}}) + \frac{1}{\sqrt{a_1}} + \frac{1}{\sqrt{a_2}} + \dots + \frac{1}{\sqrt{a_{12}}} \geq 36.$$

□

P 1.34. If $a_1, a_2, \dots, a_n$ ($n \geq 3$) are positive numbers so that $a_1 + a_2 + \dots + a_n = 1$, then

$$\left(\frac{1}{\sqrt{a_1}} - \sqrt{a_1}\right) \left(\frac{1}{\sqrt{a_2}} - \sqrt{a_2}\right) \dots \left(\frac{1}{\sqrt{a_n}} - \sqrt{a_n}\right) \geq \left(\sqrt{n} - \frac{1}{\sqrt{n}}\right)^n.$$

(Vasile C., 2006)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = \frac{1}{n},$$

where

$$f(u) = \ln \left(\frac{1}{\sqrt{u}} - \sqrt{u} \right) = \ln(1 - u) - \frac{1}{2} \ln u, \quad u \in (0, 1).$$

From

$$f'(u) = \frac{-1}{1-u} - \frac{1}{2u}, \quad f''(u) = \frac{1-2u-u^2}{2u^2(1-u)^2},$$

it follows that $f''(u) \geq 0$ for $u \in (0, \sqrt{2} - 1]$. Since

$$s = \frac{1}{n} \leq \frac{1}{3} < \sqrt{2} - 1,$$

f is convex on $(0, s]$. Thus, we can apply the LHCF-Theorem.

First Solution. By the LHCF-Theorem, it suffices to show that

$$f(x) + (n-1)f(y) \geq nf\left(\frac{1}{n}\right)$$

for all $x, y > 0$ so that $x + (n-1)y = 1$; that is, to show that

$$\left(\frac{1}{\sqrt{x}} - \sqrt{x} \right) \left(\frac{1}{\sqrt{y}} - \sqrt{y} \right)^{n-1} \geq \left(\sqrt{n} - \frac{1}{\sqrt{n}} \right)^n.$$

Write this inequality as

$$n^{n/2}(1-y)^{n-1} \geq (n-1)^{n-1}x^{1/2}y^{(n-3)/2}.$$

By squaring, this inequality becomes as follows:

$$n^n(1-y)^{2n-2} \geq (n-1)^{2n-2}xy^{n-3},$$

$$(2-2y)^{2n-2} \geq \frac{(2n-2)^{2n-2}}{n^n}xy^{n-3},$$

$$\left[n \cdot \frac{1}{n} + x + (n-3)y \right]^{2n-2} \geq [n+1+(n-3)]^{n+1+(n-3)} \cdot \frac{1}{n^n} \cdot x \cdot y^{n-3}.$$

The last inequality follows from the AM-GM inequality. The proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n = 1/n$.

Second Solution. By the LHCF-Theorem and Note 2, it suffices to prove that $H(x, y) \geq 0$ for $x, y > 0$ so that $x + (n-1)y = 1$, where

$$H(x, y) = \frac{f'(x) - f'(y)}{x - y}.$$

We have

$$\begin{aligned} H(x, y) &= \frac{1 - x - y - xy}{2xy(1 - x)(1 - y)} = \frac{n(y + 1) - y - 3}{2x(1 - x)(1 - y)} \\ &\geq \frac{3(y + 1) - y - 3}{2x(1 - x)(1 - y)} = \frac{y}{x(1 - x)(1 - y)} > 0. \end{aligned}$$

Remark 1. We may write the inequality in P 1.34 in the form

$$\prod_{i=1}^n \left(\frac{1}{\sqrt{a_i}} - 1 \right) \cdot \prod_{i=1}^n (1 + \sqrt{a_i}) \geq \left(\sqrt{n} - \frac{1}{\sqrt{n}} \right)^n.$$

On the other hand, by the AM-GM inequality and the Cauchy-Schwarz inequality, we have

$$\prod_{i=1}^n (1 + \sqrt{a_i}) \leq \left(1 + \frac{1}{n} \sum_{i=1}^n \sqrt{a_i} \right)^n \leq \left(1 + \sqrt{\frac{1}{n} \sum_{i=1}^n a_i} \right)^n = \left(1 + \frac{1}{\sqrt{n}} \right)^n.$$

Thus, the following statement follows:

- If $a_1, a_2, \dots, a_n$ ($n \geq 3$) are positive real numbers so that $a_1 + a_2 + \dots + a_n = 1$, then

$$\left(\frac{1}{\sqrt{a_1}} - 1 \right) \left(\frac{1}{\sqrt{a_2}} - 1 \right) \cdots \left(\frac{1}{\sqrt{a_n}} - 1 \right) \geq (\sqrt{n} - 1)^n,$$

with equality for $a_1 = a_2 = \dots = a_n = 1/n$.

Remark 2. By squaring, the inequality in P 1.34 becomes

$$\prod_{i=1}^n \frac{(1 - a_i)^2}{a_i} \geq \frac{(n - 1)^{2n}}{n^n}.$$

On the other hand, since the function $f(x) = \ln \frac{1+x}{1-x}$ is convex on $(0, 1)$, by Jensen's inequality we have

$$\prod_{i=1}^n \left(\frac{1 + a_i}{1 - a_i} \right) \geq \left(\frac{1 + \frac{a_1 + a_2 + \dots + a_n}{n}}{1 - \frac{a_1 + a_2 + \dots + a_n}{n}} \right)^n = \left(\frac{n + 1}{n - 1} \right)^n.$$

Multiplying these inequalities yields the following result (*Kee-Wai Lau, 2000*):

- If $a_1, a_2, \dots, a_n$ ($n \geq 3$) are positive real numbers so that $a_1 + a_2 + \dots + a_n = 1$, then

$$\left(\frac{1}{a_1} - a_1 \right) \left(\frac{1}{a_2} - a_2 \right) \cdots \left(\frac{1}{a_n} - a_n \right) \geq \left(n - \frac{1}{n} \right)^n,$$

with equality for $a_1 = a_2 = \dots = a_n = 1/n$.

□

P 1.35. Let $a_1, a_2, \dots, a_n$ be positive real numbers so that $a_1 + a_2 + \dots + a_n = n$. If

$$0 < k \leq \left(1 + \frac{2\sqrt{n-1}}{n}\right)^2,$$

then

$$\left(ka_1 + \frac{1}{a_1}\right) \left(ka_2 + \frac{1}{a_2}\right) \cdots \left(ka_n + \frac{1}{a_n}\right) \geq (k+1)^n.$$

(Vasile C., 2006)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \dots + a_n}{n} = 1,$$

where

$$f(u) = \ln \left(ku + \frac{1}{u} \right), \quad u \in (0, n).$$

We have

$$f'(u) = \frac{ku^2 - 1}{u(ku^2 + 1)}, \quad f''(u) = \frac{1 + 4ku^2 - k^2u^4}{u^2(ku^2 + 1)^2}.$$

For $u \in (0, 1]$, we get $f''(u) > 0$ since

$$1 + 4ku^2 - k^2u^4 > ku^2(4 - ku^2) \geq ku^2(4 - k) \geq 0.$$

Therefore, f is convex on $(0, s]$. By the LHCF-Theorem and Note 2, it suffices to prove that $H(x, y) \geq 0$ for $x, y > 0$ so that $x + (n-1)y = n$, where

$$H(x, y) = \frac{f'(x) - f'(y)}{x - y}.$$

Since

$$H(x, y) = \frac{1 + k(x+y)^2 - k^2x^2y^2}{xy(kx^2 + 1)(ky^2 + 1)} > \frac{k[(x+y)^2 - kx^2y^2]}{xy(kx^2 + 1)(ky^2 + 1)},$$

it suffices to show that

$$x + y \geq \sqrt{k} xy.$$

Indeed, by the Cauchy-Schwarz inequality, we have

$$(x + y)[(n-1)y + x] \geq (\sqrt{n-1} + 1)^2 xy,$$

hence

$$x + y \geq \frac{1}{n}(\sqrt{n-1} + 1)^2 xy = \left(1 + \frac{2\sqrt{n-1}}{n}\right) xy \geq \sqrt{k} xy.$$

The equality holds for $a_1 = a_2 = \dots = a_n = 1$.

□

P 1.36. If a, b, c, d are nonzero real numbers so that

$$a, b, c, d \geq \frac{-1}{2}, \quad a + b + c + d = 4,$$

then

$$3 \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} \right) + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \geq 16.$$

Solution. Write the inequality as

$$f(a) + f(b) + f(c) + f(d) \geq 4f(s), \quad s = \frac{a + b + c + d}{4} = 1,$$

where

$$f(u) = \frac{3}{u^2} + \frac{1}{u}, \quad u \in \mathbb{I} = \left[\frac{-1}{2}, \frac{11}{2} \right] \setminus \{0\},$$

is convex on $\mathbb{I}_{\geq s}$ (because $3/u^2$ and $1/u$ are convex). By the RHCf-Theorem, Note 1 and Note 3, it suffices to prove that $h(x, y) \geq 0$ for $x, y \in \mathbb{I}$ so that

$$x + 3y = 4,$$

where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

Indeed, we have

$$g(u) = -\frac{4}{u} - \frac{3}{u^2},$$

$$h(x, y) = \frac{4xy + 3x + 3y}{x^2y^2} = \frac{2(1 + 2x)(6 - x)}{3x^2y^2} \geq 0.$$

In accordance with Note 4, the equality holds for $a = b = c = d = 1$, and also for

$$a = \frac{-1}{2}, \quad b = c = d = \frac{3}{2}$$

(or any cyclic permutation).

□

P 1.37. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that $a_1^2 + a_2^2 + \dots + a_n^2 = n$, then

$$a_1^3 + a_2^3 + \dots + a_n^3 - n + \sqrt{\frac{n}{n-1}} (a_1 + a_2 + \dots + a_n - n) \geq 0.$$

(Vasile C., 2007)

Solution. Replacing each a_i by $\sqrt{a_i}$, we have to prove that

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s),$$

where

$$s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1$$

and

$$f(u) = u\sqrt{u} + k\sqrt{u}, \quad k = \sqrt{\frac{n}{n-1}}, \quad u \in [0, n].$$

For $u \geq 1$, we have

$$f''(u) = \frac{3u - k}{4u\sqrt{u}} \geq \frac{3 - k}{4u\sqrt{u}} > 0.$$

Therefore, f is convex on $[s, n]$. According to the RHCF-Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for $x, y \geq 0$ so that $x + (n-1)y = n$. Since

$$g(u) = \frac{f(u) - f(1)}{u - 1} = 1 + \frac{u + k}{\sqrt{u} + 1}$$

and

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{\sqrt{x} + \sqrt{y} + \sqrt{xy} - k}{(\sqrt{x} + \sqrt{y})(\sqrt{x} + 1)(\sqrt{y} + 1)},$$

we need to show that

$$\sqrt{x} + \sqrt{y} + \sqrt{xy} \geq k.$$

Since

$$\sqrt{x} + \sqrt{y} + \sqrt{xy} \geq \sqrt{x} + \sqrt{y} \geq \sqrt{x + y},$$

it suffices to show that

$$x + y \geq k^2.$$

Indeed, we have

$$x + y \geq \frac{x}{n-1} + y = \frac{n}{n-1} = k^2.$$

In accordance with Note 4, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = 0, \quad a_2 = \cdots = a_n = \sqrt{\frac{n}{n-1}},$$

(or any cyclic permutation).

□

P 1.38. If a, b, c, d, e are nonnegative real numbers so that $a^2 + b^2 + c^2 + d^2 + e^2 = 5$, then

$$\frac{1}{7-2a} + \frac{1}{7-2b} + \frac{1}{7-2c} + \frac{1}{7-2d} + \frac{1}{7-2e} \leq 1.$$

(Vasile C., 2010)

Solution. Replacing a, b, c, d, e by $\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}, \sqrt{e}$, we have to prove that

$$f(a) + f(b) + f(c) + f(d) + f(e) \geq 5f(s),$$

where

$$s = \frac{a + b + c + d + e}{5} = 1$$

and

$$f(u) = \frac{1}{2\sqrt{u} - 7}, \quad u \in [0, 5].$$

For $u \in [0, 1]$, we have

$$f''(u) = \frac{7 - 6\sqrt{u}}{2u\sqrt{u}(7 - 2\sqrt{u})^3} > 0.$$

Therefore, f is convex on $[0, s]$. According to the LHCF-Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for $x, y \geq 0$ so that $x + 4y = 5$. Since

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{-2}{5(7 - 2\sqrt{u})(1 + \sqrt{u})}$$

and

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{2(5 - 2\sqrt{x} - 2\sqrt{y})}{(\sqrt{x} + \sqrt{y})(1 + \sqrt{x})(1 + \sqrt{y})(7 - 2\sqrt{x})(7 - 2\sqrt{y})},$$

we need to show that

$$\sqrt{x} + \sqrt{y} \leq \frac{5}{2}.$$

Indeed, by the Cauchy-Schwarz inequality, we have

$$(\sqrt{x} + \sqrt{y})^2 \leq \left(1 + \frac{1}{4}\right)(x + 4y) = \frac{25}{4}.$$

The proof is completed. The equality holds for $a = b = c = d = e = 1$, and also for

$$a = 2, \quad b = c = d = e = \frac{1}{2}$$

(or any cyclic permutation).

Remark In the same manner, we can prove the following generalization:

• Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1^2 + a_2^2 + \dots + a_n^2 = n$. If $k \geq 1 + \frac{n}{\sqrt{n-1}}$, then

$$\frac{1}{k - a_1} + \frac{1}{k - a_2} + \dots + \frac{1}{k - a_n} \leq \frac{n}{k - 1},$$

with equality for $a_1 = a_2 = \dots = a_n = 1$. If $k = 1 + \frac{n}{\sqrt{n-1}}$, then the equality holds also for

$$a_1 = \sqrt{n-1}, \quad a_2 = \dots = a_n = \frac{1}{\sqrt{n-1}}$$

(or any cyclic permutation).

□

P 1.39. Let $0 \leq a_1, a_2, \dots, a_n < k$ so that $a_1^2 + a_2^2 + \dots + a_n^2 = n$. If

$$1 < k \leq 1 + \sqrt{\frac{n}{n-1}},$$

then

$$\frac{1}{k-a_1} + \frac{1}{k-a_2} + \dots + \frac{1}{k-a_n} \geq \frac{n}{k-1}.$$

(Vasile C., 2010)

Solution. Replacing $a_1, a_2, \dots, a_n$ by $\sqrt{a_1}, \sqrt{a_2}, \dots, \sqrt{a_n}$, we have to prove that

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq nf(s),$$

where

$$s = \frac{a_1 + a_2 + \dots + a_n}{n} = 1$$

and

$$f(u) = \frac{1}{k - \sqrt{u}}, \quad u \in [0, k^2).$$

From

$$f''(u) = \frac{3\sqrt{u} - k}{4u\sqrt{u}(k - \sqrt{u})^3},$$

it follows that f is convex on $[s, k^2)$. According to the RHCF-Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for all $x, y \in [0, k^2)$ so that $x + (n-1)y = n$. Since

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{1}{(k-1)(k - \sqrt{u})(1 + \sqrt{u})}$$

and

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{\sqrt{x} + \sqrt{y} + 1 - k}{(k-1)(\sqrt{x} + \sqrt{y})(1 + \sqrt{x})(1 + \sqrt{y})(k - \sqrt{x})(k - \sqrt{y})},$$

we need to show that

$$\sqrt{x} + \sqrt{y} \geq k - 1.$$

Indeed,

$$\sqrt{x} + \sqrt{y} \geq \sqrt{x+y} \geq \sqrt{\frac{x}{n-1} + y} = \sqrt{\frac{n}{n-1}} \geq k - 1.$$

The proof is completed. The equality holds for $a_1 = a_2 = \dots = a_n = 1$, and also for

$$a_1 = 0, \quad a_2 = \dots = a_n = \sqrt{\frac{n}{n-1}}$$

(or any cyclic permutation).

□

P 1.40. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\sqrt{1 + \frac{48a}{b+c}} + \sqrt{1 + \frac{48b}{c+a}} + \sqrt{1 + \frac{48c}{a+b}} \geq 15.$$

(Vasile C., 2005)

Solution. Due to homogeneity, we may assume that $a + b + c = 1$. Thus, we need to show that

$$f(a) + f(b) + f(c) \geq 3f(s),$$

where

$$s = \frac{a+b+c}{3} = \frac{1}{3}$$

and

$$f(u) = \sqrt{\frac{1+47u}{1-u}}, \quad u \in [0, 1).$$

From

$$f''(u) = \frac{48(47u-11)}{\sqrt{(1-u)^5(1+47u)^3}},$$

it follows that f is convex on $[s, 1)$. By the RHCF-Theorem, it suffices to show that

$$f(x) + 2f(y) \geq 3f\left(\frac{1}{3}\right)$$

for $x, y \geq 0$ so that $x + 2y = 1$; that is,

$$\sqrt{\frac{1+47x}{1-x}} + 2\sqrt{\frac{49-47x}{1+x}} \geq 15.$$

Setting

$$t = \sqrt{\frac{49-47x}{1+x}}, \quad 1 < t \leq 7,$$

the inequality turns into

$$\sqrt{\frac{1175-23t^2}{t^2-1}} \geq 15-2t.$$

By squaring, this inequality becomes

$$350 - 15t - 61t^2 + 15t^3 - t^4 \geq 0,$$

$$(5-t)^2(2+t)(7-t) \geq 0.$$

The original inequality is an equality for $a = b = c$, and also for $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 1.41. If a, b, c are nonnegative real numbers, then

$$\sqrt{\frac{3a^2}{7a^2 + 5(b+c)^2}} + \sqrt{\frac{3b^2}{7b^2 + 5(c+a)^2}} + \sqrt{\frac{3c^2}{7c^2 + 5(a+b)^2}} \leq 1.$$

(Vasile C., 2008)

Solution. Due to homogeneity, we may assume that $a + b + c = 3$. Thus, we need to show that

$$f(a) + f(b) + f(c) \geq 3f(s),$$

where

$$s = \frac{a + b + c}{3} = 1$$

and

$$f(u) = -\sqrt{\frac{3u^2}{7u^2 + 5(3-u)^2}} = \frac{-u}{\sqrt{4u^2 - 10u + 15}}, \quad u \in [0, 3].$$

From

$$f''(u) = \frac{5(-8u^2 + 41u - 30)}{(4u^2 - 10u + 15)^{5/2}} \geq \frac{5(-8u^2 + 38u - 30)}{(4u^2 - 10u + 15)^{5/2}} = \frac{10(u-1)(15-4u)}{(4u^2 - 10u + 15)^{5/2}},$$

it follows that f is convex on $[s, 3]$. By the RHCF-Theorem, it suffices to prove the original homogeneous inequality for $b = c = 0$ and $b = c = 1$. For the nontrivial case $b = c = 1$, we need to show that

$$\sqrt{\frac{3a^2}{7a^2 + 20}} + 2\sqrt{\frac{3}{5a^2 + 10a + 12}} \leq 1.$$

By squaring two times, the inequality becomes

$$a(5a^3 + 10a^2 + 16a + 50) \geq 3a\sqrt{(7a^2 + 20)(5a^2 + 10a + 12)},$$

$$a^2(5a^6 + 20a^5 - 11a^4 + 38a^3 - 80a^2 - 40a + 68) \geq 0,$$

$$a^2(a-1)^2(5a^4 + 30a^3 + 44a^2 + 96a + 68) \geq 0.$$

The last inequality is clearly true.

The equality holds for $a = b = c$, and also for $a = 0$ and $b = c$ (or any cyclic permutation). $\square$

P 1.42. If a, b, c are nonnegative real numbers, then

$$\sqrt{\frac{a^2}{a^2 + 2(b+c)^2}} + \sqrt{\frac{b^2}{b^2 + 2(c+a)^2}} + \sqrt{\frac{c^2}{c^2 + 2(a+b)^2}} \geq 1.$$

(Vasile C., 2008)

Solution. Due to homogeneity, we may assume that $a + b + c = 3$. Thus, we need to show that

$$f(a) + f(b) + f(c) \geq 3f(s),$$

where

$$s = \frac{a + b + c}{3} = 1$$

and

$$f(u) = \sqrt{\frac{3u^2}{u^2 + 2(3-u)^2}} = \frac{u}{\sqrt{u^2 - 4u + 6}}, \quad u \in [0, 3].$$

From

$$f''(u) = \frac{2(2u^2 - 11u + 12)}{(u^2 - 4u + 6)^{5/2}} \geq \frac{2(-11u + 12)}{(u^2 - 4u + 6)^{5/2}},$$

it follows that f is convex on $[0, s]$. By the LHCF-Theorem, it suffices to prove the original homogeneous inequality for $b = c = 0$ and $b = c = 1$. For the nontrivial case $b = c = 1$, the inequality has the form

$$\frac{a}{\sqrt{a^2 + 8}} + \frac{2}{\sqrt{2a^2 + 4a + 3}} \geq 1.$$

By squaring, the inequality becomes

$$a\sqrt{(a^2 + 8)(2a^2 + 4a + 3)} \geq 3a^2 + 8a - 2.$$

For the nontrivial case $3a^2 + 8a - 2 > 0$, by squaring both sides we get

$$a^6 + 2a^5 + 5a^4 - 8a^3 - 14a^2 + 16a - 2 \geq 0,$$

$$(a - 1)^2[a^4 + 4a^3 + 9a^2 + 4a + (3a^2 + 8a - 2)] \geq 0.$$

The equality holds for $a = b = c$, and also for $b = c = 0$ (or any cyclic permutation).

□

P 1.43. Let a, b, c be nonnegative real numbers, no two of which are zero. If

$$k \geq k_0, \quad k_0 = \frac{\ln 3}{\ln 2} - 1 \approx 0.585,$$

then

$$\left(\frac{2a}{b+c}\right)^k + \left(\frac{2b}{c+a}\right)^k + \left(\frac{2c}{a+b}\right)^k \geq 3.$$

(Vasile C., 2005)

Solution. For $k = 1$, the inequality is just the well known Nesbitt's inequality

$$\frac{2a}{b+c} + \frac{2b}{c+a} + \frac{2c}{a+b} \geq 3.$$

For $k \geq 1$, the inequality follows from Nesbitt's inequality and Jensens's inequality applied to the convex function $f(u) = u^k$:

$$\left(\frac{2a}{b+c}\right)^k + \left(\frac{2b}{c+a}\right)^k + \left(\frac{2c}{a+b}\right)^k \geq 3 \left(\frac{\frac{2a}{b+c} + \frac{2b}{c+a} + \frac{2c}{a+b}}{3}\right)^k \geq 3.$$

Consider now that

$$k_0 \leq k < 1.$$

Due to homogeneity, we may assume that $a + b + c = 1$. Thus, we need to show that

$$f(a) + f(b) + f(c) \geq 3f(s),$$

where

$$s = \frac{a+b+c}{3} = \frac{1}{3}$$

and

$$f(u) = \left(\frac{2u}{1-u}\right)^k, \quad u \in [0, 1).$$

From

$$f''(u) = \frac{4k}{(1-u)^4} \left(\frac{2u}{1-u}\right)^{k-2} (2u+k-1),$$

it follows that f is convex on $[s, 1)$ (because $u \geq s = 1/3$ involves $2u+k-1 \geq 2/3+k-1 = k-1/3 > 0$). By the RHCF-Theorem, it suffices to prove the original homogeneous inequality for $b = c = 1$ and $a \in [0, 1]$; that is, to show that $h(a) \geq 3$, where

$$h(a) = a^k + 2 \left(\frac{2}{a+1}\right)^k, \quad a \in [0, 1].$$

For $a \in (0, 1]$, the derivative

$$h'(a) = ka^{k-1} - k \left(\frac{2}{a+1}\right)^{k+1}$$

has the same sign as

$$g(a) = (k-1) \ln a - (k+1) \ln \frac{2}{a+1}.$$

From

$$g'(a) = \frac{2ka+k-1}{a(a+1)},$$

it follows that $g'(a_0) = 0$ for $a_0 = (1-k)/(2k) < 1$, $g'(a) < 0$ for $a \in (0, a_0)$ and $g'(a) > 0$ for $a \in (a_0, 1]$. Consequently, g is strictly decreasing on $(0, a_0]$ and strictly increasing on $(a_0, 1]$. Since $g(0_+) = \infty$ and $g(1) = 0$, there exists $a_1 \in (0, a_0)$ so that $g(a_1) = 0$, $g(a) > 0$ for $a \in (0, a_1)$ and $g(a) < 0$ for $a \in (a_1, 1]$; therefore, $h(a)$ is strictly increasing on $[0, a_1]$ and strictly decreasing on $[a_1, 1]$. As a result,

$$h(a) \geq \min\{h(0), h(1)\}.$$

Since $h(0) = 2^{k+1} \geq 3$ and $h(1) = 3$, we get $h(a) \geq 3$. The proof is completed. The equality holds for $a = b = c$. If $k = k_0$, then the equality holds also for $a = 0$ and $b = c$ (or any cyclic permutation).

Remark. For $k = 2/3$, we can give the following solution (based on the AM-GM inequality):

$$\begin{aligned} \sum \left(\frac{2a}{b+c} \right)^{2/3} &= \sum \frac{2a}{\sqrt[3]{2a \cdot (b+c) \cdot (b+c)}} \\ &\geq \sum \frac{6a}{2a + (b+c) + (b+c)} = 3. \end{aligned}$$

□

P 1.44. If $a, b, c \in [1, 7 + 4\sqrt{3}]$, then

$$\sqrt{\frac{2a}{b+c}} + \sqrt{\frac{2b}{c+a}} + \sqrt{\frac{2c}{a+b}} \geq 3.$$

(Vasile C., 2007)

Solution. Denoting

$$s = \frac{a+b+c}{3}, \quad 1 \leq s \leq 7 + 4\sqrt{3},$$

we need to show that

$$f(a) + f(b) + f(c) \geq 3f(s),$$

where

$$f(u) = \sqrt{\frac{2u}{3s-u}}, \quad 1 \leq u < 3s.$$

For $u \geq s$, we have

$$f''(u) = 3s \left(\frac{3s-u}{2u} \right)^{3/2} \frac{4u-3s}{(3s-u)^4} > 0.$$

Therefore, $f(u)$ is convex for $u \geq s$. By the RHCF-Theorem, it suffices to prove the original inequality for $b = c$; that is,

$$\sqrt{\frac{a}{b}} + 2\sqrt{\frac{2b}{a+b}} \geq 3.$$

Putting $t = \sqrt{\frac{b}{a}}$, the condition $a, b \in [1, 7 + 4\sqrt{3}]$ involves

$$2 - \sqrt{3} \leq t \leq 2 + \sqrt{3}.$$

We need to show that

$$2\sqrt{\frac{2t^2}{t^2+1}} \geq 3 - \frac{1}{t}.$$

This is true if

$$\frac{8t^2}{t^2+1} \geq \left(3 - \frac{1}{t}\right)^2,$$

which is equivalent to the obvious inequality

$$(t-1)^2(t-2+\sqrt{3})(t-2-\sqrt{3}) \leq 0.$$

The equality holds for $a = b = c$, and also for $a = 1$, and $b = c = 7 + 4\sqrt{3}$ (or any cyclic permutation).

□

P 1.45. Let a, b, c be nonnegative real numbers so that $a + b + c = 3$. If

$$0 < k \leq k_0, \quad k_0 = \frac{\ln 2}{\ln 3 - \ln 2} \approx 1.71,$$

then

$$a^k(b+c) + b^k(c+a) + c^k(a+b) \leq 6.$$

Solution. For $0 < k \leq 1$, the inequality follows from Jensens's inequality applied to the convex function $f(u) = -u^k$:

$$\begin{aligned} (b+c)a^k + (c+a)b^k + (a+b)c^k &\leq 2(a+b+c) \left[\frac{(b+c)a + (c+a)b + (a+b)c}{2(a+b+c)} \right]^k \\ &= 6 \left(\frac{ab+bc+ca}{3} \right)^k \leq 6 \left(\frac{a+b+c}{3} \right)^{2k} = 6. \end{aligned}$$

Consider now that

$$1 < k \leq k_0,$$

and write the inequality as

$$f(a) + f(b) + f(c) \geq 3f(s),$$

where

$$s = \frac{a+b+c}{3} = 1$$

and

$$f(u) = u^k(u-3), \quad u \in [0, 3].$$

For $u \geq 1$, we have

$$f''(u) = ku^{k-2}[(k+1)u - 3k + 3] \geq ku^{k-2}[(k+1) - 3k + 3] = 2k(2-k)u^{k-2} > 0;$$

therefore, f is convex on $[1, s]$. By the RHCF-Theorem, it suffices to consider the case $a \leq b = c$. So, we only need to prove the homogeneous inequality

$$a^k(b+c) + b^k(c+a) + c^k(a+b) \leq 6 \left(\frac{a+b+c}{3} \right)^{k+1}$$

for $b = c = 1$ and $a \in [0, 1]$; that is, to show that $g(a) \geq 0$ for $a \geq 0$, where

$$g(a) = 3 \left(\frac{a+2}{3} \right)^{k+1} - a^k - a - 1.$$

We have

$$g'(a) = (k+1) \left(\frac{a+2}{3} \right)^k - ka^{k-1} - 1, \quad \frac{1}{k}g''(a) = \frac{k+1}{3} \left(\frac{a+2}{3} \right)^{k-1} - \frac{k-1}{a^{2-k}}.$$

Since g'' is strictly increasing, $g''(0_+) = -\infty$ and $g''(1) = 2k(2-k)/3 > 0$, there exists $a_1 \in (0, 1)$ so that $g''(a_1) = 0$, $g''(a) < 0$ for $a \in (0, a_1)$, $g''(a) > 0$ for $a \in (a_1, 1]$. Therefore, g' is strictly decreasing on $[0, a_1]$ and strictly increasing on $[a_1, 1]$. Since

$$g'(0) = (k+1)(2/3)^k - 1 \geq (k+1)(2/3)^{k_0} - 1 = \frac{k+1}{2} - 1 = \frac{k-1}{2} > 0,$$

$$g'(1) = 0,$$

there exists $a_2 \in (0, a_1)$ so that $g'(a_2) = 0$, $g'(a) > 0$ for $a \in [0, a_2)$, $g'(a) < 0$ for $a \in (a_2, 1]$. Thus, g is strictly increasing on $[0, a_2]$ and strictly decreasing on $[a_2, 1]$; consequently,

$$g(a) \geq \min\{g(0), g(1)\}.$$

From

$$g(0) = 3(2/3)^{k+1} - 1 \geq 3(2/3)^{k_0+1} - 1 = 1 - 1 = 0, \quad g(1) = 0,$$

we get $g(a) \geq 0$. This completes the proof. The equality holds for $a = b = c = 1$. If $k = k_0$, then the equality holds also for $a = 0$ and $b = c = 3/2$ (or any cyclic permutation).

Remark 1. Using the Cauchy-Schwarz inequality and the inequality in P 1.45, we get

$$\sum \frac{a}{b^k + c^k} \geq \frac{(a + b + c)^2}{\sum a(b^k + c^k)} = \frac{9}{\sum a^k(b + c)} \geq \frac{3}{2}.$$

Thus, the following statement holds.

- Let a, b, c be nonnegative real numbers so that $a + b + c = 3$. If

$$0 < k \leq k_0, \quad k_0 = \frac{\ln 2}{\ln 3 - \ln 2} \approx 1.71,$$

then

$$\frac{a}{b^k + c^k} + \frac{b}{c^k + a^k} + \frac{c}{a^k + b^k} \geq \frac{3}{2},$$

with equality for $a = b = c = 1$. If $k = k_0$, then the equality holds also for $a = 0$ and $b = c = 3/2$ (or any cyclic permutation).

Remark 2. Also, the following statement holds:

- Let a, b, c be nonnegative real numbers so that $a + b + c = 3$. If

$$k \geq k_1, \quad k_1 = \frac{\ln 9 - \ln 8}{\ln 3 - \ln 2} \approx 0.2905,$$

then

$$\frac{a^k}{b + c} + \frac{b^k}{c + a} + \frac{c^k}{a + b} \geq \frac{3}{2},$$

with equality for $a = b = c = 1$. If $k = k_1$, then the equality holds also for $a = 0$ and $b = c = 3/2$ (or any cyclic permutation).

For $k_1 \leq k \leq 2$, the inequality can be proved using the Cauchy-Schwarz inequality and the inequality in P 1.45, as follows:

$$\sum \frac{a^k}{b + c} \geq \frac{(a + b + c)^2}{\sum a^{2-k}(b + c)} = \frac{9}{\sum a^{2-k}(b + c)} \geq \frac{3}{2}.$$

For $k \geq 2$, the inequality can be deduced from the Cauchy-Schwarz inequality and Bernoulli's inequality, as follows:

$$\sum \frac{a^k}{b+c} \geq \frac{(\sum a^{k/2})^2}{\sum(b+c)} = \frac{(\sum a^{k/2})^2}{6},$$

$$\sum a^{k/2} \geq \sum \left[1 + \frac{k}{2}(a-1) \right] = 3.$$

□

P 1.46. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$\sqrt{a} + \sqrt{b} + \sqrt{c} - 3 \geq 13 \left(\sqrt{\frac{a+b}{2}} + \sqrt{\frac{b+c}{2}} + \sqrt{\frac{c+a}{2}} - 3 \right).$$

(Vasile C., 2008)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) \geq 3f(s), \quad s = \frac{a+b+c}{3} = 1,$$

where

$$f(u) = \sqrt{u} - 13\sqrt{\frac{3-u}{2}}, \quad u \in [0, 3].$$

For $u \in [1, 3]$, we have

$$4f''(u) = -u^{-3/2} + \frac{13}{4} \left(\frac{3-u}{2} \right)^{-3/2} \geq -1 + \frac{13}{4} > 0.$$

Therefore, f is convex on $[s, 3]$. By the RHCF-Theorem, it suffices to consider only the case $a \leq b = c$. Write the original inequality in the homogeneous form

$$\sqrt{a} + \sqrt{b} + \sqrt{c} - 3\sqrt{\frac{a+b+c}{3}} \geq 13 \left(\sqrt{\frac{a+b}{2}} + \sqrt{\frac{b+c}{2}} + \sqrt{\frac{c+a}{2}} - 3\sqrt{\frac{a+b+c}{3}} \right).$$

Due to homogeneity, we may assume that $b = c = 1$. Moreover, it is convenient to use the notation $\sqrt{a} = x$. Thus, we need to show that $g(x) \geq 0$ for $x \in [0, 1]$, where

$$g(x) = x - 11 + 36\sqrt{\frac{x^2+2}{3}} - 26\sqrt{\frac{x^2+1}{2}}.$$

We have

$$g'(x) = 1 + 12x\sqrt{\frac{3}{x^2+2}} - 13x\sqrt{\frac{2}{x^2+1}},$$

$$g''(x) = \frac{13}{2} \left(\frac{2}{x^2+1} \right)^{3/2} \left[\left(m \cdot \frac{x^2+1}{x^2+2} \right)^{3/2} - 1 \right],$$

where

$$m = \frac{6\sqrt[3]{52}}{13} \approx 1.72.$$

Clearly, $g''(x)$ has the same sign as $h(x)$, where

$$h(x) = m \cdot \frac{x^2 + 1}{x^2 + 2} - 1.$$

Since h is strictly increasing,

$$h(0) = \frac{m}{2} - 1 < 0, \quad h(1) = \frac{2m}{3} - 1 > 0,$$

there is $x_1 \in (0, 1)$ so that $h(x_1) = 0$, $h(x) < 0$ for $x \in [0, x_1)$ and $h(x) > 0$ for $x \in (x_1, 1]$. Therefore, g' is strictly decreasing on $[0, x_1]$ and strictly increasing on $[x_1, 1]$. Since $g'(0) = 1$ and $g'(1) = 0$, there is $x_2 \in (0, x_1)$ so that $g'(x_2) = 0$, $g'(x) > 0$ for $x \in (0, x_2)$ and $g'(x) < 0$ for $x \in (x_2, 1)$. Thus, $g(x)$ is strictly increasing on $[0, x_2]$ and strictly decreasing on $[x_2, 1]$. From

$$g(0) = -11 + 12\sqrt{6} - 13\sqrt{2} > 0$$

and $g(1) = 0$, it follows that $g(x) \geq 0$ for $x \in [0, 1]$. This completes the proof. The equality holds for $a = b = c = 1$.

Remark. Similarly, we can prove the following generalizations:

- Let a, b, c be nonnegative real numbers so that $a + b + c = 3$. If $k \geq k_0$, where

$$k_0 = \frac{\sqrt{6} - 2}{\sqrt{6} - \sqrt{2} - 1} = (2 + \sqrt{2})(2 + \sqrt{3}) \approx 12.74,$$

then

$$\sqrt{a} + \sqrt{b} + \sqrt{c} - 3 \geq k \left(\sqrt{\frac{a+b}{2}} + \sqrt{\frac{b+c}{2}} + \sqrt{\frac{c+a}{2}} - 3 \right),$$

with equality for $a = b = c = 1$. If $k = k_0$, then the equality holds also for $a = 0$ and $b = c = 3/2$ (or any cyclic permutation).

- Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$. If $k \geq k_0$, where

$$k_0 = \frac{\sqrt{n} - \sqrt{n-1}}{\sqrt{n} - \sqrt{n-2} - \frac{1}{\sqrt{n-1}}},$$

then

$$\sqrt{a_1} + \sqrt{a_2} + \dots + \sqrt{a_n} - n \geq k \left(\sqrt{\frac{n-a_1}{n-1}} + \sqrt{\frac{n-a_2}{n-1}} + \dots + \sqrt{\frac{n-a_n}{n-1}} - n \right),$$

with equality for $a_1 = a_2 = \dots = a_n = 1$. If $k = k_0$, then the equality holds also for $a_1 = 0$ and $a_2 = a_3 = \dots = a_n = \frac{n}{n-1}$ (or any cyclic permutation).

□

P 1.47. Let a, b, c be nonnegative real numbers so that $a + b + c = 3$. If $k > 2$, then

$$a^k + b^k + c^k + 3 \geq 2 \left(\frac{a+b}{2} \right)^k + 2 \left(\frac{b+c}{2} \right)^k + 2 \left(\frac{c+a}{2} \right)^k.$$

Solution. Write the inequality as

$$f(a) + f(b) + f(c) \geq 3f(s), \quad s = \frac{a+b+c}{3} = 1,$$

where

$$f(u) = u^k - 2 \left(\frac{3-u}{2} \right)^k, \quad u \in [0, 3].$$

For $u \geq 1$, we have

$$\frac{f''(u)}{k(k-1)} = u^{k-2} - \frac{1}{2} \left(\frac{3-u}{2} \right)^{k-2} \geq 1 - \frac{1}{2} > 0.$$

Therefore, f is convex on $[s, 3]$. By the RHCF-Theorem, it suffices to consider only the case $a \leq b = c$. Write the original inequality in the homogeneous form

$$a^k + b^k + c^k + 3 \left(\frac{a+b+c}{3} \right)^k \geq 2 \left(\frac{a+b}{2} \right)^k + 2 \left(\frac{b+c}{2} \right)^k + 2 \left(\frac{c+a}{2} \right)^k.$$

Due to homogeneity, we may assume that $b = c = 1$. Thus, we need to prove that

$$a^k + 3 \left(\frac{a+2}{3} \right)^k \geq 4 \left(\frac{a+1}{2} \right)^k$$

for $a \in [0, 1]$. Substituting

$$a^k = t, \quad t \in [0, 1],$$

we need to show that $g(t) \geq 0$, where

$$g(t) = t + 3 \left(\frac{t^{1/k} + 2}{3} \right)^k - 4 \left(\frac{t^{1/k} + 1}{2} \right)^k.$$

We have

$$\begin{aligned} g'(t) &= 1 + t^{1/k-1} \left(\frac{t^{1/k} + 2}{3} \right)^{k-1} - 2t^{1/k-1} \left(\frac{t^{1/k} + 1}{2} \right)^{k-1}, \\ \frac{kt^{2-1/k}}{k-1} g''(t) &= \left(\frac{t^{1/k} + 1}{2} \right)^{k-2} - \frac{2}{3} \left(\frac{t^{1/k} + 2}{3} \right)^{k-2}. \end{aligned}$$

Setting

$$m = \left(\frac{2}{3} \right)^{\frac{1}{k-2}}, \quad 0 < m < 1,$$

we see that $g''(t)$ has the same sign as $h(t)$, where

$$h(t) = 6 \left(\frac{t^{1/k} + 1}{2} - m \frac{t^{1/k} + 2}{3} \right) = (3 - 2m)t^{1/k} + 3 - 4m$$

is strictly increasing. There are two cases to consider: $0 < m \leq 3/4$ and $3/4 < m < 1$.

Case 1: $0 < m \leq 3/4$. Since $h(0) = 3 - 4m \geq 0$, we have $h(t) > 0$ for $t \in (0, 1]$, hence g' is strictly increasing on $(0, 1]$. From $g'(1) = 0$, it follows that $g'(t) < 0$ for $t \in (0, 1)$, hence g is strictly decreasing on $[0, 1]$. Since $g(1) = 0$, we get $g(t) > 0$ for $t \in [0, 1)$.

Case 2: $3/4 < m < 1$. From $m > 3/4$, we get

$$2^{2k-3} > 3^{k-1}.$$

Since $h(0) = 3 - 4m < 0$ and $h(1) = 3(1 - m) > 0$, there is $t_1 \in (0, 1)$ so that $h(t_1) = 0$, $h(t) < 0$ for $t \in [0, t_1)$ and $h(t) > 0$ for $t \in (t_1, 1]$. Thus, $g'(t)$ is strictly decreasing on $(0, t_1]$ and strictly increasing on $[t_1, 1]$. Since $g'(0_+) = +\infty$ and $g'(1) = 0$, there exists $t_2 \in (0, t_1)$ so that $g'(t_2) = 0$, $g'(t) > 0$ for $t \in (0, t_2)$ and $g'(t) < 0$ for $t \in (t_2, 1)$. Therefore, $g(t)$ is strictly increasing on $[0, t_2]$ and strictly decreasing on $[t_2, 1]$. Since

$$g(0) = \frac{2^{2k-2} - 3^{k-1}}{2^k 3^{k-1}} > 0$$

and $g(1) = 0$, we have $g(t) \geq 0$ for $t \in [0, 1]$.

The equality holds for $a = b = c = 1$.

Remark 1. The inequality in P 1.47 is Popoviciu's inequality

$$f(a) + f(b) + f(c) + 3f\left(\frac{a+b+c}{3}\right) \geq 2f\left(\frac{a+b}{2}\right) + 2f\left(\frac{b+c}{2}\right) + 2f\left(\frac{c+a}{2}\right)$$

applied to the convex function $f(x) = x^k$ defined on $[0, \infty)$.

Remark 2. In the same manner, we can prove the following refinements (Vasile C., 2008):

- Let a, b, c be nonnegative real numbers so that $a + b + c = 3$. If $k > 2$ and $m \leq m_0$, where

$$m_0 = \frac{2^k(3^{k-1} - 2^{k-1})}{6^{k-1} + 3^{k-1} - 2^{2k-1}} > 2,$$

then

$$a^k + b^k + c^k - 3 \geq m \left[\left(\frac{a+b}{2}\right)^k + \left(\frac{b+c}{2}\right)^k + \left(\frac{c+a}{2}\right)^k - 3 \right],$$

with equality for $a = b = c = 1$. If $m = m_0$, then the equality holds also for $a = 0$ and $b = c = 3/2$ (or any cyclic permutation).

- Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$. If $k > 2$ and $m \leq m_1$, where

$$m_1 = \frac{\frac{1}{(n-1)^{k-1}} - \frac{1}{n^{k-1}}}{\frac{1}{(n-1)^k} + \frac{(n-2)^k}{(n-1)^{2k-1}} - \frac{1}{n^{k-1}}} > n - 1,$$

then

$$a_1^k + a_2^k + \dots + a_n^k - n \geq m \left[\left(\frac{n-a_1}{n-1}\right)^k + \left(\frac{n-a_2}{n-1}\right)^k + \dots + \left(\frac{n-a_n}{n-1}\right)^k - n \right],$$

with equality for $a_1 = a_2 = \dots = a_n = 1$. If $m = m_1$, then the equality holds also for $a_1 = 0$ and $a_2 = a_3 = \dots = a_n = \frac{n}{n-1}$ (or any cyclic permutation).

□

P 1.48. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$, then

$$\sqrt{a_1} + \sqrt{a_2} + \dots + \sqrt{a_n} + n(k-1) \leq k \left(\sqrt{\frac{n-a_1}{n-1}} + \sqrt{\frac{n-a_2}{n-1}} + \dots + \sqrt{\frac{n-a_n}{n-1}} \right),$$

where

$$k = (\sqrt{n} - 1)(\sqrt{n} + \sqrt{n-1}).$$

(Vasile C., 2008)

Solution. For $n = 2$, the inequality is an identity. Consider further that $n \geq 3$. We will show first that

$$n-1 < k < 2(n-1).$$

The left inequality reduces to

$$(\sqrt{n} - 1)(\sqrt{n-1} - 1) > 0,$$

while the right inequality is equivalent to

$$(\sqrt{n} - 1)(\sqrt{n} - \sqrt{n-1} + 2) > 0.$$

Write the inequality as

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \dots + a_n}{n} = 1,$$

where

$$f(u) = -\sqrt{u} + k\sqrt{\frac{n-u}{n-1}}, \quad u \in [0, n].$$

For $u \leq 1$, we have

$$\begin{aligned} 4f''(u) &= u^{-3/2} - \frac{k}{\sqrt{n-1}}(n-u)^{-3/2} \geq 1 - \frac{k}{\sqrt{n-1}}(n-1)^{-3/2} \\ &= 1 - \frac{k}{(n-1)^2} \geq 1 - \frac{k}{2(n-1)} > 0. \end{aligned}$$

Therefore, f is convex on $[0, s]$. By the LHCF-Theorem, it suffices to consider the case

$$a_1 \geq a_2 = \dots = a_n.$$

Write the original inequality in the homogeneous form

$$\sum \sqrt{a_i} + n(k-1)\sqrt{\frac{a_1 + a_2 + \dots + a_n}{n}} \leq k \sum \sqrt{\frac{a_2 + \dots + a_n}{n-1}}.$$

Do to homogeneity, we need to prove this inequality for $a_2 = \dots = a_n = 1$ and $\sqrt{a_1} = x \geq 1$; that is, to show that $g(x) \leq 0$ for $x \geq 1$, where

$$g(x) = x + n - 1 - k + (k-1)\sqrt{n(x^2 + n - 1)} - k\sqrt{(n-1)(x^2 + n - 2)}.$$

We have

$$g'(x) = 1 + (k-1)\sqrt{\frac{nx^2}{x^2 + n - 1}} - k\sqrt{\frac{(n-1)x^2}{x^2 + n - 2}},$$

$$g''(x) = \frac{k(n-2)\sqrt{n-1}}{(x^2+n-2)^{3/2}} \left[\left(m \cdot \frac{x^2+n-2}{x^2+n-1} \right)^{3/2} - 1 \right],$$

where

$$m = \sqrt[3]{\frac{(k-1)^2 n(n-1)}{k^2(n-2)^2}}.$$

Clearly, $g''(x)$ has the same sign as $h(x)$, where

$$h(x) = \frac{m(x^2+n-2)}{x^2+n-1} - 1 = m \left(1 - \frac{1}{x^2+n-1} \right) - 1.$$

We have

$$h(1) = \frac{m(n-1)}{n} - 1, \quad \lim_{x \rightarrow \infty} h(x) = m - 1.$$

We will show that $h(1) < 0$ and $\lim_{x \rightarrow \infty} h(x) > 0$; that is, to show that

$$1 < m < \frac{n}{n-1}.$$

The inequality $m > 1$ is equivalent to

$$1 - \frac{1}{k} > \frac{n-2}{\sqrt{n(n-1)}},$$

which is true since

$$1 - \frac{1}{k} > 1 - \frac{1}{n-1} = \frac{n-2}{n-1} > \frac{n-2}{\sqrt{n(n-1)}}.$$

The inequality $m < \frac{n}{n-1}$ is equivalent to

$$1 - \frac{1}{k} < \frac{n(n-2)}{(n-1)^2},$$

which is also true because

$$1 - \frac{1}{k} < 1 - \frac{1}{2(n-1)} = \frac{2n-3}{2(n-1)} \leq \frac{n(n-2)}{(n-1)^2}.$$

Since h is strictly increasing on $[1, \infty)$, $h(1) < 0$ and $\lim_{x \rightarrow \infty} h(x) > 0$, there is $x_1 \in (1, \infty)$ so that $h(x_1) = 0$, $h(x) < 0$ for $x \in [1, x_1)$ and $h(x) > 0$ for $x \in (x_1, \infty)$. Therefore, g' is strictly decreasing on $[1, x_1]$ and strictly increasing on $[x_1, \infty)$. Since $g'(1) = 0$ and $\lim_{x \rightarrow \infty} g'(x) = 0$, it follows that $g'(x) < 0$ for $x \in (1, \infty)$. Thus, $g(x)$ is strictly decreasing on $[1, \infty)$, hence $g(x) \leq g(1) = 0$.

The equality holds for $a_1 = a_2 = \dots = a_n = 1$, and also for

$$a_1 = n, \quad a_2 = a_3 = \dots = a_n = 0$$

(or any cyclic permutation).

Remark. Since $k > n-1$ for $n \geq 3$, the inequality in P 1.48 is sharper than Popoviciu's inequality applied to the convex function $f(x) = -\sqrt{x}$, $x \geq 0$:

$$\sqrt{a_1} + \sqrt{a_2} + \dots + \sqrt{a_n} + n(n-2) \leq (n-1) \left(\sqrt{\frac{n-a_1}{n-1}} + \sqrt{\frac{n-a_2}{n-1}} + \dots + \sqrt{\frac{n-a_n}{n-1}} \right).$$

□

P 1.49. If a, b, c are the lengths of the sides of a triangle so that $a + b + c = 3$, then

$$\frac{1}{a+b-c} + \frac{1}{b+c-a} + \frac{1}{c+a-b} - 3 \geq 4(2 + \sqrt{3}) \left(\frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{c+a} - 3 \right).$$

(Vasile C., 2008)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) \geq 3f(s), \quad s = \frac{a+b+c}{3} = 1,$$

where

$$f(u) = \frac{1}{3-2u} - \frac{4k}{3-u}, \quad k = 2(2 + \sqrt{3}) \approx 7.464, \quad u \in [0, 3/2).$$

For $u \geq 1$, we have

$$f''(u) = \frac{8}{(3-2u)^3} - \frac{8k}{(3-u)^3} > 8 \left[\left(\frac{1}{3-2u} \right)^3 - \left(\frac{2}{3-u} \right)^3 \right].$$

Since

$$\frac{1}{3-2u} \geq \frac{2}{3-u}, \quad u \in [1, 3/2),$$

it follows that f is convex on $[s, 3/2)$. By the RHCF-Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for $x, y \in [0, 3/2)$ so that $x + 2y = 3$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{2}{3-2u} - \frac{2k}{3-u}$$

and

$$\begin{aligned} h(x, y) &= \frac{g(x) - g(y)}{x - y} = \frac{2}{(3-2x)(3-2y)} - \frac{k}{(3-x)(3-y)} \\ &= \frac{2}{(2y-x)x} - \frac{k}{2y(x+y)} \\ &= \frac{kx^2 - 2(k-2)xy + 4y^2}{2xy(x+y)(2y-x)} \\ &= \frac{[(\sqrt{3}+1)x - 2y]^2}{2xy(x+y)(2y-x)} \geq 0. \end{aligned}$$

According to Note 4, the equality holds for $a = b = c = 1$, and also for

$$a = 3(2 - \sqrt{3}), \quad b = c = \frac{3(\sqrt{3} - 1)}{2}$$

(or any cyclic permutation).

□

P 1.50. Let $a_1, a_2, \dots, a_5$ be nonnegative numbers so that $a_1 + a_2 + a_3 + a_4 + a_5 \leq 5$. If

$$k \geq k_0, \quad k_0 = \frac{29 + \sqrt{761}}{10} \approx 5.66,$$

then

$$\sum \frac{1}{ka_1^2 + a_2 + a_3 + a_4 + a_5} \geq \frac{5}{k+4}.$$

(Vasile C., 2006)

Solution. Since each term of the left hand side of the inequality decreases by increasing any number a_i , it suffices to consider the case

$$a_1 + a_2 + a_3 + a_4 + a_5 = 5,$$

when the desired inequality can be written as

$$f(a_1) + f(a_2) + f(a_3) + f(a_4) + f(a_5) \geq 5f(s),$$

where

$$s = \frac{a_1 + a_2 + a_3 + a_4 + a_5}{5} = 1$$

and

$$f(u) = \frac{1}{ku^2 - u + 5}, \quad u \in [0, 5].$$

For $u \geq 1$, we have

$$\begin{aligned} f''(u) &= \frac{2[3ku(ku-1) - 5k+1]}{(ku^2 - u + 5)^3} \\ &\geq \frac{2[3k(k-1) - 5k+1]}{(ku^2 - u + 5)^3} \\ &= \frac{2[k(3k-8) + 1]}{(ku^2 - u + 5)^3} > 0; \end{aligned}$$

therefore, f is convex on $[s, 5]$. By the RHCF-Theorem, it suffices to show that

$$\frac{1}{kx^2 - x + 5} + \frac{4}{ky^2 - y + 5} \geq \frac{5}{k+4}$$

for

$$0 \leq x \leq 1 \leq y, \quad x + 4y = 5.$$

Write this inequality as follows:

$$\begin{aligned} \frac{1}{kx^2 - x + 5} - \frac{1}{k+4} + 4 \left[\frac{1}{ky^2 - y + 5} - \frac{1}{k+4} \right] &\geq 0, \\ \frac{(x-1)(1-k-kx)}{kx^2 - x + 5} + \frac{4(y-1)(1-k-ky)}{ky^2 - y + 5} &\geq 0. \end{aligned}$$

Since

$$4(y-1) = 1-x,$$

the inequality is equivalent to

$$(x-1) \left(\frac{1-k-kx}{kx^2-x+5} - \frac{1-k-ky}{ky^2-y+5} \right) \geq 0,$$

$$\frac{5(x-1)^2 g(x, y, k)}{4(kx^2-x+5)(ky^2-y+5)} \geq 0,$$

where

$$g(x, y, k) = k^2 xy + k(k-1)(x+y) - 6k + 1.$$

For fixed x and y , let $h(k) = g(x, y, k)$. Since

$$\begin{aligned} h'(k) &= 2kxy + (2k-1)(x+y) - 6 \geq (2k-1)(x+y) - 6 \\ &\geq (2k-1) \left(x + \frac{y}{4} \right) - 6 = \frac{10k-29}{4} > 0, \end{aligned}$$

it suffices to show that $g(x, y, k_0) \geq 0$. We have

$$\begin{aligned} g(x, y, k_0) &= k_0^2 xy + k_0(k_0-1)(x+y) - 6k_0 + 1 \\ &= -4k_0^2 y^2 + k_0(2k_0+3)y + 5k_0^2 - 11k_0 + 1. \end{aligned}$$

Since

$$5k_0^2 - 29k_0 + 4 = 0,$$

we get

$$g(x, y, k_0) = (5-4y) \left(k_0^2 y + k_0^2 - \frac{11k_0-1}{5} \right) = x \left(k_0^2 y + k_0^2 - \frac{11k_0-1}{5} \right).$$

It suffices to show that

$$k_0^2 - \frac{11k_0-1}{5} \geq 0.$$

Indeed,

$$k_0^2 - \frac{11k_0-1}{5} = \frac{k_0(5k_0-11)+1}{5} > 0.$$

The equality holds for $a_1 = a_2 = a_3 = a_4 = a_5 = 1$. If $k = k_0$, then the equality holds also for

$$a_1 = 0, \quad a_2 = a_3 = a_4 = a_5 = \frac{5}{4}$$

(or any cyclic permutation).

Remark. In the same manner, we can prove the following statement:

- Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n \leq n$. If

$$k \geq k_0, \quad k_0 = \frac{n^2 + n - 1 + \sqrt{n^4 + 2n^3 - 5n^2 + 2n + 1}}{2n},$$

then

$$\sum \frac{1}{ka_1^2 + a_2 + \dots + a_n} \geq \frac{n}{k+n-1},$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$. If $k = k_0$, then the equality holds also for

$$a_1 = 0, \quad a_2 = \cdots = a_n = \frac{n}{n-1}$$

(or any cyclic permutation).

□

P 1.51. Let $a_1, a_2, \dots, a_5$ be nonnegative numbers so that $a_1 + a_2 + a_3 + a_4 + a_5 \leq 5$. If

$$0 < k \leq k_0, \quad k_0 = \frac{11 - \sqrt{101}}{10} \approx 0.095,$$

then

$$\sum \frac{1}{ka_1^2 + a_2 + a_3 + a_4 + a_5} \geq \frac{5}{k+4}.$$

(Vasile C., 2006)

Solution. As shown at the preceding P 1.50, it suffices to consider the case

$$a_1 + a_2 + a_3 + a_4 + a_5 = 5,$$

when the desired inequality can be written as

$$f(a_1) + f(a_2) + f(a_3) + f(a_4) + f(a_5) \geq 5f(s),$$

where

$$s = \frac{a_1 + a_2 + a_3 + a_4 + a_5}{5} = 1,$$

and

$$f(u) = \frac{1}{ku^2 - u + 5}, \quad u \in [0, 5].$$

For $u \in [0, 1]$, we have

$$u(ku - 1) - (k - 1) = (1 - u)(1 - ku) \geq 0,$$

hence

$$\begin{aligned} f''(u) &= \frac{2[3ku(ku - 1) - 5k + 1]}{(ku^2 - u + 5)^3} \\ &\geq \frac{2[3k(k - 1) - 5k + 1]}{(ku^2 - u + 5)^3} \\ &= \frac{2[(1 - 8k) + 3k^2]}{(ku^2 - u + 5)^3} > 0; \end{aligned}$$

therefore, f is convex on $[0, s]$. By the LHCF-Theorem, it suffices to show that

$$\frac{1}{kx^2 - x + 5} + \frac{4}{ky^2 - y + 5} \geq \frac{5}{k+4}$$

for

$$x \geq 1 \geq y \geq 0, \quad x + 4y = 5.$$

Write this inequality as follows:

$$\begin{aligned} \frac{1}{kx^2 - x + 5} - \frac{1}{k + 4} + 4 \left[\frac{1}{ky^2 - y + 5} - \frac{1}{k + 4} \right] &\geq 0, \\ \frac{(x-1)(1-k-kx)}{kx^2 - x + 5} + \frac{4(y-1)(1-k-ky)}{ky^2 - y + 5} &\geq 0. \end{aligned}$$

Since

$$4(y-1) = 1-x,$$

the inequality is equivalent to

$$\begin{aligned} (x-1) \left(\frac{1-k-kx}{kx^2 - x + 5} - \frac{1-k-ky}{ky^2 - y + 5} \right) &\geq 0, \\ \frac{5(x-1)^2 g(x, y, k)}{4(kx^2 - x + 5)(ky^2 - y + 5)} &\geq 0, \end{aligned}$$

where

$$g(x, y, k) = k^2 xy - k(1-k)(x+y) - 6k + 1.$$

For fixed x and y , let $h(k) = g(x, y, k)$. Since

$$\begin{aligned} h'(k) &= 2kxy - (1-2k)(x+y) - 6 \leq 2kxy - 6 \\ &\leq \frac{k(x+4y)^2}{8} - 6 = \frac{25k}{8} - 6 < 0, \end{aligned}$$

it suffices to show that $g(x, y, k_0) \geq 0$. We have

$$\begin{aligned} g(x, y, k_0) &= k_0^2 xy + k_0(k_0 - 1)(x+y) - 6k_0 + 1 \\ &= -4k_0^2 y^2 + k_0(2k_0 + 3)y + 5k_0^2 - 11k_0 + 1. \end{aligned}$$

Since

$$5k_0^2 - 11k_0 + 1 = 0,$$

we get

$$g(x, y, k_0) = k_0 y(-4k_0 y + 2k_0 + 3) \geq k_0 y(-4k_0 + 2k_0 + 3) = k_0(3 - 2k_0)y \geq 0.$$

The equality holds for $a_1 = a_2 = a_3 = a_4 = a_5 = 1$. If $k = k_0$, then the equality holds also for

$$a_1 = 5, \quad a_2 = a_3 = a_4 = a_5 = 0$$

(or any cyclic permutation).

Remark. Similarly, we can prove the following statement:

- Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n \leq n$. If

$$0 \leq k \leq k_0, \quad k_0 = \frac{2n+1 - \sqrt{4n^2+1}}{2n},$$

then

$$\sum \frac{1}{ka_1^2 + a_2 + \cdots + a_n} \geq \frac{n}{k + n - 1},$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$. If $k = k_0$, then the equality holds also for

$$a_1 = n, \quad a_2 = \cdots = a_n = 0$$

(or any cyclic permutation).

□

P 1.52. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n \leq n$. If

$$0 < k \leq \frac{1}{n+1},$$

then

$$\frac{a_1}{ka_1^2 + a_2 + \cdots + a_n} + \frac{a_2}{a_1 + ka_2^2 + \cdots + a_n} + \cdots + \frac{a_n}{a_1 + a_2 + \cdots + ka_n^2} \geq \frac{n}{k + n - 1}.$$

(Vasile C., 2006)

Solution. Using the notation

$$x_1 = \frac{a_1}{s}, \quad x_2 = \frac{a_2}{s}, \quad \dots, \quad x_n = \frac{a_n}{s},$$

where

$$s = \frac{a_1 + a_2 + \cdots + a_n}{n} \leq 1,$$

we need to show that $x_1 + x_2 + \cdots + x_n = n$ involves

$$\frac{x_1}{ksx_1^2 + x_2 + \cdots + x_n} + \cdots + \frac{x_n}{x_1 + x_2 + \cdots + ksx_n^2} \geq \frac{n}{k + n - 1}.$$

Since $s \leq 1$, it suffices to prove the inequality for $s = 1$; that is, to show that

$$\frac{a_1}{ka_1^2 - a_1 + n} + \frac{a_2}{ka_2^2 - a_2 + n} + \cdots + \frac{a_n}{ka_n^2 - a_n + n} \geq \frac{n}{k + n - 1}$$

for

$$a_1 + a_2 + \cdots + a_n = n.$$

Write the desired inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s),$$

where

$$s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1$$

and

$$f(u) = \frac{u}{u^2 - u + n}, \quad u \in [0, n].$$

We have

$$f'(u) = \frac{n - ku^2}{(ku^2 - u + n)^2}, \quad f''(u) = \frac{f_1(u)}{(u^2 - u + n)^3},$$

where

$$f_1(u) = k^2u^3 - 3knu + n.$$

For $u \in [0, 1]$, we have

$$\begin{aligned} f_1(u) &\geq -3knu + n \geq -3kn + n \\ &\geq -\frac{3n}{n+1} + n = \frac{n(n-2)}{n+1} \geq 0. \end{aligned}$$

Since $f''(u) > 0$, it follows that f is convex on $[0, s]$. By the LHCF-Theorem, we only need to show that

$$\frac{x}{kx^2 - x + n} + \frac{(n-1)y}{ky^2 - y + n} \geq \frac{n}{k + n - 1}$$

for all nonnegative x, y which satisfy $x + (n-1)y = n$. Write this inequality as follows:

$$\frac{x}{kx^2 - x + n} - \frac{1}{k + n - 1} + (n-1) \left[\frac{y}{ky^2 - y + n} - \frac{1}{k + n - 1} \right] \geq 0,$$

$$(x-1) \left(\frac{n-kx}{kx^2 - x + n} - \frac{n-ky}{ky^2 - y + n} \right) \geq 0,$$

$$\frac{(x-1)^2 h(x, y)}{(kx^2 - x + n)(ky^2 - y + n)} \geq 0,$$

where

$$h(x, y) = k^2xy - kn(x + y) + n - nk.$$

We need to show that $h(x, y) \geq 0$. Indeed,

$$\begin{aligned} h(x, y) &= ky[n(k+n-2) - k(n-1)y] + n[1 - k(n+1)] \\ &= ky[n(n-2) + kx] + n[1 - k(n+1)] \geq 0. \end{aligned}$$

The equality holds for $a_1 = a_2 = \dots = a_n = 1$. If $k = \frac{1}{n+1}$, then the equality holds also for

$$a_1 = n, \quad a_2 = a_3 = \dots = a_n = 0$$

(or any cyclic permutation).

□

P 1.53. If $a_1, a_2, a_3, a_4, a_5 \leq \frac{7}{2}$ so that $a_1 + a_2 + a_3 + a_4 + a_5 = 5$, then

$$\frac{a_1}{a_1^2 - a_1 + 5} + \frac{a_2}{a_2^2 - a_2 + 5} + \frac{a_3}{a_3^2 - a_3 + 5} + \frac{a_4}{a_4^2 - a_4 + 5} + \frac{a_5}{a_5^2 - a_5 + 5} \leq 1.$$

(Vasile C., 2006)

Solution. Write the desired inequality as

$$f(a_1) + f(a_2) + f(a_3) + f(a_4) + f(a_5) \geq 5f(s),$$

where

$$s = \frac{a_1 + a_2 + a_3 + a_4 + a_5}{5} = 1$$

and

$$f(u) = \frac{-u}{u^2 - u + 5}, \quad u \leq \frac{7}{2}.$$

For $u \in \left[1, \frac{7}{2}\right]$, we have

$$\begin{aligned} f''(u) &= \frac{-u^3 + 15u - 5}{(u^2 - u + 5)^3} \\ &= \frac{(2u + 9)(u - 1)(7 - 2u) + 43 - 7u}{4(u^2 - u + 5)^3} > 0. \end{aligned}$$

Thus, f is convex on $\left[s, \frac{7}{2}\right]$. By the RHCF-Theorem, it suffices to show that

$$\frac{x}{x^2 - x + 5} + \frac{4y}{y^2 - y + 5} \leq 1$$

for all nonnegative $x, y \leq \frac{7}{2}$ which satisfy $x + 4y = 5$. Write this inequality as follows:

$$\frac{x}{x^2 - x + 5} - \frac{1}{5} + 4 \left(\frac{y}{y^2 - y + 5} - \frac{1}{5} \right) \leq 0,$$

$$(x - 1) \left(\frac{5 - x}{x^2 - x + 5} - \frac{5 - y}{y^2 - y + 5} \right) \leq 0,$$

$$\frac{(x - 1)^2 [5(x + y) - xy]}{(x^2 - x + 5)(y^2 - y + 5)} \geq 0,$$

$$\frac{(x - 1)^2 [(x + 4y)(x + y) - xy]}{(x^2 - x + 5)(y^2 - y + 5)} \geq 0,$$

$$\frac{(x - 1)^2 (x + 2y)^2}{(x^2 - x + 5)(y^2 - y + 5)} \geq 0.$$

The equality holds for $a_1 = a_2 = a_3 = a_4 = a_5 = 1$, and also for

$$a_1 = -5, \quad a_2 = a_3 = a_4 = a_5 = \frac{5}{2}$$

(or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization:

- Let $a_1, a_2, \dots, a_n \leq \sqrt{3}$ so that $a_1 + a_2 + \dots + a_n \leq n$. If

$$k = \frac{n^2 + 2n - 2 - 2\sqrt{(n-1)(2n^2-1)}}{n},$$

then

$$\frac{a_1}{ka_1^2 - a_1 + n} + \frac{a_2}{ka_2^2 - a_2 + n} + \dots + \frac{a_n}{ka_n^2 - a_n + n} \leq \frac{n}{k-1+n},$$

with equality for $a_1 = a_2 = \dots = a_n = 1$, and also for

$$a_1 = \frac{n(k-n+2)}{2k}, \quad a_2 = \dots = a_n = \frac{n(k+n-2)}{2k(n-1)}$$

(or any cyclic permutation).

□

P 1.54. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n \geq n$. If

$$0 < k \leq \frac{1}{1 + \frac{1}{4(n-1)^2}},$$

then

$$\frac{a_1^2}{ka_1^2 + a_2 + \dots + a_n} + \frac{a_2^2}{a_1 + ka_2^2 + \dots + a_n} + \dots + \frac{a_n^2}{a_1 + a_2 + \dots + ka_n^2} \geq \frac{n}{k+n-1}.$$

(Vasile C., 2006)

Solution. Using the substitution

$$x_1 = \frac{a_1}{s}, \quad x_2 = \frac{a_2}{s}, \quad \dots, \quad x_n = \frac{a_n}{s},$$

where

$$s = \frac{a_1 + a_2 + \dots + a_n}{n} \geq 1,$$

we need to show that $x_1 + x_2 + \dots + x_n = n$ involves

$$\frac{x_1^2}{kx_1^2 + (x_2 + \dots + x_n)/s} + \dots + \frac{x_n^2}{(x_1 + \dots + x_{n-1})/s + kx_n^2} \geq \frac{n}{k+n-1}.$$

Since $s \geq 1$, it suffices to prove the inequality for $s = 1$; that is, to show that

$$\frac{a_1^2}{ka_1^2 - a_1 + n} + \frac{a_2^2}{ka_2^2 - a_2 + n} + \dots + \frac{a_n^2}{ka_n^2 - a_n + n} \geq \frac{n}{k+n-1}$$

for

$$a_1 + a_2 + \dots + a_n = n.$$

Write the desired inequality as

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq nf(s),$$

where

$$s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1$$

and

$$f(u) = \frac{u^2}{u^2 - u + n}, \quad u \in [0, n].$$

We have

$$f'(u) = \frac{u(2n - u)}{(ku^2 - u + n)^2}, \quad f''(u) = \frac{2f_1(u)}{(u^2 - u + n)^3},$$

where

$$f_1(u) = ku^3 - 3knu^2 + n^2.$$

For $u \in [0, 1]$ and $n \geq 3$, we have

$$f_1(u) \geq -3knu^2 + n^2 \geq -3kn + n^2 > -3n + n^2 \geq 0.$$

Also, for $u \in [0, 1]$ and $n = 2$, we have

$$\begin{aligned} f_1(u) &= 4 - ku^2(6 - u) \geq 4 - \frac{4}{5}u^2(6 - u) \\ &\geq 4 - \frac{4}{5}u(6 - u) = \frac{4(1 - u)(5 - u)}{5} \geq 0. \end{aligned}$$

Since $f''(u) \geq 0$ for $u \in [0, 1]$, it follows that f is convex on $[0, s]$. By the LHCF-Theorem, we need to show that

$$\frac{x^2}{kx^2 - x + n} + \frac{(n - 1)y^2}{ky^2 - y + n} \geq \frac{n}{k + n - 1}$$

for all nonnegative x, y which satisfy $x + (n - 1)y = n$. Write this inequality as follows:

$$\begin{aligned} \frac{x^2}{kx^2 - x + n} - \frac{1}{k + n - 1} + (n - 1) \left[\frac{y^2}{ky^2 - y + n} - \frac{1}{k + n - 1} \right] &\geq 0, \\ \frac{(x - 1)(nx - x + n)}{kx^2 - x + 5} + \frac{4(y - 1)(ny - y + n)}{ky^2 - y + 5} &\geq 0, \\ (x - 1) \left(\frac{nx - x + n}{kx^2 - x + n} - \frac{ny - y + n}{ky^2 - y + n} \right) &\geq 0, \\ \frac{(x - 1)^2 h(x, y)}{(kx^2 - x + n)(ky^2 - y + n)} &\geq 0, \end{aligned}$$

where

$$h(x, y) = n^2 - kn(x + y) - k(n - 1)xy.$$

Since

$$0 < k \leq k_0, \quad k_0 = \frac{1}{1 + \frac{1}{4(n-1)^2}},$$

we have

$$\begin{aligned} h(x, y) &\geq n^2 - k_0n(x + y) - k_0(n - 1)xy \\ &= (n - 1)^2 k_0 y^2 - n k_0 y + n^2 (1 - k_0) \\ &= k_0 \left[(n - 1)y - \frac{n}{2(n - 1)} \right]^2 \geq 0. \end{aligned}$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $k = k_0$, then the equality holds also for

$$a_1 = \frac{n(2n-3)}{2(n-1)}, \quad a_2 = a_3 = \cdots = a_n = \frac{n}{2(n-1)^2}$$

(or any cyclic permutation).

□

P 1.55. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n \leq n$. If $k \geq n-1$, then

$$\frac{a_1^2}{ka_1^2 + a_2 + \cdots + a_n} + \frac{a_2^2}{a_1 + ka_2^2 + \cdots + a_n} + \cdots + \frac{a_n^2}{a_1 + a_2 + \cdots + ka_n^2} \leq \frac{n}{k+n-1}.$$

(Vasile C., 2006)

Solution. Using the notation

$$x_1 = \frac{a_1}{s}, \quad x_2 = \frac{a_2}{s}, \quad \dots, \quad x_n = \frac{a_n}{s},$$

where

$$s = \frac{a_1 + a_2 + \cdots + a_n}{n} \leq 1,$$

we need to show that $x_1 + x_2 + \cdots + x_n = n$ involves

$$\frac{x_1^2}{kx_1^2 + (x_2 + \cdots + x_n)/s} + \cdots + \frac{x_n^2}{(x_1 + \cdots + x_{n-1})/s + kx_n^2} \leq \frac{n}{k+n-1}.$$

Since $s \leq 1$, it suffices to prove the inequality for $s = 1$; that is, to show that

$$\frac{a_1^2}{ka_1^2 - a_1 + n} + \frac{a_2^2}{ka_2^2 - a_2 + n} + \cdots + \frac{a_n^2}{ka_n^2 - a_n + n} \leq \frac{n}{k+n-1}$$

for

$$a_1 + a_2 + \cdots + a_n = n.$$

Write the desired inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{-u^2}{u^2 - u + n}, \quad u \in [0, n].$$

We have

$$f'(u) = \frac{u(u-2n)}{(ku^2 - u + n)^2}, \quad f''(u) = \frac{2f_1(u)}{(u^2 - u + n)^3},$$

where

$$f_1(u) = -ku^3 + 3knu^2 - n^2.$$

For $u \in [1, n]$, we have

$$\begin{aligned} f_1(u) &\geq -knu^2 + 3knu^2 - n^2 = 2knu^2 - n^2 \\ &\geq 2kn - n^2 \geq 2(n-1)n - n^2 = n(n-2) \geq 0. \end{aligned}$$

Since $f''(u) \geq 0$ for $u \in [1, n]$, it follows that f is convex on $[s, n]$. By the RHCF-Theorem, it suffices to show that

$$\frac{x^2}{kx^2 - x + n} + \frac{(n-1)y^2}{ky^2 - y + n} \leq \frac{n}{k + n - 1}$$

for all nonnegative x, y which satisfy $x + (n-1)y = n$. As shown in the proof of the preceding P 1.54, we only need to show that $h(x, y) \geq 0$, where

$$h(x, y) = kn(x + y) + k(n-1)xy - n^2.$$

Since $k \geq n-1$, we have

$$\begin{aligned} h(x, y) &\geq n(n-1)(x + y) + (n-1)^2xy - n^2 \\ &= -(n-1)^3y^2 + n(n-1)y + n^2(n-2) \\ &= [n - (n-1)y][n(n-2) + (n-1)^2y] \\ &= x[n(n-2) + (n-1)^2y] \geq 0. \end{aligned}$$

The equality holds for $a_1 = a_2 = \dots = a_n = 1$. If $k = n-1$, then the equality holds also for

$$a_1 = 0, \quad a_2 = a_3 = \dots = a_n = \frac{n}{n-1}$$

(or any cyclic permutation).

□

P 1.56. Let $a_1, a_2, \dots, a_n \in [0, n]$ so that $a_1 + a_2 + \dots + a_n \geq n$. If $0 < k \leq \frac{1}{n}$, then

$$\frac{a_1 - 1}{ka_1^2 + a_2 + \dots + a_n} + \frac{a_2 - 1}{a_1 + ka_2^2 + \dots + a_n} + \dots + \frac{a_n - 1}{a_1 + a_2 + \dots + ka_n^2} \geq 0.$$

(Vasile C., 2006)

Solution. Let

$$s = \frac{a_1 + a_2 + \dots + a_n}{n}, \quad s \geq 1.$$

Case 1: $s > 1$ Without loss of generality, assume that

$$a_1 \geq \dots \geq a_j > 1 \geq a_{j+1} \dots \geq a_n, \quad j \in \{1, 2, \dots, n\}.$$

Clearly, there are $b_1, b_2, \dots, b_n$ so that $b_1 + b_2 + \dots + b_n = n$ and

$$a_1 \geq b_1 \geq 1, \dots, a_j \geq b_j \geq 1, b_{j+1} = a_{j+1}, \dots, b_n = a_n.$$

Write the desired inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq 0,$$

where

$$f(u) = \frac{u-1}{ku^2 - u + ns}, \quad u \in [0, n],$$

$$f'(u) = \frac{f_1(u)}{(ku^2 - u + ns)^2}, \quad f_1(u) = k(-u^2 + 2u) + ns - 1.$$

For $u \in [1, n)$, we have

$$\begin{aligned} f_1(u) &\geq k(-nu + 2u) + ns - 1 = -k(n-2)u + ns - 1 \\ &\geq -k(n-2)n + ns - 1 \geq -(n-2) + ns - 1 = n(s-1) + 1 > 0. \end{aligned}$$

Consequently, f is strictly increasing on $[1, n]$ and

$$f(b_1) \leq f(a_1), \dots, f(b_j) \leq f(a_j), f(b_{j+1}) = f(a_{j+1}), \dots, f(b_n) = f(a_n).$$

Since

$$f(b_1) + f(b_2) + \cdots + f(b_n) \leq f(a_1) + f(a_2) + \cdots + f(a_n),$$

it suffices to show that $f(b_1) + f(b_2) + \cdots + f(b_n) \geq 0$ for $b_1 + b_2 + \cdots + b_n = n$. This inequality is proved at Case 2.

Case 2: $s = 1$. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{u-1}{ku^2 - u + n}, \quad u \in [0, n],$$

$$f''(u) = \frac{2g(u)}{(ku^2 - u + n)^3}, \quad g(u) = k^2u^3 - 3k^2u^2 - 3k(n-1)u + kn + n - 1.$$

We will show that $f''(u) \geq 0$ for $u \in [0, 1]$. From

$$g'(u) = 3k^2u(u-2) - 3k(n-1),$$

it follows that $g'(u) < 0$, g is decreasing, hence

$$\begin{aligned} g(u) &\geq g(1) = -2k^2 - (2n-3)k + n - 1 \\ &\geq \frac{-2}{n^2} - \frac{2n-3}{n} + n - 1 \\ &= \frac{(n-1)^3 - 1}{n^2} \geq 0. \end{aligned}$$

Thus, f is convex on $[0, s]$. By the LHCF-Theorem, it suffices to show that

$$\frac{x-1}{kx^2 - x + n} + \frac{(n-1)(y-1)}{ky^2 - y + n} \geq 0$$

for all nonnegative real x, y so that $x + (n-1)y = n$. Since $(n-1)(y-1) = 1-x$, we have

$$\begin{aligned}
 \frac{x-1}{kx^2-x+n} + \frac{(n-1)(y-1)}{ky^2-y+n} &= (x-1) \left(\frac{1}{kx^2-x+n} - \frac{1}{ky^2-y+n} \right) \\
 &= \frac{(x-1)(x-y)(1-kx-ky)}{(kx^2-x+n)(ky^2-y+n)} \\
 &= \frac{n(x-1)^2(1-kx-ky)}{(n-1)(kx^2-x+n)(ky^2-y+n)} \\
 &\geq \frac{n(x-1)^2(1-\frac{x+y}{n})}{(n-1)(kx^2-x+n)(ky^2-y+n)} \\
 &= \frac{(n-2)y(x-1)^2}{(n-1)(kx^2-x+n)(ky^2-y+n)} \geq 0.
 \end{aligned}$$

The proof is completed. The equality holds for $a_1 = a_2 = \dots = a_n = 1$. If $k = \frac{1}{n}$, then the equality holds also for

$$a_1 = n, \quad a_2 = a_3 = \dots = a_n = 0.$$

□

P 1.57. If a, b, c are positive real numbers so that $abc = 1$, then

$$\sqrt{a^2 - a + 1} + \sqrt{b^2 - b + 1} + \sqrt{c^2 - c + 1} \geq a + b + c.$$

Solution. Using the substitution

$$a = e^x, \quad b = e^y, \quad c = e^z,$$

we need to show that

$$f(x) + f(y) + f(z) \geq 3f(s), \quad s = \frac{x+y+z}{3} = 0,$$

where

$$f(u) = \sqrt{e^{2u} - e^u + 1} - e^u, \quad u \in \mathbb{I} = \mathbb{R}.$$

We claim that f is convex on $\mathbb{I}_{\geq s}$. Since

$$e^{-u} f''(u) = \frac{4e^{3u} - 6e^{2u} + 9e^u - 2}{4(e^{2u} - e^u + 1)^{3/2}} - 1,$$

we need to show that $4x^3 - 6x^2 + 9x - 2 > 0$ and

$$(4x^3 - 6x^2 + 9x - 2)^2 \geq 16(x^2 - x + 1)^3,$$

where $x = e^u \geq 1$. Indeed,

$$4x^3 - 6x^2 + 9x - 2 = x(x-3)^2 + (3x^3 - 2) > 0$$

and

$$(4x^3 - 6x^2 + 9x - 2)^2 - 16(x^2 - x + 1)^3 = 12x^3(x - 1) + 9x^2 + 12(x - 1) > 0.$$

By the RHCF-Theorem, it suffices to prove the original inequality for

$$b = c := t, \quad a = 1/t^2, \quad t > 0,$$

that is,

$$\begin{aligned} \frac{\sqrt{t^4 - t^2 + 1}}{t^2} + 2\sqrt{t^2 - t + 1} &\geq \frac{1}{t^2} + 2t, \\ \frac{t^2 - 1}{\sqrt{t^4 - t^2 + 1} + 1} + \frac{2(1 - t)}{\sqrt{t^2 - t + 1} + t} &\geq 0. \end{aligned}$$

Since

$$\frac{t^2 - 1}{\sqrt{t^4 - t^2 + 1}} \geq \frac{t^2 - 1}{t^2 + 1},$$

it suffices to show that

$$\frac{t^2 - 1}{t^2 + 1} + \frac{2(1 - t)}{\sqrt{t^2 - t + 1} + t} \geq 0,$$

which is equivalent to

$$\begin{aligned} (t - 1) \left[\frac{t + 1}{t^2 + 1} - \frac{2}{\sqrt{t^2 - t + 1} + t} \right] &\geq 0, \\ (t - 1) \left[(t + 1)\sqrt{t^2 - t + 1} - t^2 + t - 2 \right] &\geq 0, \\ \frac{(t - 1)^2(3t^2 - 2t + 3)}{(t + 1)\sqrt{t^2 - t + 1} + t^2 - t + 2} &\geq 0. \end{aligned}$$

The equality holds for $a = b = c = 1$.

□

P 1.58. If $a, b, c, d \geq \frac{1}{1 + \sqrt{6}}$ so that $abcd = 1$, then

$$\frac{1}{a + 2} + \frac{1}{b + 2} + \frac{1}{c + 2} + \frac{1}{d + 2} \leq \frac{4}{3}.$$

(Vasile C., 2005)

Solution. Using the notation

$$a = e^x, \quad b = e^y, \quad c = e^z, \quad d = e^w,$$

we need to show that

$$f(x) + f(y) + f(z) + f(w) \geq 4f(s), \quad s = \frac{x + y + z + w}{4} = 0,$$

where

$$f(u) = \frac{-1}{e^u + 2}, \quad u \in \mathbb{I} = \mathbb{R}.$$

For $u \leq 0$, we have

$$f''(u) = \frac{e^u(2 - e^u)}{(e^u + 2)^3} > 0,$$

hence f is convex on $\mathbb{I}_{\leq s}$. By the LHCF-Theorem, it suffices to prove the original inequality for

$$b = c = d := t, \quad a = 1/t^3, \quad t \geq \frac{1}{1 + \sqrt{6}};$$

that is,

$$\frac{t^3}{2t^3 + 1} + \frac{3}{t + 2} \leq \frac{4}{3},$$

which is equivalent to the obvious inequality

$$(t - 1)^2(5t^2 + 2t - 1) \geq 0.$$

According to Note 4, the equality holds for $a = b = c = d = 1$, and also for

$$a = 19 + 9\sqrt{6}, \quad b = c = d = \frac{1}{1 + \sqrt{6}}$$

(or any cyclic permutation).

□

P 1.59. If a, b, c are positive real numbers so that $abc = 1$, then

$$a^2 + b^2 + c^2 - 3 \geq 2(ab + bc + ca - a - b - c).$$

Solution. Using the substitution

$$a = e^x, \quad b = e^y, \quad c = e^z,$$

we need to show that

$$f(x) + f(y) + f(z) \geq 3f(s), \quad s = \frac{x + y + z}{3} = 0,$$

where

$$f(u) = e^{2u} - 1 + 2(e^u - e^{-u}), \quad u \in \mathbb{R} = \mathbb{R}.$$

For $u \geq 0$, we have

$$f''(u) = 4e^{2u} + 2(e^u - e^{-u}) > 0,$$

hence f is convex on $\mathbb{I}_{\geq s}$. By the RHCF-Theorem, it suffices to prove the original inequality for $b = c := t$ and $a = 1/t^2$, where $t > 0$; that is, to show that

$$4t^5 - 3t^4 - 4t^3 + 2t^2 + 1 \geq 0,$$

which is equivalent to

$$(t - 1)^2(4t^3 + 5t^2 + 2t + 1) \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 1.60. If a, b, c are positive real numbers so that $abc = 1$, then

$$a^2 + b^2 + c^2 - 3 \geq 18(a + b + c - ab - bc - ca).$$

Solution. Using the substitution

$$a = e^x, \quad b = e^y, \quad c = e^z,$$

we need to show that

$$f(x) + f(y) + f(z) \geq 3f(s), \quad s = \frac{x + y + z}{3} = 0,$$

where

$$f(u) = e^{2u} - 1 - 18(e^u - e^{-u}), \quad u \in \mathbb{R}.$$

For $u \leq 0$, we have

$$f''(u) = 4e^{2u} + 18(e^{-u} - e^u) > 0,$$

hence f is convex on $\mathbb{I}_{\leq s}$. By the LHCF-Theorem, it suffices to prove the original inequality for $b = c := t$ and $a = 1/t^2$, where $t > 0$. Since

$$a^2 + b^2 + c^2 - 3 = \frac{1}{t^4} + 2t^2 - 3 = \frac{(t^2 - 1)^2(2t^2 + 1)}{t^4}$$

and

$$a + b + c - ab - bc - ca = \frac{-(t^4 - 2t^3 + 2t - 1)}{t^2} = \frac{-(t - 1)^3(t + 1)}{t^2},$$

we get

$$a^2 + b^2 + c^2 - 3 - 18(a + b + c - ab - bc - ca) = \frac{(t - 1)^2(2t - 1)^2(t + 1)(5t + 1)}{t^4} \geq 0.$$

The equality holds for $a = b = c = 1$, and also for $a = 4$ and $b = c = 1/2$ (or any cyclic permutation). □

P 1.61. If $a_1, a_2, \dots, a_n$ are positive real numbers so that $a_1 a_2 \cdots a_n = 1$, then

$$a_1^2 + a_2^2 + \cdots + a_n^2 - n \geq 6\sqrt{3} \left(a_1 + a_2 + \cdots + a_n - \frac{1}{a_1} - \frac{1}{a_2} - \cdots - \frac{1}{a_n} \right).$$

Solution. Using the notation $a_i = e^{x_i}$ for $i = 1, 2, \dots, n$, we need to show that

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s), \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

where

$$f(u) = e^{2u} - 1 - 6\sqrt{3} (e^u - e^{-u}), \quad u \in \mathbb{I} = \mathbb{R}.$$

For $u \leq 0$, we have

$$f''(u) = 4e^{2u} + 6\sqrt{3}(e^{-u} - e^u) > 0,$$

hence f is convex on $\mathbb{I}_{\leq s}$. By the LHCF-Theorem and Note 2, it suffices to show that $H(x, y) \geq 0$ for $x, y \in \mathbb{R}$ so that $x + (n-1)y = 0$, where

$$H(x, y) = \frac{f'(x) - f'(y)}{x - y}.$$

From

$$f'(u) = 2e^{2u} - 6\sqrt{3}(e^u + e^{-u}),$$

we get

$$H(x, y) = \frac{2(e^x - e^y)}{x - y} \left(e^x + e^y - 3\sqrt{3} + 3\sqrt{3} e^{-x-y} \right).$$

Since $(e^x - e^y)/(x - y) > 0$, we need to prove that

$$e^x + e^y + 3\sqrt{3} e^{-x-y} \geq 3\sqrt{3}.$$

Indeed, by the AM-GM inequality, we have

$$e^x + e^y + 3\sqrt{3} e^{-x-y} \geq 3\sqrt[3]{e^x \cdot e^y \cdot 3\sqrt{3} e^{-x-y}} = 3\sqrt{3}.$$

The proof is completed. The equality holds for $a_1 = a_2 = \dots = a_n = 1$.

□

P 1.62. If $a_1, a_2, \dots, a_n$ ($n \geq 4$) are positive real numbers so that $a_1 a_2 \dots a_n = 1$, then

$$(n-1)(a_1^2 + a_2^2 + \dots + a_n^2) + n(n+3) \geq (2n+2)(a_1 + a_2 + \dots + a_n).$$

Solution. Using the substitutions $a_i = e^{x_i}$ for $i = 1, 2, \dots, n$, we need to show that

$$f(x_1) + f(x_2) + \dots + f(x_n) \geq n f(s), \quad s = \frac{x_1 + x_2 + \dots + x_n}{n} = 0,$$

where

$$f(u) = (n-1)e^{2u} - (2n+2)e^u, \quad u \in \mathbb{I} = \mathbb{R}.$$

For $u \geq 0$, we have

$$\begin{aligned} f''(u) &= 4(n-1)e^{2u} - (2n+2)e^u \\ &= 2e^u[2(n-1)e^u - n - 1] \\ &\geq 2e^u[2(n-1) - n - 1] = 2(n-3)e^u > 0. \end{aligned}$$

Therefore, f is convex on $\mathbb{I}_{\geq s}$. By the RHCF-Theorem and Note 2, it suffices to show that $H(x, y) \geq 0$ for $x, y \in \mathbb{R}$ so that $x + (n-1)y = 0$, where

$$H(x, y) = \frac{f'(x) - f'(y)}{x - y}.$$

From

$$f'(u) = 2(n-1)e^{2u} - (2n+2)e^u,$$

we get

$$H(x, y) = \frac{2(e^x - e^y)}{x - y} [(n-1)(e^x + e^y) - (n+1)].$$

Since $(e^x - e^y)/(x - y) > 0$, we need to prove that $(n-1)(e^x + e^y) \geq n+1$. Using the AM-GM inequality, we have

$$\begin{aligned} (n-1)(e^x + e^y) &= (n-1)e^x + e^y + e^y + \cdots + e^y \\ &\geq n \sqrt[n]{(n-1)e^x \cdot e^y \cdot e^y \cdots e^y} \\ &= n \sqrt[n]{(n-1)e^{x+(n-1)y}} = n \sqrt[n]{n-1}. \end{aligned}$$

Thus, it suffices to show that

$$n \sqrt[n]{n-1} \geq n+1,$$

which is equivalent to

$$n-1 \geq \left(1 + \frac{1}{n}\right)^n.$$

This is true for $n \geq 4$, since

$$n-1 \geq 3 > \left(1 + \frac{1}{n}\right)^n.$$

The proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Remark. From the proof above, the following sharper inequality follows (*Gabriel Dospinescu and Calin Popa*):

- If $a_1, a_2, \dots, a_n$ are positive real numbers so that $a_1 a_2 \cdots a_n = 1$, then

$$a_1^2 + a_2^2 + \cdots + a_n^2 - n \geq \frac{2n \sqrt[n]{n-1}}{n-1} (a_1 + a_2 + \cdots + a_n - n).$$

□

P 1.63. Let $a_1, a_2, \dots, a_n$ ($n \geq 3$) be positive real numbers so that $a_1 a_2 \cdots a_n = 1$. If $p, q \geq 0$ so that $p + q \geq n-1$, then

$$\frac{1}{1 + pa_1 + qa_1^2} + \frac{1}{1 + pa_2 + qa_2^2} + \cdots + \frac{1}{1 + pa_n + qa_n^2} \geq \frac{n}{1 + p + q}.$$

(*Vasile C., 2007*)

Solution. Using the substitutions $a_i = e^{x_i}$ for $i = 1, 2, \dots, n$, we need to show that

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s), \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

where

$$f(u) = \frac{1}{1 + pe^u + qe^{2u}}, \quad u \in \mathbb{I} = \mathbb{R}.$$

For $u \geq 0$, we have

$$\begin{aligned} f''(u) &= \frac{e^u[4q^2e^{3u} + 3pqe^{2u} + (p^2 - 4q)e^u - p]}{(1 + pe^u + qe^{2u})^3} \\ &\geq \frac{e^{2u}[4q^2 + 3pq + (p^2 - 4q) - p]}{(1 + pe^u + qe^{2u})^3} \\ &= \frac{e^{2u}[(p + 2q)(p + q - 2) + 2q^2 + p]}{(1 + pe^u + qe^{2u})^3} > 0, \end{aligned}$$

therefore f is convex on $\mathbb{I}_{\geq s}$. By the RHCF-Theorem, it suffices to prove the original inequality for

$$a_1 = 1/t^{n-1}, \quad a_2 = \cdots = a_n = t, \quad t > 0.$$

Write this inequality as

$$\frac{t^{2n-2}}{t^{2n-2} + pt^{n-1} + q} + \frac{n-1}{1 + pt + qt^2} \geq \frac{n}{1 + p + q}.$$

Applying the Cauchy-Schwarz inequality, it suffices to prove that

$$\frac{(t^{n-1} + n - 1)^2}{(t^{2n-2} + pt^{n-1} + q) + (n-1)(1 + pt + qt^2)} \geq \frac{n}{1 + p + q},$$

which is equivalent to

$$pB + qC \geq A,$$

where

$$A = (n-1)(t^{n-1} - 1)^2 \geq 0,$$

$$B = (t^{n-1} - 1)^2 + nE = \frac{A}{n-1} + nE, \quad E = t^{n-1} + n - 2 - (n-1)t,$$

$$C = (t^{n-1} - 1)^2 + nF = \frac{A}{n-1} + nF, \quad F = 2t^{n-1} + n - 3 - (n-1)t^2.$$

By the AM-GM inequality applied to $n-1$ positive numbers, we have $E \geq 0$ and $F \geq 0$ for $n \geq 3$. Since $A \geq 0$ and $p + q \geq n-1$, we have

$$pB + qC - A \geq pB + qC - \frac{(p+q)A}{n-1} = n(pE + qF) \geq 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Remark 1. For $p = 2k$ and $q = k^2$, we get the following result:

• Let $a_1, a_2, \dots, a_n$ ($n \geq 3$) be positive real numbers so that $a_1 a_2 \cdots a_n = 1$. If $k \geq \sqrt{n} - 1$, then

$$\frac{1}{(1 + ka_1)^2} + \frac{1}{(1 + ka_2)^2} + \cdots + \frac{1}{(1 + ka_n)^2} \geq \frac{n}{(1 + k)^2},$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$.

In addition, for $n = 4$ and $k = 1$, we get the known inequality (Vasile C., 1999):

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{1}{(1+d)^2} \geq 1,$$

where $a, b, c, d > 0$ so that $abcd = 1$.

Remark 2. For $p + q = n - 1$ ($n \geq 3$), we get the beautiful inequality

$$\frac{1}{1 + pa_1 + qa_1^2} + \frac{1}{1 + pa_2 + qa_2^2} + \cdots + \frac{1}{1 + pa_n + qa_n^2} \geq 1,$$

which is a generalization of the following inequalities:

$$\begin{aligned} \frac{1}{1 + (n-1)a_1} + \frac{1}{1 + (n-1)a_2} + \cdots + \frac{1}{1 + (n-1)a_n} &\geq 1, \\ \frac{1}{[1 + (\sqrt{n}-1)a_1]^2} + \frac{1}{[1 + (\sqrt{n}-1)a_2]^2} + \cdots + \frac{1}{[1 + (\sqrt{n}-1)a_n]^2} &\geq 1, \\ \frac{1}{2 + (n-1)(a_1 + a_1^2)} + \frac{1}{2 + (n-1)(a_2 + a_2^2)} + \cdots + \frac{1}{2 + (n-1)(a_n + a_n^2)} &\geq \frac{1}{2}. \end{aligned}$$

□

P 1.64. Let a, b, c, d be positive real numbers so that $abcd = 1$. If p and q are nonnegative real numbers so that $p + q = 3$, then

$$\frac{1}{1 + pa + qa^3} + \frac{1}{1 + pb + qb^3} + \frac{1}{1 + pc + qc^3} + \frac{1}{1 + pd + qd^3} \geq 1.$$

(Vasile C., 2007)

Solution. Using the notation

$$a = e^x, \quad b = e^y, \quad c = e^z, \quad d = e^w,$$

we need to show that

$$f(x) + f(y) + f(z) + f(w) \geq 4f(s), \quad s = \frac{x + y + z + w}{4} = 0,$$

where

$$f(u) = \frac{1}{1 + pe^u + qe^{3u}}, \quad u \in \mathbb{I} = \mathbb{R}.$$

We will show that $f''(u) > 0$ for $u \geq 0$, hence f is convex on $\mathbb{I}_{\geq s}$. Since

$$f''(u) = \frac{th(t)}{(1 + pt + qt^3)^3},$$

where

$$h(t) = 9q^2t^5 + 2pqt^3 - 9qt^2 + p^2t - p, \quad t = e^u,$$

we need to show that $h(t) \geq 0$ for $t \geq 1$. Indeed, we have

$$h(t) \geq 9q^2t^3 + 2pqt^3 - 9qt^2 + p^2t - pt = tg(t),$$

where

$$\begin{aligned} g(t) &= (9q^2 + 2pq)t^2 - 9qt + p^2 - p \\ &\geq (9q^2 + 2pq)(2t - 1) - 9qt + p^2 - p \\ &= q(18q + 4p - 9)t - 9q^2 - 2pq + p^2 - p \\ &\geq q(18q + 4p - 9) - 9q^2 - 2pq + p^2 - p \\ &= p^2 + 2pq + 9q^2 - p - 9q \\ &= p^2 + 2pq + 9q^2 - \frac{(p + 9q)(p + q)}{3} \\ &= \frac{2(p - q)^2 + 16q^2}{3} \geq 0. \end{aligned}$$

By the RHCF-Theorem, it suffices to prove the original inequality for

$$b = c = d = t, \quad a = 1/t^3, \quad t > 0;$$

that is,

$$\begin{aligned} \frac{t^9}{t^9 + pt^6 + q} + \frac{3}{1 + pt + qt^3} &\geq 1, \\ \frac{3}{1 + pt + qt^3} &\geq \frac{pt^6 + q}{t^9 + pt^6 + q}, \\ (3 - pq)t^9 - p^2t^7 + 2pt^6 - q^2t^3 - pqt + 2q &\geq 0, \\ [(p + q)^2 - 3pq]t^9 - 3p^2t^7 + 2p(p + q)t^6 - 3q^2t^3 - 3pqt + 2q(p + q) &\geq 0, \\ Ap^2 + Bq^2 &\geq Cpq, \end{aligned}$$

where

$$\begin{aligned} A &= t^9 - 3t^7 + 2t^6 = t^6(t - 1)^2(t + 2) \geq 0, \\ B &= t^9 - 3t^3 + 2 = (t^3 - 1)^2(t^3 + 2) \geq 0, \\ C &= t^9 - 2t^6 + 3t - 2. \end{aligned}$$

Since $A \geq 0$ and $B \geq 0$, it suffices to consider the case $C \geq 0$. Since

$$Ap^2 + Bq^2 \geq 2\sqrt{AB}pq,$$

we only need to show that $4AB \geq C^2$. From

$$t^3 - 3t + 2 = (t - 1)^2(t + 2) \geq 0,$$

we get $3t - 2 \leq t^3$. Therefore

$$C \leq t^9 - 2t^6 + t^3 = t^3(t^3 - 1)^2,$$

hence

$$\begin{aligned} 4AB - C^2 &\geq 4AB - t^6(t^3 - 1)^4 \\ &= t^6(t - 1)^2(t^3 - 1)^2[4(t + 2)(t^3 + 2) - (t^2 + t + 1)^2] \\ &= t^6(t - 1)^2(t^3 - 1)^2(3t^4 + 6t^3 - 3t^2 + 6t + 15) \geq 0. \end{aligned}$$

The proof is completed. The inequality holds for $a = b = c = d = 1$.

Remark 1. For $p = 1$ and $p = 2$, we get the following nice inequalities:

$$\begin{aligned} \frac{1}{1 + a + 2a^3} + \frac{1}{1 + b + 2b^3} + \frac{1}{1 + c + 2c^3} + \frac{1}{1 + d + 2d^3} &\geq 1, \\ \frac{1}{1 + 2a + a^3} + \frac{1}{1 + 2b + b^3} + \frac{1}{1 + 2c + c^3} + \frac{1}{1 + 2d + d^3} &\geq 1. \end{aligned}$$

Remark 2. Similarly, we can prove the following generalizations:

• Let a, b, c, d be positive real numbers so that $abcd = 1$. If p and q are nonnegative real numbers so that $p + q \geq 3$, then

$$\frac{1}{1 + pa + qa^3} + \frac{1}{1 + pb + qb^3} + \frac{1}{1 + pc + qc^3} + \frac{1}{1 + pd + qd^3} \geq \frac{4}{1 + p + q}.$$

• Let $a_1, a_2, \dots, a_n$ ($n \geq 4$) be positive real numbers so that $a_1 a_2 \cdots a_n = 1$. If $p, q, r \geq 0$ so that $p + q + r \geq n - 1$, then

$$\sum_{i=1}^n \frac{1}{1 + pa_i + qa_i^2 + ra_i^3} \geq \frac{n}{1 + p + q + r}.$$

For $n = 4$ and $p + q + r = 3$, we get the beautiful inequality

$$\sum_{i=1}^4 \frac{1}{1 + pa_i + qa_i^2 + ra_i^3} \geq 1.$$

Since

$$a_i^2 \leq \frac{a_i + a_i^3}{2},$$

the best inequality with respect to q if for $q = 0$:

$$\sum_{i=1}^4 \frac{1}{1 + pa_i + ra_i^3} \geq 1, \quad p + r = 3.$$

□

P 1.65. If $a_1, a_2, \dots, a_n$ are positive real numbers so that $a_1 a_2 \cdots a_n = 1$, then

$$\frac{1}{1 + a_1 + \cdots + a_1^{n-1}} + \frac{1}{1 + a_2 + \cdots + a_2^{n-1}} + \cdots + \frac{1}{1 + a_n + \cdots + a_n^{n-1}} \geq 1.$$

(Vasile C., 2007)

Solution. Using the substitution $a_i = e^{x_i}$ for $i = 1, 2, \dots, n$, and denoting

$$f(u) = \frac{1}{1 + e^u + \dots + e^{(n-1)u}}, \quad u \in \mathbb{R},$$

we need to show that

$$f(x_1) + f(x_2) + \dots + f(x_n) \geq nf(s),$$

where $s = \frac{x_1 + x_2 + \dots + x_n}{n} = 0$. In addition, by RHCF-Theorem, if f is convex for $u \geq s = 0$, then we only need to show that

$$f(x) + (n-1)f(y) \geq 1$$

for all $x, y \in \mathbb{R}$ so that $x \leq 0 \leq y$ and $x + (n-1)y = 0$. Denoting $k = n-1$ and $t = e^y$ (hence $t \geq 1$), the inequality becomes

$$\frac{t^{k^2}}{1 + t^k + \dots + t^{k^2}} + \frac{k}{1 + t + \dots + t^k} \geq 1. \quad (*)$$

First, we will show by induction that $f(u)$ is convex for $u \geq 0$. Setting $t = e^u$, the necessary and sufficient condition $f''(u) \geq 0$ for $u \geq 0$ (hence $t \geq 1$) is equivalent to

$$2A^2 \geq B(1 + C),$$

where

$$A = t + 2t^2 + \dots + (n-1)t^{n-1}, \quad B = t + 4t^2 + \dots + (n-1)^2t^{n-1}, \quad C = t + t^2 + \dots + t^{n-1}.$$

For $n = 2$, the inequality reduces to $t(t-1) \geq 0$, which is true. Assuming that the inequality is true for n (where $n \geq 2$), we will show that it is true for $n+1$. Using the induction hypothesis, we need to show that $2A^2 \geq B(1 + C)$ involves

$$2(A + nt^n)^2 \geq (B + n^2t^n)(1 + C + t^n),$$

which is equivalent to

$$2A^2 - B(1 + C) + t^n[n^2(t^n - 1) + D] \geq 0,$$

where

$$D = 4nA - B - n^2C = \sum_{i=1}^{n-1} b_i t^i, \quad b_i = 3n^2 - (2n-i)^2.$$

Since $2A^2 - B(1 + C) \geq 0$, it suffices to show that $D \geq 0$. Since

$$b_1 < b_2 < \dots < b_{n-1}, \quad t \leq t^2 \leq \dots \leq t^{n-1},$$

we may apply Chebyshev's inequality to get

$$D \geq \frac{1}{n}(b_1 + b_2 + \dots + b_{n-1})(t + t^2 + \dots + t^{n-1}).$$

Thus, it suffices to show that $b_1 + b_2 + \dots + b_{n-1} \geq 0$. Indeed,

$$b_1 + b_2 + \dots + b_{n-1} = \sum_{i=1}^{n-1} [3n^2 - (2n-i)^2] = \frac{n(n-1)(4n+1)}{6} > 0.$$

To finish the proof, we need to prove (*). For the nontrivial case $t > 1$, the inequality can be written as follows:

$$\begin{aligned} \frac{k}{1+t+\dots+t^k} &\geq \frac{1+t^k+\dots+t^{(k-1)k}}{1+t^k+\dots+t^{k^2}}, & \frac{k(t-1)}{t^{k+1}-1} &\geq \frac{t^{k^2}-1}{t^k-1} \cdot \frac{t^k-1}{t^{(k+1)k}-1}, & k \frac{t^{(k+1)k}-1}{t^{k+1}-1} &\geq \frac{t^{k^2}-1}{t-1}, \\ k [1+t^{k+1}+t^{2(k+1)}+\dots+t^{(k-1)(k+1)}] &\geq \frac{t^k-1}{t-1} \cdot \frac{t^{k^2}-1}{t^k-1}, \\ k [1 \cdot 1+t \cdot t^k+\dots+t^{k-1} \cdot t^{(k-1)k}] &\geq (1+t+\dots+t^{k-1}) [1+t^k+\dots+t^{(k-1)k}]. \end{aligned}$$

Since $1 < t < \dots < t^{k-1}$ and $1 < t^k < \dots < t^{(k-1)k}$, the last inequality follows from Chebyshev's inequality.

The proof is completed. The equality occurs for $a_1 = a_2 = \dots = a_n = 1$.

Remark. Actually, the following generalization holds:

• Let $a_1, a_2, \dots, a_n$ be positive numbers so that $a_1 a_2 \dots a_n = 1$, and let $k_1, k_2, \dots, k_m \geq 0$ so that $k_1 + k_2 + \dots + k_m \geq n - 1$. If $m \leq n - 1$, then

$$\sum_{i=1}^n \frac{1}{1+k_1 a_i + k_2 a_i^2 + \dots + k_m a_i^m} \geq \frac{n}{1+k_1 + k_2 + \dots + k_m}.$$

In addition, since

$$a_i^k \leq \frac{(m-k)a_i + (k-1)a_i^m}{m-1}, \quad k = 2, 3, \dots, m-1$$

(by the AM-GM inequality applied to $m-1$ positive numbers), the best inequality with respect to $k_2, \dots, k_{m-1}$ is for $k_2 = 0, \dots, k_{m-1} = 0$; that is,

$$\sum_{i=1}^n \frac{1}{1+k_1 a_i + k_m a_i^m} \geq \frac{n}{1+k_1 + k_m}, \quad k_1 + k_m \geq n-1, \quad 1 \leq m \leq n-1.$$

If $k_1 + k_m = n-1$, then

$$\sum_{i=1}^n \frac{1}{1+k_1 a_i + k_m a_i^m} \geq 1, \quad 1 \leq m \leq n-1,$$

therefore

$$\sum_{i=1}^n \frac{1}{1+k_1 a_i + k_{n-1} a_i^{n-1}} \geq 1, \quad k_1 + k_{n-1} = n-1.$$

For $k_1 = 1$ and $k_1 = n-2$, we get the following strong inequalities:

$$\begin{aligned} \sum_{i=1}^n \frac{1}{1+a_i + (n-2)a_i^{n-1}} &\geq 1, \\ \sum_{i=1}^n \frac{1}{1+(n-2)a_i + a_i^{n-1}} &\geq 1. \end{aligned}$$

□

P 1.66. Let $a_1, a_2, \dots, a_n$ be positive real numbers so that $a_1 a_2 \cdots a_n = 1$. If

$$k \geq n^2 - 1,$$

then

$$\frac{1}{\sqrt{1+ka_1}} + \frac{1}{\sqrt{1+ka_2}} + \cdots + \frac{1}{\sqrt{1+ka_n}} \geq \frac{n}{\sqrt{1+k}}.$$

Solution. Using the substitutions $a_i = e^{x_i}$ for $i = 1, 2, \dots, n$, we need to show that

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s), \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

where

$$f(u) = \frac{1}{\sqrt{1+ke^u}}, \quad u \in \mathbb{I} = \mathbb{R}.$$

For $u \geq 0$, we have

$$f''(u) = \frac{ke^u(ke^u - 2)}{4(1+ke^u)^{5/2}} \geq \frac{ke^u(k-2)}{4(1+ke^u)^{5/2}} > 0.$$

Therefore, f is convex on $\mathbb{I}_{\geq s}$. By the RHCF-Theorem, it suffices to prove the original inequality for

$$a_1 = 1/t^{n-1}, \quad a_2 = \cdots = a_n = t, \quad t \geq 1.$$

Write this inequality as $h(t) \geq 0$, where

$$h(t) = \sqrt{\frac{t^{n-1}}{t^{n-1}+k}} + \frac{n-1}{\sqrt{1+kt}} - \frac{n}{\sqrt{1+k}}.$$

The derivative

$$h'(t) = \frac{(n-1)kt^{(n-3)/2}}{2(t^{n-1}+k)^{3/2}} - \frac{(n-1)k}{2(kt+1)^{3/2}}$$

has the same sign as

$$h_1(t) = t^{n/3-1}(kt+1) - t^{n-1} - k.$$

Denoting $m = n/3$ ($m \geq 2/3$), we see that

$$h_1(t) = kt^m + t^{m-1} - t^{3m-1} - k = k(t^m - 1) - t^{m-1}(t^{2m} - 1) = (t^m - 1)h_2(t),$$

where

$$h_2(t) = k - t^{m-1} - t^{2m-1}.$$

For $t > 1$, we have

$$\begin{aligned} h_2'(t) &= t^{m-2}[-m+1-(2m-1)t^m] < t^{m-2}[-m+1-(2m-1)] \\ &= -(3m-2)t^{m-2} \leq 0, \end{aligned}$$

hence $h_2(t)$ is strictly decreasing for $t \geq 1$. Since

$$h_2(1) = k - 2 > 0, \quad \lim_{t \rightarrow \infty} h_2(t) = -\infty,$$

there exists $t_1 > 1$ so that $h_2(t_1) = 0$, $h_2(t) > 0$ for $t \in [1, t_1)$, $h_2(t) < 0$ for $t \in (t_1, \infty)$. Since $h_2(t)$, $h_1(t)$ and $h'(t)$ has the same sign for $t > 1$, $h(t)$ is strictly increasing for $t \in [1, t_1]$ and strictly decreasing for $t \in [t_1, \infty)$; this yields

$$h(t) \geq \min\{h(1), h(\infty)\}.$$

From $h(1) = 0$ and $h(\infty) = 1 - \frac{n}{\sqrt{1+k}} \geq 0$, it follows that $h(t) \geq 0$ for all $t \geq 1$. The proof is completed. The equality holds for $a_1 = a_2 = \dots = a_n = 1$.

Remark. The following generalization holds (Vasile C., 2005):

• Let $a_1, a_2, \dots, a_n$ be positive real numbers so that $a_1 a_2 \dots a_n = 1$. If k and m are positive numbers so that

$$m \leq n - 1, \quad k \geq n^{1/m} - 1,$$

then

$$\frac{1}{(1 + ka_1)^m} + \frac{1}{(1 + ka_2)^m} + \dots + \frac{1}{(1 + ka_n)^m} \geq \frac{n}{(1 + k)^m},$$

with equality for $a_1 = a_2 = \dots = a_n = 1$.

For $0 < m \leq n - 1$ and $k = n^{1/m} - 1$, we get the beautiful inequality

$$\frac{1}{(1 + ka_1)^m} + \frac{1}{(1 + ka_2)^m} + \dots + \frac{1}{(1 + ka_n)^m} \geq 1.$$

□

P 1.67. Let $a_1, a_2, \dots, a_n$ be positive real numbers so that $a_1 a_2 \dots a_n = 1$. If $p, q \geq 0$ so that $0 < p + q \leq \frac{1}{n-1}$, then

$$\frac{1}{1 + pa_1 + qa_1^2} + \frac{1}{1 + pa_2 + qa_2^2} + \dots + \frac{1}{1 + pa_n + qa_n^2} \leq \frac{n}{1 + p + q}.$$

(Vasile C., 2007)

Solution. Using the notation $a_i = e^{x_i}$ for $i = 1, 2, \dots, n$, we need to show that

$$f(x_1) + f(x_2) + \dots + f(x_n) \geq nf(s), \quad s = \frac{x_1 + x_2 + \dots + x_n}{n} = 0,$$

where

$$f(u) = \frac{-1}{1 + pe^u + qe^{2u}}, \quad u \in \mathbb{I} = \mathbb{R}.$$

For $u \leq 0$, we have

$$\begin{aligned} f''(u) &= \frac{e^u[-4q^2e^{3u} - 3pqe^{2u} + (4q - p^2)e^u + p]}{(1 + pe^u + qe^{2u})^3} \\ &= \frac{e^{2u}[-4q^2e^{2u} - 3pqe^u + (4q - p^2) + pe^{-u}]}{(1 + pe^u + qe^{2u})^3} \\ &\geq \frac{e^{2u}[-4q^2 - 3pq + (4q - p^2) + p]}{(1 + pe^u + qe^{2u})^3} \\ &= \frac{e^{2u}[(p + 4q)(1 - p - q) + 2pq]}{(1 + pe^u + qe^{2u})^3} \geq 0, \end{aligned}$$

therefore f is convex on $\mathbb{I}_{\leq s}$. By the LHCF-Theorem, it suffices to prove the original inequality for

$$a_1 = 1/t^{n-1}, \quad a_2 = \cdots = a_n = t, \quad t > 0.$$

Write this inequality as

$$\frac{t^{2n-2}}{t^{2n-2} + pt^{n-1} + q} + \frac{n-1}{1+pt+qt^2} \leq \frac{n}{1+p+q},$$

$$p^2A + q^2B + pqC \leq pD + qE,$$

where

$$A = t^{n-1}(t^n - nt + n - 1), \quad B = t^{2n} - nt^2 + n - 1,$$

$$C = t^{2n-1} + t^{2n} - nt^{n+1} + (n-1)t^{n-1} - nt + n - 1,$$

$$D = t^{n-1}[(n-1)t^n + 1 - nt^{n-1}], \quad E = (n-1)t^{2n} + 1 - nt^{2n-2}.$$

Applying the AM-GM inequality to n positive numbers yields $D \geq 0$ and $E \geq 0$. Since $(n-1)(p+q) \leq 1$ involves $pD + qE \geq (n-1)(p+q)(pD + qE)$, it suffices to show that

$$p^2A + q^2B + pqC \leq (n-1)(p+q)(pD + qE).$$

Write this inequality as

$$p^2A_1 + q^2B_1 + pqC_1 \geq 0,$$

where

$$A_1 = (n-1)D - A = nt^n[(n-2)t^{n-1} + 1 - (n-1)t^{n-2}],$$

$$B_1 = (n-1)E - B = nt^2[(n-2)t^{2n-2} + 1 - (n-1)t^{2n-4}],$$

$$C_1 = (n-1)(D + E) - C = nt[(n-2)(t^{2n-1} + t^{2n-2}) - 2(n-1)t^{2n-3} + t^n + 1].$$

Applying the AM-GM inequality to $n-1$ nonnegative numbers yields $A_1 \geq 0$ and $B_1 \geq 0$. So, it suffices to show that $C_1 \geq 0$. Indeed, we have

$$(n-2)(t^{2n-1} + t^{2n-2}) - 2(n-1)t^{2n-3} + t^n + 1 = A_2 + B_2 + C_2,$$

where

$$A_2 = (n-2)t^{2n-1} + t - (n-1)t^{2n-3} \geq 0,$$

$$B_2 = (n-2)t^{2n-2} + t^{n-1} - (n-1)t^{2n-3} \geq 0,$$

$$C_2 = t^n - t^{n-1} - t + 1 = (t-1)(t^{n-1} - 1) \geq 0.$$

The inequalities $A_2 \geq 0$ and $B_2 \geq 0$ follow by applying the AM-GM inequality to $n-1$ nonnegative numbers.

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Remark 1. For $p+q = \frac{1}{n-1}$, we get the inequality

$$\frac{1}{1+pa_1+qa_1^2} + \frac{1}{1+pa_2+qa_2^2} + \cdots + \frac{1}{1+pa_n+qa_n^2} \leq n-1,$$

which is a generalization of the following inequalities:

$$\frac{1}{n-1+a_1} + \frac{1}{n-1+a_2} + \cdots + \frac{1}{n-1+a_n} \leq 1,$$

$$\frac{1}{2n-2+a_1+a_1^2} + \frac{1}{2n-2+a_2+a_2^2} + \cdots + \frac{1}{2n-2+a_n+a_n^2} \leq \frac{1}{2}.$$

Remark 2. For

$$p = \frac{4n-3}{2(n-1)(2n-1)}, \quad q = \frac{1}{2(n-1)(2n-1)},$$

we get the inequality

$$\frac{1}{(a_1+2n-2)(a_1+2n-1)} + \cdots + \frac{1}{(a_n+2n-2)(a_n+2n-1)} \leq \frac{1}{4n-2},$$

which is equivalent to

$$\frac{1}{a_1+2n-2} + \cdots + \frac{1}{a_n+2n-2} \leq \frac{1}{4n-2} + \frac{1}{a_1+2n-1} + \cdots + \frac{1}{a_n+2n-1}.$$

Remark 3. For $p = 2k$ and $q = k^2$, we get the following statement:

- Let $a_1, a_2, \dots, a_n$ be positive real numbers so that $a_1 a_2 \cdots a_n = 1$. If

$$0 < k \leq \sqrt{\frac{n}{n-1}} - 1,$$

then

$$\frac{1}{(1+ka_1)^2} + \frac{1}{(1+ka_2)^2} + \cdots + \frac{1}{(1+ka_n)^2} \leq \frac{n}{(1+k)^2},$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 1.68. Let $a_1, a_2, \dots, a_n$ ($n \geq 3$) be positive real numbers so that $a_1 a_2 \cdots a_n = 1$. If

$$0 < k \leq \frac{2n-1}{(n-1)^2},$$

then

$$\frac{1}{\sqrt{1+ka_1}} + \frac{1}{\sqrt{1+ka_2}} + \cdots + \frac{1}{\sqrt{1+ka_n}} \leq \frac{n}{\sqrt{1+k}}.$$

Solution. Using the substitutions $a_i = e^{x_i}$ for $i = 1, 2, \dots, n$, we need to show that

$$f(x_1) + f(x_2) + \dots + f(x_n) \geq nf(s), \quad s = \frac{x_1 + x_2 + \dots + x_n}{n} = 0,$$

where

$$f(u) = \frac{-1}{\sqrt{1 + ke^u}}, \quad u \in \mathbb{I} = \mathbb{R}.$$

For $u \leq 0$, we have

$$f''(u) = \frac{ke^u(2 - ke^u)}{4(1 + ke^u)^{5/2}} \geq \frac{ke^u(2 - k)}{4(1 + ke^u)^{5/2}} > 0.$$

Therefore, f is convex on $\mathbb{I}_{\leq s}$. By the LHCF-Theorem, it suffices to prove the original inequality for

$$a_1 = 1/t^{n-1}, \quad a_2 = \dots = a_n = t. \quad 0 < t \leq 1.$$

Write this inequality as $h(t) \leq 0$, where

$$h(t) = \sqrt{\frac{t^{n-1}}{t^{n-1} + k}} + \frac{n-1}{\sqrt{1+kt}} - \frac{n}{\sqrt{1+k}}.$$

The derivative

$$h'(t) = \frac{(n-1)kt^{(n-3)/2}}{2(t^{n-1} + k)^{3/2}} - \frac{(n-1)k}{2(kt+1)^{3/2}}$$

has the same sign as

$$h_1(t) = t^{n/3-1}(kt+1) - t^{n-1} - k.$$

Denoting $m = n/3$, $m \geq 1$, we see that

$$h_1(t) = kt^m + t^{m-1} - t^{3m-1} - k = -k(1 - t^m) + t^{m-1}(1 - t^{2m}) = (1 - t^m)h_2(t),$$

where

$$h_2(t) = t^{m-1} + t^{2m-1} - k$$

is strictly increasing for $t \in [0, 1]$. There are two possible cases: $h_2(0) \geq 0$ and $h_2(0) < 0$.

Case 1: $h_2(0) \geq 0$. This case is possible only for $m = 1$ and $k \leq 1$, when $h_2(t) = t + 1 - k > 0$ for $t \in (0, 1]$. Also, we have $h_1(t) > 0$ and $h'(t) > 0$ for $t \in (0, 1)$. Therefore, h is strictly increasing on $[0, 1]$, hence $h(t) \leq h(1) = 0$.

Case 2: $h_2(0) < 0$. This case is possible for either $m = 1$ ($n = 3$) and $1 < k \leq 5/4$, or $m > 1$ ($n \geq 4$). Since $h_2(1) = 2 - k > 0$, there exists $t_1 \in (0, 1)$ so that $h_2(t_1) = 0$, $h_2(t) < 0$ for $t \in (0, t_1)$, and $h_2(t) > 0$ for $t \in (t_1, 1)$. Since h' has the same sign as h_2 on $(0, 1)$, it follows that h is strictly decreasing on $[0, t_1]$ and strictly increasing on $[t_1, 1]$. Therefore, $h(t) \leq \max\{h(0), h(1)\}$. Since $h(0) = n - 1 - \frac{n}{\sqrt{1+k}} \leq 0$ and $h(1) = 0$, we have $h(t) \leq 0$ for all $t \in (0, 1]$.

The equality holds for $a_1 = a_2 = \dots = a_n = 1$.

Remark. The following generalization holds (Vasile C., 2005):

• Let $a_1, a_2, \dots, a_n$ ($n \geq 3$) be positive real numbers so that $a_1 a_2 \dots a_n = 1$. If k and m are positive numbers so that

$$m \geq \frac{1}{n-1}, \quad k \leq \left(\frac{n}{n-1} \right)^{1/m} - 1,$$

then

$$\frac{1}{(1+ka_1)^m} + \frac{1}{(1+ka_2)^m} + \cdots + \frac{1}{(1+ka_n)^m} \leq \frac{n}{(1+k)^m},$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$.

For $n \geq 3$, $m \geq \frac{1}{n-1}$ and $k = \left(\frac{n}{n-1}\right)^{1/m} - 1$, we get the beautiful inequality

$$\frac{1}{(1+ka_1)^m} + \frac{1}{(1+ka_2)^m} + \cdots + \frac{1}{(1+ka_n)^m} \leq n-1.$$

□

P 1.69. If $a_1, a_2, \dots, a_n$ are positive real numbers so that $a_1 a_2 \cdots a_n = 1$, then

$$\sqrt{a_1^4 + \frac{2n-1}{(n-1)^2}} + \sqrt{a_2^4 + \frac{2n-1}{(n-1)^2}} + \cdots + \sqrt{a_n^4 + \frac{2n-1}{(n-1)^2}} \geq \frac{1}{n-1}(a_1 + a_2 + \cdots + a_n)^2.$$

(Vasile C., 2006)

Solution. According to the preceding P 1.68, the following inequality holds

$$\sum \frac{1}{\sqrt{1 + \frac{2n-1}{(n-1)^2} a_1^{-4}}} \leq n-1.$$

On the other hand, by the Cauchy-Schwarz inequality

$$\left(\sum \frac{1}{\sqrt{1 + \frac{2n-1}{(n-1)^2} a_1^{-4}}} \right) \left(\sum a_1^2 \sqrt{1 + \frac{2n-1}{(n-1)^2} a_1^{-4}} \right) \geq \left(\sum a_1 \right)^2.$$

From these inequalities, we get

$$(n-1) \left(\sum a_1^2 \sqrt{1 + \frac{2n-1}{(n-1)^2} a_1^{-4}} \right) \geq \left(\sum a_1 \right)^2,$$

which is the desired inequality.

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 1.70. If $a_1, a_2, \dots, a_n$ are positive real numbers so that $a_1 a_2 \cdots a_n = 1$, then

$$a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1} + n(n-2) \geq (n-1) \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right).$$

Solution. Using the notation $a_i = e^{x_i}$ for $i = 1, 2, \dots, n$, we need to show that

$$f(x_1) + f(x_2) + \dots + f(x_n) \geq nf(s), \quad s = \frac{x_1 + x_2 + \dots + x_n}{n} = 0,$$

where

$$f(u) = e^{(n-1)u} - (n-1)e^{-u}, \quad u \in \mathbb{I} = \mathbb{R}.$$

For $u \geq 0$, we have

$$f''(u) = (n-1)^2 e^{(n-1)u} - (n-1)e^{-u} = (n-1)e^{-u}[(n-1)e^{nu} - 1] \geq 0;$$

therefore, f is convex on $\mathbb{I}_{\geq s}$. By the RHCF-Theorem and Note 2, it suffices to show that $H(x, y) \geq 0$ for $x, y \in \mathbb{R}$ so that $x + (n-1)y = 0$, where

$$H(x, y) = \frac{f'(x) - f'(y)}{x - y}.$$

From

$$f'(u) = (n-1)[e^{(n-1)u} + e^{-u}],$$

we get

$$\begin{aligned} H(x, y) &= \frac{(n-1)(e^x - e^y)}{x - y} [e^{(n-2)x} + e^{(n-3)x+y} + \dots + e^{x+(n-3)y} + e^{(n-2)y} - e^{-x-y}] \\ &= \frac{(n-1)(e^x - e^y)}{x - y} [e^{(n-2)x} + e^{(n-3)x+y} + \dots + e^{x+(n-3)y}]. \end{aligned}$$

Since $(e^x - e^y)/(x - y) > 0$, we have $H(x, y) > 0$.

The equality holds for $a_1 = a_2 = \dots = a_n = 1$.

□

P 1.71. Let $a_1, a_2, \dots, a_n$ be positive real numbers so that $a_1 a_2 \dots a_n = 1$. If $k \geq n$, then

$$a_1^k + a_2^k + \dots + a_n^k + kn \geq (k+1) \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right).$$

(Vasile C., 2006)

Solution. Using the notations $a_i = e^{x_i}$ for $i = 1, 2, \dots, n$, we need to show that

$$f(x_1) + f(x_2) + \dots + f(x_n) \geq nf(s), \quad s = \frac{x_1 + x_2 + \dots + x_n}{n} = 0,$$

where

$$f(u) = e^{ku} - (k+1)e^{-u}, \quad u \in \mathbb{I} = \mathbb{R}.$$

For $u \geq 0$, we have

$$f''(u) = k^2 e^{ku} - (k+1)e^{-u} = e^{-u} [k^2 e^{(k+1)u} - k - 1] \geq e^{-u} (k^2 - k - 1) > 0;$$

therefore, f is convex on $\mathbb{I}_{\geq s}$. By the RHCF-Theorem, it suffices to prove the original inequality for $a_1 \leq 1 \leq a_2 = \dots = a_n$; that is, to show that

$$a^k + (n-1)b^k - \frac{k+1}{a} - \frac{(k+1)(n-1)}{b} + kn \geq 0$$

for

$$ab^{n-1} = 1, \quad 0 < a \leq 1 \leq b.$$

By the weighted AM-GM inequality, we have

$$a^k + (kn - k - 1) \geq [1 + (kn - k - 1)]a^{\frac{k}{1+(kn-k-1)}} = \frac{k(n-1)}{b}.$$

Thus, we still have to show that

$$(n-1) \left(b^k - \frac{1}{b} \right) - (k+1) \left(\frac{1}{a} - 1 \right) \geq 0,$$

which is equivalent to $h(b) \geq 0$ for $b \geq 1$, where

$$h(b) = (n-1)(b^{k+1} - 1) - (k+1)(b^n - b).$$

Since

$$\begin{aligned} \frac{h'(b)}{k+1} &= (n-1)b^k - nb^{n-1} + 1 \geq (n-1)b^n - nb^{n-1} + 1 \\ &= nb^{n-1}(b-1) - (b^n - 1) \\ &= (b-1) [(b^{n-1} - b^{n-2}) + (b^{n-1} - b^{n-3}) + \dots + (b^{n-1} - 1)] \geq 0, \end{aligned}$$

h is increasing on $[1, \infty)$, hence $h(b) \geq h(1) = 0$. The proof is completed. The equality holds for $a_1 = a_2 = \dots = a_n = 1$.

□

P 1.72. If $a_1, a_2, \dots, a_n$ are positive real numbers so that $a_1 a_2 \dots a_n = 1$, then

$$\left(1 - \frac{1}{n}\right)^{a_1} + \left(1 - \frac{1}{n}\right)^{a_2} + \dots + \left(1 - \frac{1}{n}\right)^{a_n} \leq n-1.$$

(Vasile C., 2006)

Solution. Let

$$k = \frac{n}{n-1}, \quad k > 1,$$

and

$$m = \ln k, \quad 0 < m \leq \ln 2 < 1.$$

Using the substitutions $a_i = e^{x_i}$ for $i = 1, 2, \dots, n$, we need to show that

$$f(x_1) + f(x_2) + \dots + f(x_n) \geq nf(s), \quad s = \frac{x_1 + x_2 + \dots + x_n}{n} = 0,$$

where

$$f(u) = -k^{-e^u}, \quad u \in \mathbb{I} = \mathbb{R}.$$

From

$$f''(u) = me^u k^{-e^u} (1 - me^u),$$

it follows that $f''(u) > 0$ for $u \leq 0$, since

$$1 - me^u \geq 1 - m \geq 1 - \ln 2 > 0.$$

Therefore, f is convex on $\mathbb{I}_{\leq s}$. By the LHCF-Theorem, it suffices to prove the original inequality for

$$a_2 = \cdots = a_n := t, \quad a_1 = t^{-n+1}, \quad 0 < t \leq 1.$$

Write this inequality as

$$h(t) \leq n - 1,$$

where

$$h(t) = k^{-t^{-n+1}} + (n - 1)k^{-t}, \quad t \in (0, 1].$$

We have

$$h'(t) = (n - 1)mt^{-n}k^{-t^{-n+1}}h_1(t), \quad h_1(t) = 1 - t^n k^{t^{-n+1}-t},$$

$$h'_1(t) = k^{t^{-n+1}-t}h_2(t), \quad h_2(t) = m(n - 1 + t^n) - nt^{n-1}.$$

Since

$$h'_2(t) = nt^{n-2}(mt - n + 1) \leq nt^{n-2}(m - n + 1) \leq nt^{n-2}(m - 1) < 0,$$

h_2 is strictly decreasing on $[0, 1]$. From

$$h_2(0) = (n - 1)m > 0, \quad h_2(1) = n(m - 1) < 0,$$

it follows that there is $t_1 \in (0, 1)$ so that $h_2(t_1) = 0$, $h_2(t) > 0$ for $t \in [0, t_1)$ and $h_2(t) < 0$ for $t \in (t_1, 1]$. Therefore, h_1 is strictly increasing on $(0, t_1]$ and strictly decreasing on $[t_1, 1]$. Since $h_1(0_+) = -\infty$ and $h_1(1) = 0$, there is $t_2 \in (0, t_1)$ so that $h_1(t_2) = 0$, $h_1(t) < 0$ for $t \in (0, t_2)$, $h_1(t) > 0$ for $t \in (t_2, 1)$. Thus, h is strictly decreasing on $(0, t_2]$ and strictly increasing on $[t_2, 1]$. Since $h(0_+) = n - 1$ and $h(1) = n - 1$, we have $h(t) \leq n - 1$ for all $t \in (0, 1]$. This completes the proof. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 1.73. If a, b, c are positive real numbers so that $abc = 1$, then

$$\frac{1}{1 + \sqrt{1 + 3a}} + \frac{1}{1 + \sqrt{1 + 3b}} + \frac{1}{1 + \sqrt{1 + 3c}} \leq 1.$$

(Vasile C., 2008)

Solution. Write the inequality as

$$\frac{\sqrt{1+3a}-1}{3a} + \frac{\sqrt{1+3b}-1}{3b} + \frac{\sqrt{1+3c}-1}{3c} \leq 1,$$

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 3 \geq \sqrt{\frac{1}{a^2} + \frac{3}{a}} + \sqrt{\frac{1}{b^2} + \frac{3}{b}} + \sqrt{\frac{1}{c^2} + \frac{3}{c}}.$$

Replacing a, b, c by $1/a, 1/b, 1/c$, respectively, we need to prove that $abc = 1$ involves

$$a + b + c + 3 \geq \sqrt{a^2 + 3a} + \sqrt{b^2 + 3b} + \sqrt{c^2 + 3c}. \quad (*)$$

Using the notation

$$a = e^x, \quad b = e^y, \quad c = e^z,$$

we need to show that

$$f(x) + f(y) + f(z) \geq 3f(s), \quad s = \frac{x+y+z}{3} = 0,$$

where

$$f(u) = e^u - \sqrt{e^{2u} + 3e^u}, \quad u \in \mathbb{I} = \mathbb{R}.$$

We have

$$f''(u) = t \left[1 - \frac{4t^2 + 18t + 9}{4(t+3)\sqrt{t(t+3)}} \right], \quad t = e^u \geq 1.$$

For $u \geq 0$, which involves $t \geq 1$, from

$$16t(t+3)^3 - (4t^2 + 18t + 9)^2 = 9(4t^2 + 12t - 9) > 0,$$

it follows that $f'' > 0$, hence f is convex on $\mathbb{I}_{\geq s}$. By the RHCF-Theorem, it suffices to prove the inequality (*) for $b = c$. Thus, we need to show that

$$a - \sqrt{a^2 + 3a} + 2(b - \sqrt{b^2 + 3b}) + 3 \geq 0$$

for $ab^2 = 1$. Write this inequality as

$$2b^3 + 3b^2 + 1 \geq \sqrt{3b^2 + 1} + 2b^2\sqrt{b^2 + 3b}.$$

Squaring and dividing by b^2 , the inequality becomes

$$9b^2 + 4b + 3 \geq 4\sqrt{(b^2 + 3b)(3b^2 + 1)}.$$

Since

$$2\sqrt{(b^2 + 3b)(3b^2 + 1)} \leq (b^2 + 3b) + (3b^2 + 1) = 4b^2 + 3b + 1,$$

it suffices to show that

$$9b^2 + 4b + 3 \geq 2(4b^2 + 3b + 1),$$

which is equivalent to $(b-1)^2 \geq 0$. The equality holds for $a = b = c = 1$.

Remark. In the same manner, we can prove the following generalization:

- Let $a_1, a_2, \dots, a_n$ be positive real numbers so that $a_1 a_2 \cdots a_n = 1$. If

$$0 < k \leq \frac{4n}{(n-1)^2},$$

then

$$\frac{1}{1 + \sqrt{1 + ka_1}} + \frac{1}{1 + \sqrt{1 + ka_2}} + \cdots + \frac{1}{1 + \sqrt{1 + ka_n}} \leq \frac{n}{1 + \sqrt{1 + k}}.$$

□

P 1.74. If $a_1, a_2, \dots, a_n$ are positive real numbers so that $a_1 a_2 \cdots a_n = 1$, then

$$\frac{1}{1 + \sqrt{1 + 4n(n-1)a_1}} + \frac{1}{1 + \sqrt{1 + 4n(n-1)a_2}} + \cdots + \frac{1}{1 + \sqrt{1 + 4n(n-1)a_n}} \geq \frac{1}{2}.$$

(Vasile C., 2008)

Solution. Denote

$$k = 4n(n-1), \quad k \geq 8,$$

and write the inequality as follows:

$$\frac{\sqrt{1 + ka_1} - 1}{ka_1} + \frac{\sqrt{1 + ka_2} - 1}{ka_2} + \cdots + \frac{\sqrt{1 + ka_n} - 1}{ka_n} \geq \frac{1}{2},$$

$$\sqrt{\frac{1}{a_1^2} + \frac{k}{a_1}} + \sqrt{\frac{1}{a_2^2} + \frac{k}{a_2}} + \cdots + \sqrt{\frac{1}{a_n^2} + \frac{k}{a_n}} \geq \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} + \frac{k}{2}.$$

Replacing $a_1, a_2, \dots, a_n$ by $1/a_1, 1/a_2, \dots, 1/a_n$, we need to prove that $a_1 a_2 \cdots a_n = 1$ implies

$$\sqrt{a_1^2 + ka_1} + \sqrt{a_2^2 + ka_2} + \cdots + \sqrt{a_n^2 + ka_n} \geq a_1 + a_2 + \cdots + a_n + \frac{k}{2}. \quad (*)$$

Using the substitutions $a_i = e^{x_i}$ for $i = 1, 2, \dots, n$, we need to show that

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s), \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

where

$$f(u) = \sqrt{e^{2u} + ke^u} - e^u, \quad u \in \mathbb{I} = \mathbb{R}.$$

We will show that $f''(u) > 0$ for $u \leq 0$. Indeed, denoting $t = e^u$, $t \in (0, 1]$, we have

$$f''(u) = t \left[\frac{4t^2 + 6kt + k^2}{4(t+k)\sqrt{t(t+k)}} - 1 \right] > 0$$

because

$$(4t^2 + 6kt + k^2)^2 - 16t(t+k)^3 = k^2(k^2 - 4kt - 4t^2) \geq k^2(k^2 - 4k - 4) > 0.$$

Thus, f is convex on $\mathbb{I}_{\leq s}$. By the LHCF-Theorem, it suffices to prove the inequality (*) for $a_2 = a_3 = \dots = a_n$; that is, to show that

$$\sqrt{a^2 + ka} - a + (n-1) \left(\sqrt{b^2 + kb} - b \right) \geq n \left(\sqrt{1+k} - 1 \right),$$

for all positive a, b satisfying $ab^{n-1} = 1$. Write this inequality as

$$\sqrt{kb^{n-1} + 1} + (n-1)\sqrt{kb^{2n-1} + b^{2n}} \geq (n-1)b^n + 2n(n-1)b^{n-1} + 1.$$

By Minkowski's inequality, we have

$$\begin{aligned} & \sqrt{kb^{n-1} + 1} + (n-1)\sqrt{kb^{2n-1} + b^{2n}} \geq \\ & \geq \sqrt{kb^{n-1}[1 + (n-1)b^{n/2}]^2 + [1 + (n-1)b^n]^2}. \end{aligned}$$

Thus, it suffices to show that

$$kb^{n-1}[1 + (n-1)b^{n/2}]^2 + [1 + (n-1)b^n]^2 \geq [(n-1)b^n + 2n(n-1)b^{n-1} + 1]^2,$$

which is equivalent to

$$4n(n-1)^2 b^{\frac{3n-2}{2}} \left[2 + (n-2)b^{\frac{n}{2}} - nb^{\frac{n-2}{2}} \right] \geq 0.$$

This inequality follows immediately by the AM-GM inequality applied to n positive numbers.

The equality holds for $a_1 = a_2 = \dots = a_n = 1$.

□

P 1.75. If a, b, c are positive real numbers so that $abc = 1$, then

$$\frac{a^6}{1+2a^5} + \frac{b^6}{1+2b^5} + \frac{c^6}{1+2c^5} \geq 1.$$

(Vasile C., 2008)

Solution. Using the substitution

$$a = e^x, \quad b = e^y, \quad c = e^z,$$

we need to show that

$$f(x) + f(y) + f(z) \geq 3f(s), \quad s = \frac{x+y+z}{3} = 0,$$

where

$$f(u) = \frac{e^{6u}}{1+2e^{5u}}, \quad u \in \mathbb{I} = \mathbb{R}.$$

For $u \leq 0$, which involves $w = e^u \in (0, 1]$, we have

$$f''(u) = \frac{2w^6(2-w^5)(9-2w^5)}{(1+2w^5)^3} > 0.$$

Therefore, f is convex on $\mathbb{I}_{\leq s}$. By the LHCF-Theorem, it suffices to prove the original inequality for $b = c$ and $ab^2 = 1$; that is,

$$\frac{1}{b^2(b^{10} + 2)} + \frac{2b^6}{1 + 2b^5} \geq 1.$$

Since

$$1 + 2b^5 \leq 1 + b^4 + b^6,$$

it suffices to show that

$$\frac{1}{x(x^5 + 2)} + \frac{2x^3}{1 + x^2 + x^3} \geq 1, \quad x = \sqrt{b}.$$

This inequality can be written as follows:

$$\begin{aligned} x^3(x^6 - x^5 - x^3 + 2x - 1) + (x - 1)^2 &\geq 0, \\ x^3(x - 1)^2(x^4 + x^3 + x^2 - 1) + (x - 1)^2 &\geq 0, \\ (x - 1)^2[x^7 + x^5 + (x^6 - x^3 + 1)] &\geq 0. \end{aligned}$$

The equality holds for $a = b = c = 1$.

□

P 1.76. If a, b, c are positive real numbers so that $abc = 1$, then

$$\sqrt{25a^2 + 144} + \sqrt{25b^2 + 144} + \sqrt{25c^2 + 144} \leq 5(a + b + c) + 24.$$

(Vasile C., 2008)

Solution. Using the notation

$$a = e^x, \quad b = e^y, \quad c = e^z,$$

we need to show that

$$f(x) + f(y) + f(z) \geq 3f(s), \quad s = \frac{x + y + z}{3} = 0,$$

where

$$f(u) = 5e^u - \sqrt{25e^{2u} + 144}, \quad u \in \mathbb{R}.$$

We will show that $f(u)$ is convex for $u \leq 0$. From

$$f''(u) = 5w \left[1 - \frac{5w(25w^2 + 288)}{(25w^2 + 144)^{3/2}} \right], \quad w = e^u \in (0, 1],$$

we need to show that

$$(25w^2 + 144)^3 \geq 25w^2(25w^2 + 288)^2.$$

Setting $25w^2 = 144z$, we have $z \in \left(0, \frac{25}{144}\right]$ and

$$\begin{aligned} (25w^2 + 144)^3 - 25w^2(25w^2 + 288)^2 &= 144^3(z + 1)^3 - 144^3z(z + 2)^2 \\ &= 144^3(1 - z - z^2) > 0. \end{aligned}$$

By the LHCF-Theorem, it suffices to prove the original inequality for

$$a = t^2, \quad b = c = 1/t, \quad t > 0;$$

that is,

$$5t^3 + 24t + 10 \geq \sqrt{25t^6 + 144t^2} + 2\sqrt{25 + 144t^2}.$$

Squaring and dividing by $4t$ give

$$60t^3 + 25t^2 - 36t + 120 \geq \sqrt{(25t^4 + 144)(144t^2 + 25)}.$$

Squaring again and dividing by 120, the inequality becomes

$$25t^5 - 36t^4 + 105t^3 - 112t^2 - 72t + 90 \geq 0,$$

$$(t - 1)^2(25t^3 + 14t^2 + 108t + 90) \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 1.77. If a, b, c are positive real numbers so that $abc = 1$, then

$$\sqrt{16a^2 + 9} + \sqrt{16b^2 + 9} + \sqrt{16c^2 + 9} \geq 4(a + b + c) + 3.$$

(Vasile C., 2008)

Solution. Using the substitution

$$a = e^x, \quad b = e^y, \quad c = e^z,$$

we need to show that

$$f(x) + f(y) + f(z) \geq 3f(s), \quad s = \frac{x + y + z}{3} = 0,$$

where

$$f(u) = \sqrt{16e^{2u} + 9} - 4e^u, \quad u \in \mathbb{R}.$$

We will show that $f(u)$ is convex for $u \geq 0$. From

$$f''(u) = 4w \left[\frac{4w(16w^2 + 18)}{(16w^2 + 9)^{3/2}} - 1 \right], \quad w = e^u \geq 1,$$

we need to show that

$$16w^2(16w^2 + 18)^2 \geq (16w^2 + 9)^3.$$

Setting $16w^2 = 9z$, we have $z \geq \frac{16}{9}$ and

$$\begin{aligned} 16w^2(16w^2 + 18)^2 - (16w^2 + 9)^3 &= 729z(z + 2)^2 - 729(z + 1)^3 \\ &= 729(z^2 + z - 1) > 0. \end{aligned}$$

By the RHCF-Theorem, it suffices to prove the original inequality for

$$a = t^2, \quad b = c = 1/t, \quad t > 0;$$

that is,

$$\sqrt{16t^6 + 9t^2} + 2\sqrt{16 + 9t^2} \geq 4t^3 + 3t + 8.$$

Squaring and dividing by $4t$ give

$$\sqrt{(16t^4 + 9)(9t^2 + 16)} \geq 6t^3 + 16t^2 - 9t + 12.$$

Squaring again and dividing by $12t$, the inequality becomes

$$9t^5 - 16t^4 + 9t^3 + 12t^2 - 32t + 18 \geq 0,$$

$$(t - 1)^2(9t^3 + 2t^2 + 4t + 18) \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 1.78. *If ABC is a triangle, then*

$$\sin A \left(2 \sin \frac{A}{2} - 1 \right) + \sin B \left(2 \sin \frac{B}{2} - 1 \right) + \sin C \left(2 \sin \frac{C}{2} - 1 \right) \geq 0.$$

(Lorian Saceanu, 2015)

Solution. Write the inequality as

$$f(A) + f(B) + f(C) \geq 3f(s), \quad s = \frac{A + B + C}{3} = \frac{\pi}{3},$$

where

$$f(u) = \sin u \left(2 \sin \frac{u}{2} - 1 \right) = \cos \frac{u}{2} - \cos \frac{3u}{2} - \sin u, \quad u \in \mathbb{I} = [0, \pi].$$

We will show that f is convex on $\mathbb{I}_{\leq s}$. Indeed, for $u \in [0, \pi/3]$, we have

$$\begin{aligned} f''(u) &= \cos \frac{u}{2} \left(2 + 2 \sin \frac{u}{2} - 9 \sin^2 \frac{u}{2} \right) \geq \cos \frac{u}{2} \left(2 + 2 \sin \frac{u}{2} - 12 \sin^2 \frac{u}{2} \right) \\ &= 2 \cos \frac{u}{2} \left(1 + 3 \sin \frac{u}{2} \right) \left(1 - 2 \sin \frac{u}{2} \right) \geq 0. \end{aligned}$$

By the LHCF-Theorem, it suffices to prove the original inequality for $B = C$, when it transforms into

$$\sin 2B(2 \cos B - 1) + 2 \sin B \left(2 \sin \frac{B}{2} - 1 \right) \geq 0,$$

$$\sin B \sin \frac{B}{2} \left(\sin \frac{B}{2} + 1 \right) \left(2 \sin \frac{B}{2} - 1 \right)^2 \geq 0.$$

The equality occurs for an equilateral triangle, and for a degenerate triangle with $A = \pi$ and $B = C = 0$ (or any cyclic permutation).

Remark. Based on this inequality, we can prove the following statement:

- If ABC is a triangle, then

$$\sin 2A(2 \cos A - 1) + \sin 2B(2 \cos B - 1) + \sin 2C(2 \cos C - 1) \geq 0,$$

with equality for an equilateral triangle, for a degenerate triangle with $A = 0$ and $B = C = \pi/2$ (or any cyclic permutation), and for a degenerate triangle with $A = \pi$ and $B = C = 0$ (or any cyclic permutation).

If ABC is an acute or right triangle, then this inequality follows by replacing A , B and C with $\pi - 2A$, $\pi - 2B$ and $\pi - 2C$ in the inequality from P 1.78. Consider now that

$$A > \frac{\pi}{2} > B \geq C \geq 0.$$

The inequality is true for $B \leq \pi/3$, because

$$\sin 2A(2 \cos A - 1) \geq 0, \quad \sin 2B(2 \cos B - 1) \geq 0, \quad \sin 2C(2 \cos C - 1) \geq 0.$$

Consider further that

$$\frac{2\pi}{3} > A > \frac{\pi}{2} > B > \frac{\pi}{3} > C \geq 0.$$

From

$$1 - 2 \cos A > 1 - 2 \cos B,$$

it follows that

$$(-\sin 2A)(1 - 2 \cos A) > (-\sin 2A)(1 - 2 \cos B).$$

Therefore it suffices to

$$(-\sin 2A)(1 - 2 \cos B) + \sin 2B(2 \cos B - 1) + \sin 2C(2 \cos C - 1) \geq 0,$$

which is equivalent to

$$(\sin 2A + \sin 2B)(2 \cos B - 1) + \sin 2C(2 \cos C - 1) \geq 0,$$

$$2 \sin C \cos(A - B)(2 \cos B - 1) + 2 \sin C \cos C(2 \cos C - 1) \geq 0.$$

This inequality is true if

$$\cos(A - B)(2 \cos B - 1) + \cos C(2 \cos C - 1) \geq 0,$$

which can be written as

$$\cos C(2 \cos C - 1) \geq \cos(A - B)(1 - 2 \cos B).$$

Since

$$C < A - B < \frac{2\pi}{3} - \frac{\pi}{3} = \frac{\pi}{3},$$

we have $\cos C > \cos(A - B)$. Therefore, it suffices to show that

$$2 \cos C - 1 \geq 1 - 2 \cos B,$$

which is equivalent to

$$\cos B + \cos C \geq 1.$$

From $B + C < \pi/2$, we get $\cos B > \cos(\pi/2 - C) = \sin C$, hence

$$\cos B + \cos C > \sin C + \cos C = \sqrt{1 + \sin 2C} \geq 1.$$

□

P 1.79. If ABC is an acute or right triangle, then

$$\sin 2A \left(1 - 2 \sin \frac{A}{2}\right) + \sin 2B \left(1 - 2 \sin \frac{B}{2}\right) + \sin 2C \left(1 - 2 \sin \frac{C}{2}\right) \geq 0.$$

(Vasile C., 2015)

Solution. Write the inequality as

$$f(A) + f(B) + f(C) \geq 3f(s), \quad s = \frac{A + B + C}{3} = \frac{\pi}{3},$$

where

$$f(u) = \sin 2u \left(1 - 2 \sin \frac{u}{2}\right) = \sin 2u - \cos \frac{3u}{2} + \cos \frac{5u}{2}, \quad u \in \mathbb{I} = [0, \pi/2].$$

We will show that f is convex on $[s, \pi/2]$. From

$$f''(u) = -4 \sin 2u + \frac{9}{4} \cos \frac{3u}{2} - \frac{25}{4} \cos \frac{5u}{2}$$

and

$$\cos \frac{3u}{2} - \cos \frac{5u}{2} = 2 \sin \frac{u}{2} \sin 2u \geq 0,$$

we get

$$\begin{aligned} f''(u) &\geq -4 \sin 2u + \frac{9}{4} \cos \frac{5u}{2} - \frac{25}{4} \cos \frac{5u}{2} \\ &= -4 \left[\sin 2u + \sin \frac{\pi - 5u}{2} \right] = 8 \sin \frac{\pi - u}{4} \cos \frac{5\pi - 9u}{4}. \end{aligned}$$

For $\pi/3 \leq u \leq \pi/2$, we have

$$\frac{\pi}{8} \leq \frac{5\pi - 9u}{4} \leq \frac{\pi}{2},$$

hence $f''(u) \geq 0$. By the RHCF-Theorem, it suffices to prove the original inequality for $B = C$, $0 \leq B \leq \pi/2$, when it becomes

$$-\sin 4B(1 - 2 \cos B) + 2 \sin 2B \left(1 - 2 \sin \frac{B}{2}\right) \geq 0,$$

$$2 \sin 2B \left[\cos 2B(2 \cos B - 1) + 1 - \sin \frac{B}{2} \right] \geq 0.$$

We need to show that

$$\cos 2B(2 \cos B - 1) + 1 - \sin \frac{B}{2} \geq 0,$$

which is equivalent to $g(t) \geq 0$, where

$$g(t) = (1 - 8t^2 + 8t^4)(1 - 4t^2) + 1 - 2t, \quad t = \sin \frac{B}{2}, \quad 0 \leq t \leq \frac{1}{\sqrt{2}}.$$

Indeed, we have

$$g(t) = 2(1 - t)^2(1 + 3t + 2t^2 - 4t^3 - 4t^4) \geq 0$$

because

$$1 + 3t + 2t^2 - 4t^3 - 4t^4 \geq 1 + 3t + 2t^2 - 2t - 2t^2 = 1 + t > 0.$$

The equality occurs for an equilateral triangle, for a degenerate triangle with $A = 0$ and $B = C = \pi/2$ (or any cyclic permutation), and for a degenerate triangle with $A = \pi$ and $B = C = 0$ (or any cyclic permutation).

Remark 1. Actually, the inequality holds also for an obtuse triangle ABC. To prove this, consider that

$$A > \frac{\pi}{2} > B \geq C \geq 0.$$

The inequality is true for $B \leq \pi/3$, because

$$\sin 2A \left(1 - 2 \sin \frac{A}{2}\right) \geq 0, \quad \sin 2B \left(1 - 2 \sin \frac{B}{2}\right) \geq 0, \quad \sin 2C \left(1 - 2 \sin \frac{C}{2}\right) \geq 0.$$

Consider further that

$$\frac{2\pi}{3} > A > \frac{\pi}{2} > B > \frac{\pi}{3} > C \geq 0.$$

From

$$2 \sin \frac{A}{2} - 1 > 2 \sin \frac{B}{2} - 1,$$

it follows that

$$(-\sin 2A) \left(2 \sin \frac{A}{2} - 1\right) > (-\sin 2A) \left(2 \sin \frac{B}{2} - 1\right).$$

Therefore it suffices to

$$(-\sin 2A) \left(2 \sin \frac{B}{2} - 1\right) + \sin 2B \left(1 - 2 \sin \frac{B}{2}\right) + \sin 2C \left(1 - 2 \sin \frac{C}{2}\right) \geq 0,$$

which is equivalent to

$$(\sin 2A + \sin 2B) \left(1 - 2 \sin \frac{B}{2}\right) + \sin 2C \left(1 - 2 \sin \frac{C}{2}\right) \geq 0,$$

$$2 \sin C \cos(A - B) \left(1 - 2 \sin \frac{B}{2}\right) + 2 \sin C \cos C \left(1 - 2 \sin \frac{C}{2}\right) \geq 0.$$

This inequality is true if

$$\cos(A - B) \left(1 - 2 \sin \frac{B}{2}\right) + \cos C \left(1 - 2 \sin \frac{C}{2}\right) \geq 0,$$

which can be written as

$$\cos C \left(1 - 2 \sin \frac{C}{2}\right) \geq \cos(A - B) \left(2 \sin \frac{B}{2} - 1\right).$$

Since

$$C < A - B < \frac{2\pi}{3} - \frac{\pi}{3} = \frac{\pi}{3},$$

we have $\cos C > \cos(A - B)$. Therefore, it suffices to show that

$$1 - 2 \sin \frac{C}{2} \geq 2 \sin \frac{B}{2} - 1,$$

which is equivalent to

$$\begin{aligned} \sin \frac{B}{2} + \sin \frac{C}{2} &\leq 1, \\ 2 \sin \frac{B+C}{4} \cos \frac{B-C}{4} &\leq 1. \end{aligned}$$

This is true since

$$2 \sin \frac{B+C}{4} < 2 \sin \frac{\pi}{8} < 1, \quad \cos \frac{B-C}{4} < 1.$$

Remark 2. Replacing A , B and C in P 1.79 by $\pi - 2A$, $\pi - 2B$ and $\pi - 2C$, respectively, we get the following inequality for an acute or right triangle ABC:

$$\sin 4A(2 \cos A - 1) + \sin 4B(2 \cos B - 1) + \sin 4C(2 \cos C - 1) \geq 0,$$

with equality for an equilateral triangle, for a triangle with $A = \pi/2$ and $B = C = \pi/4$ (or any cyclic permutation), and for a degenerate triangle with $A = 0$ and $B = C = \pi/2$ (or any cyclic permutation). □

P 1.80. If a, b, c, d are real numbers so that $a + b + c + d = 4$, then

$$\frac{a}{a^2 - a + 4} + \frac{b}{b^2 - b + 4} + \frac{c}{c^2 - c + 4} + \frac{d}{d^2 - d + 4} \leq 1.$$

(Sqing, 2015)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) + f(d) \geq 4f(s), \quad s = \frac{a + b + c + d}{4} = 1,$$

where

$$f(u) = \frac{-u}{u^2 - u + 4}, \quad u \in \mathbb{R}.$$

We see that

$$f(u) - f(2) = \frac{(u-2)^2}{3(u^2 - u + 4)} \geq 0.$$

From

$$f''(u) = \frac{2(-u^3 + 12u - 4)}{(u^2 - u + 4)^3},$$

it follows that f is convex on $[1, 2]$. Define the function

$$f_0(u) = \begin{cases} f(u), & u \leq 2 \\ f(2), & u > 2 \end{cases}.$$

Since $f_0(u) \leq f(u)$ for $u \in \mathbb{R}$ and $f_0(1) = f(1)$, it suffices to show that

$$f_0(a) + f_0(b) + f_0(c) + f_0(d) \geq 4f_0(s).$$

The function f_0 is convex on $[1, \infty)$ because it is differentiable on $[1, \infty)$ and its derivative

$$f'_0(u) = \begin{cases} f'(u), & u \leq 2 \\ 0, & u > 2 \end{cases}$$

is continuous and increasing on $[1, \infty)$. Therefore, by the RHCF-Theorem, we only need to show that $f_0(x) + 3f_0(y) \geq 4f_0(1)$ for all $x, y \in \mathbb{R}$ so that $x \leq 1 \leq y$ and $x + 3y = 4$. There are two cases to consider: $y \leq 2$ and $y > 2$.

Case 1: $y \leq 2$. The inequality $f_0(x) + 3f_0(y) \geq 4f_0(1)$ is equivalent to $f(x) + 3f(y) \geq 4f(1)$. According to Note 1, this is true if $h(x, y) \geq 0$ for $x + 3y = 4$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{u - 4}{4(u^2 - u + 4)},$$

$$\begin{aligned} h(x, y) &= \frac{g(x) - g(y)}{x - y} = \frac{4(x + y) - xy}{4(x^2 - x + 4)(y^2 - y + 4)} \\ &= \frac{3(y - 2)^2 + 4}{4(x^2 - x + 4)(y^2 - y + 4)} > 0. \end{aligned}$$

Case 2: $y > 2$. From $y > 2$ and $x + 3y = 4$, we get $x < -2$ and

$$f_0(x) + 3f_0(y) - 4f_0(1) = f(x) + 3f(2) - 4f(1) = \frac{-x}{x^2 - x + 4} > 0.$$

The equality holds for $a = b = c = d = 1$.

□

P 1.81. Let a, b, c be nonnegative real numbers so that $a + b + c = 2$. If

$$k_0 \leq k \leq 3, \quad k_0 = \frac{\ln 2}{\ln 3 - \ln 2} \approx 1.71,$$

then

$$a^k(b + c) + b^k(c + a) + c^k(a + b) \leq 2.$$

Solution. Write the inequality as

$$f(a) + f(b) + f(c) \leq 2,$$

where

$$f(u) = u^k(2 - u), \quad u \in [0, \infty).$$

From

$$f''(u) = ku^{k-2}[2k - 2 - (k+1)u],$$

it follows that f is convex on $\left[0, \frac{2k-2}{k+1}\right]$ and concave on $\left[\frac{2k-2}{k+1}, 2\right]$. According to LCRCF-Theorem, the sum $f(a) + f(b) + f(c)$ is maximum when either $a = 0$ or $0 < a \leq b = c$.

Case 1: $a = 0$. We need to show that

$$bc(b^{k-1} + c^{k-1}) \leq 2$$

for $b + c = 2$. Since $0 < (k-1)/2 \leq 1$, Bernoulli's inequality gives

$$\begin{aligned} b^{k-1} + c^{k-1} &= (b^2)^{(k-1)/2} + (c^2)^{(k-1)/2} \leq 1 + \frac{k-1}{2}(b^2 - 1) + 1 + \frac{k-1}{2}(c^2 - 1) \\ &= 3 - k + \frac{k-1}{2}(b^2 + c^2). \end{aligned}$$

Thus, it suffices to show that

$$(3-k)bc + \frac{k-1}{2}bc(b^2 + c^2) \leq 2.$$

Since

$$bc \leq \left(\frac{b+c}{2}\right)^2 = 1,$$

we only need to show that

$$3 - k + \frac{k-1}{2}bc(b^2 + c^2) \leq 2,$$

which is equivalent to

$$bc(b^2 + c^2) \leq 2.$$

Indeed, we have

$$8[2 - bc(b^2 + c^2)] = (b+c)^4 - 8bc(b^2 + c^2) = (b-c)^4 \geq 0.$$

Case 2: $0 < a \leq b = c$. We only need to prove the homogeneous inequality

$$a^k(b+c) + b^k(c+a) + c^k(a+b) \leq 2 \left(\frac{a+b+c}{2}\right)^{k+1}$$

for $b = c = 1$ and $0 < a \leq 1$; that is,

$$\left(1 + \frac{a}{2}\right)^{k+1} - a^k - a - 1 \geq 0.$$

Since $\left(1 + \frac{a}{2}\right)^{k+1}$ is increasing and a^k is decreasing with respect to k , it suffices consider the case $k = k_0$; that is, to prove that $g(a) \geq 0$, where

$$g(a) = \left(1 + \frac{a}{2}\right)^{k_0+1} - a^{k_0} - a - 1, \quad 0 < a \leq 1.$$

We have

$$g'(a) = \frac{k_0 + 1}{2} \left(1 + \frac{a}{2}\right)^{k_0} - k_0 a^{k_0-1} - 1,$$

$$\frac{1}{k_0} g''(a) = \frac{k_0 + 1}{4} \left(1 + \frac{a}{2}\right)^{k_0-1} - \frac{k_0 - 1}{a^{2-k_0}}.$$

Since g'' is increasing on $(0, 1]$, $g''(0_+) = -\infty$ and

$$\frac{1}{k_0} g''(1) = \frac{k_0 + 1}{4} \left(\frac{3}{2}\right)^{k_0-1} - k_0 + 1 = \frac{k_0 + 1}{3} - k_0 + 1 = \frac{2(2 - k_0)}{3} > 0,$$

there exists $a_1 \in (0, 1)$ so that $g''(a_1) = 0$, $g''(a) < 0$ for $a \in (0, a_1)$, $g''(a) > 0$ for $a \in (a_1, 1]$. Therefore, g' is strictly decreasing on $[0, a_1]$ and strictly increasing on $[a_1, 1]$. Since

$$g'(0) = \frac{k_0 - 1}{2} > 0, \quad g'(1) = \frac{k_0 + 1}{2} [(3/2)^{k_0} - 2] = 0,$$

there exists $a_2 \in (0, a_1)$ so that $g'(a_2) = 0$, $g'(a) > 0$ for $a \in [0, a_2)$, $g'(a) < 0$ for $a \in (a_2, 1)$. Thus, g is strictly increasing on $[0, a_2]$ and strictly decreasing on $[a_2, 1]$. Consequently,

$$g(a) \geq \min\{g(0), g(1)\},$$

and from

$$g(0) = 0, \quad g(1) = (3/2)^{k_0+1} - 3 = 0,$$

we get $g(a) \geq 0$.

The equality holds for $a = 0$ and $b = c$ (or any cyclic permutation). If $k = k_0$, then the equality holds also for $a = b = c$.

□

P 1.82. If $a_1, a_2, \dots, a_n$ are positive real numbers so that $a_1 + a_2 + \dots + a_n = n$, then

$$(n+1)^2 \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right) \geq 4(n+2)(a_1^2 + a_2^2 + \dots + a_n^2) + n(n^2 - 3n - 6).$$

(Vasile C., 2006)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq n(n^2 - 3n - 6),$$

where

$$f(u) = \frac{(n+1)^2}{u} - 4(n+2)u^2, \quad u \in (0, \infty).$$

From

$$f''(u) = \frac{2(n+1)^2}{u^3} - 8(n+2),$$

it follows that f is strictly convex on $(0, c]$ and strictly concave on $[c, \infty)$, where

$$c = \sqrt[3]{\frac{(n+1)^2}{4(n+2)}}.$$

In addition, $f(0+) = \infty$. According to LCRCF-Theorem and Note 6, it suffices to consider the case

$$a_1 = a_2 = \cdots = a_{n-1} = x, \quad a_n = n - (n-1)x, \quad 0 < x \leq 1,$$

when the inequality becomes as follows:

$$(n+1)^2 \left(\frac{n-1}{x} + \frac{1}{a_n} \right) \geq 4(n+2)[(n-1)x^2 + a_n^2] + n(n^2 - 3n - 6),$$

$$n(n-1)(2x-1)^2[(n+2)(n-1)x^2 - (n+2)(2n-1)x + (n+1)^2] \geq 0.$$

The last inequality is true since

$$(n-1)x^2 - (2n-1)x + \frac{(n+1)^2}{n+2} = (n-1) \left(x - \frac{2n-1}{2n-2} \right)^2 + \frac{3(n-2)}{4(n-1)(n+2)} \geq 0.$$

The equality holds for

$$a_1 = a_2 = \cdots = a_{n-1} = \frac{1}{2}, \quad a_n = \frac{n+1}{2}$$

(or any cyclic permutation).

□

P 1.83. If a, b, c, d, e are positive real numbers such that $a + b + c + d + e = 5$, then

$$27\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e}\right) \geq 4(a^3 + b^3 + c^3 + d^3 + e^3) + 115.$$

(Vasile Cîrtoaje)

Proof. Write the inequality as

$$f(a) + f(b) + f(c) + f(d) + f(e) \geq 5f(s), \quad s = \frac{a+b+c+d+e}{5} = 1,$$

where

$$f(u) = \frac{27}{u} - 4u^3, \quad 0 < u < 5.$$

From

$$f''(u) = \frac{6(9-4u^4)}{u^3},$$

it follows that f is convex on $(0, 1]$. According to LHCF-Theorem, it suffices to prove that

$$f(x) + 4f(y) \geq 5f(1)$$

for $x \geq 1 \geq y > 0$ and $x + 4y = 5$. This occurs if $h(x, y) \geq 0$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

Since

$$g(u) = -\frac{27}{u} - 4(u^2 + u + 1),$$

$$h(x, y) = \frac{A(x, y)}{xy}, \quad A(x, y) = 27 - 4xy(x + y + 1),$$

we need show that $A(x, y) \geq 0$. Indeed,

$$\begin{aligned} \frac{1}{3}A(x, y) &= 9 - 4y(4y - 5)(y - 2) = 9 - 40y + 52y^2 - 16y^3 \\ &= (1 - 2y)^2(9 - 4y) \geq 0. \end{aligned}$$

The equality holds for $a = b = c = d = e = 1$, and for $a = 3$ and $b = c = d = e = 1/2$ (or any cyclic permutation).

Generalization. If $a_1, a_2, \dots, a_n$ are positive real numbers such that

$$a_1 + a_2 + \dots + a_n = n,$$

then

$$(n+1)^2(2n-1)\left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} - n\right) \geq 27(n-1)^2(a_1^3 + a_2^3 + \dots + a_n^3 - n),$$

with equality for $a_1 = a_2 = \dots = a_n = 1$, and for

$$a_1 = \frac{2n-1}{3}, \quad a_2 = \dots = a_n = \frac{n+1}{3(n-1)}$$

(or any cyclic permutation).

□

P 1.84. If a, b, c are nonnegative real numbers so that $a + b + c = 12$, then

$$(a^2 + 10)(b^2 + 10)(c^2 + 10) \geq 13310.$$

(Vasile C., 2006)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) \geq 2 \ln 11 + \ln 110,$$

where

$$f(u) = \ln(u^2 + 10), \quad u \in [0, 12].$$

From

$$f''(u) = \frac{2(10 - u^2)}{(u^2 + 10)^2},$$

it follows that f is convex on $[0, \sqrt{10}]$ and concave on $[\sqrt{10}, 12]$. According to LCRCF-Theorem, the sum $f(a) + f(b) + f(c)$ is minimum when $a = b \leq c$. Therefore, it suffices to prove that $g(a) \geq 0$, where

$$g(a) = 2f(a) + f(c) - 2 \ln 11 - \ln 110, \quad c = 12 - 2a, \quad a \in [0, 4].$$

Since $c'(a) = -2$, we have

$$\begin{aligned} g'(a) &= 2f'(a) - 2f'(c) = 4 \left(\frac{a}{a^2 + 10} - \frac{c}{c^2 + 10} \right) \\ &= \frac{4(a - c)(10 - ac)}{(a^2 + 10)(c^2 + 10)} = \frac{24(4 - a)(5 - a)(a - 1)}{(a^2 + 10)(c^2 + 10)}. \end{aligned}$$

Therefore, $g'(a) < 0$ for $a \in [0, 1)$ and $g'(a) > 0$ for $a \in (1, 4)$, hence g is strictly decreasing on $[0, 1]$ and strictly increasing on $[1, 4]$. Thus, we have

$$g(a) \geq g(1) = 0.$$

The equality holds for $a = b = 1$ and $c = 10$ (or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

• Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = 2n(n - 1)$. If $k = (n - 1)(2n - 1)$, then

$$(a_1^2 + k)(a_2^2 + k) \cdots (a_n^2 + k) \geq k(k + 1)^n,$$

with equality for $a_1 = k$ and $a_2 = \dots = a_n = 1$ (or any cyclic permutation).

□

P 1.85. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$, then

$$(a_1^2 + 1)(a_2^2 + 1) \cdots (a_n^2 + 1) \geq \frac{(n^2 - 2n + 2)^n}{(n - 1)^{2n-2}}.$$

(Vasile C., 2006)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq \ln k, \quad k = \frac{(n^2 - 2n + 2)^n}{(n - 1)^{2n-2}},$$

where

$$f(u) = \ln(u^2 + 1), \quad u \in [0, n].$$

From

$$f''(u) = \frac{2(1 - u^2)}{(u^2 + 1)^2},$$

it follows that f is strictly convex on $[0, 1]$ and strictly concave on $[1, n]$. According to LCRCF-Theorem, it suffices to consider the case $a_1 = a_2 = \dots = a_{n-1} \leq a_n$; that is, to show that $g(x) \geq 0$, where

$$g(x) = (n - 1)f(x) + f(y) - \ln k, \quad y = n - (n - 1)x, \quad x \in [0, 1].$$

Since $y'(x) = -(n-1)$, we get

$$\begin{aligned} g'(x) &= (n-1)f'(x) - (n-1)f'(y) = (n-1)[f'(x) - f'(y)] \\ &= 2(n-1) \left(\frac{x}{x^2+1} - \frac{y}{y^2+1} \right) = \frac{2(n-1)(x-y)(1-xy)}{(x^2+1)(y^2+1)} \\ &= \frac{2n(n-1)(x-1)^2[(n-1)x-1]}{(x^2+1)(y^2+1)}. \end{aligned}$$

Therefore, $g'(x) \leq 0$ for $x \in \left[0, \frac{1}{n-1}\right]$ and $g'(x) \geq 0$ for $x \in \left[\frac{1}{n-1}, n\right]$, hence g is decreasing on $\left[0, \frac{1}{n-1}\right]$ and increasing on $\left[\frac{1}{n-1}, 1\right]$. Since $g\left(\frac{1}{n-1}\right) = 0$, the conclusion follows.

The equality holds for $a_1 = a_2 = \cdots = a_{n-1} = \frac{1}{n-1}$ and $a_n = n-1$ (or any cyclic permutation). □

P 1.86. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$(a^2 + 2)(b^2 + 2)(c^2 + 2) \leq 44.$$

(Vasile C., 2006)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) \leq \ln 44,$$

where

$$f(u) = \ln(u^2 + 2), \quad u \in [0, 3].$$

From

$$f''(u) = \frac{2(2 - u^2)}{(u^2 + 2)^2},$$

it follows that f is strictly convex on $[0, \sqrt{2}]$ and strictly concave on $[\sqrt{2}, 3]$. According to LCRCF-Theorem, the sum $f(a) + f(b) + f(c)$ is maximum for either $a = 0$ or $0 < a \leq b = c$.

Case 1: $a = 0$. We need to show that $b + c = 3$ involves

$$(b^2 + 2)(c^2 + 2) \leq 22,$$

which is equivalent to

$$bc(bc - 4) \leq 0.$$

This is true because

$$bc \leq \left(\frac{b+c}{2}\right)^2 = \frac{9}{4} < 4.$$

Case 2: $0 < a \leq b = c$. We need to show that $a + 2b = 3$ ($0 < a \leq 1$) involves

$$(a^2 + 2)(b^2 + 2)^2 \leq 44,$$

which is equivalent to $g(a) \leq 0$, where

$$g(a) = \ln(a^2 + 2) + 2\ln(b^2 + 2) - \ln 44, \quad b = \frac{3-a}{2}, \quad a \in (0, 1].$$

Since $b'(a) = -1/2$, we have

$$\begin{aligned} g'(a) &= \frac{2a}{a^2 + 2} - \frac{2b}{b^2 + 2} = \frac{2(a-b)(2-ab)}{(a^2 + 2)(b^2 + 2)} \\ &= \frac{3(a-1)(a^2 - 3a + 4)}{2(a^2 + 2)(b^2 + 2)}. \end{aligned}$$

Because

$$a^2 - 3a + 4 = (a-2)^2 + a > 0,$$

we have $g'(a) < 0$ for $a \in (0, 1)$, g is strictly decreasing on $[0, 1]$, hence it suffices to show that $g(0) \leq 0$. This reduces to $16 \cdot 22 \geq 17^2$, which is true because

$$16 \cdot 22 - 17^2 = 63 > 0.$$

The equality holds for $a = b = 0$ and $c = 3$ (or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization:

- Let a, b, c be nonnegative real numbers so that $a + b + c = 3$. If $k \geq \frac{9}{8}$, then

$$(a^2 + k)(b^2 + k)(c^2 + k) \leq k^2(k + 9),$$

with equality for $a = b = 0$ and $c = 3$ (or any cyclic permutation). If $k = 9/8$, then the equality holds also for $a = 0$ and $b = c = 3/2$ (or any cyclic permutation). □

P 1.87. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$(a^2 + 1)(b^2 + 1)(c^2 + 1) \leq \frac{169}{16}.$$

(Vasile C., 2006)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) \leq \ln 169 - \ln 16,$$

where

$$f(u) = \ln(u^2 + 1), \quad u \in [0, 3].$$

From

$$f''(u) = \frac{2(1 - u^2)}{(u^2 + 1)^2},$$

it follows that f is strictly convex on $[0, 1]$ and strictly concave on $[1, 3]$. According to LCRCF-Theorem, it suffices to consider the cases $a = 0$ and $0 < a \leq b = c$.

Case 1: $a = 0$. We need to show that $b + c = 3$ involves

$$(b^2 + 1)(c^2 + 1) \leq \frac{169}{16},$$

which is equivalent to

$$(4bc + 1)(4bc - 9) \leq 0.$$

This is true because

$$4bc \leq (b + c)^2 = 9.$$

Case 2: $0 < a \leq b = c$. We need to show that $a + 2b = 3$ ($0 < a \leq 1$) involves

$$(a^2 + 1)(b^2 + 1)^2 \leq \frac{169}{16},$$

which is equivalent to $g(a) \leq 0$, where

$$g(a) = \ln(a^2 + 1) + 2 \ln(b^2 + 1) - \ln 169 + \ln 16, \quad b = \frac{3-a}{2}, \quad a \in (0, 1].$$

Since $b'(a) = -1/2$, we have

$$\begin{aligned} g'(a) &= \frac{2a}{a^2 + 1} - \frac{2b}{b^2 + 1} = \frac{2(a-b)(1-ab)}{(a^2 + 1)(b^2 + 1)} \\ &= \frac{3(a-1)^2(a-2)}{2(a^2 + 1)(b^2 + 1)} \leq 0, \end{aligned}$$

hence g is strictly decreasing. Consequently, we have

$$g(a) < g(0) = 0.$$

The equality holds for $a = 0$ and $b = c = 3/2$ (or any cyclic permutation).

□

P 1.88. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$(2a^2 + 1)(2b^2 + 1)(2c^2 + 1) \leq \frac{121}{4}.$$

(Vasile C., 2006)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) \leq \ln 121 - \ln 4,$$

where

$$f(u) = \ln(2u^2 + 1), \quad u \in [0, 3].$$

From

$$f''(u) = \frac{4(1 - 2u^2)}{(2u^2 + 1)^2},$$

it follows that f is strictly convex on $[0, 1/\sqrt{2}]$ and strictly concave on $[1/\sqrt{2}, 3]$. By LCRCF-Theorem, it suffices to consider the cases $a = 0$ and $0 < a \leq b = c$.

Case 1: $a = 0$. We need to show that $b + c = 3$ involves

$$(2b^2 + 1)(2c^2 + 1) \leq \frac{121}{4},$$

which is equivalent to

$$(4bc + 5)(4bc - 9) \leq 0.$$

This is true because

$$4bc \leq (b + c)^2 = 9.$$

Case 2: $0 < a \leq b = c$. We need to show that $a + 2b = 3$ ($0 < a \leq 1$) involves

$$(2a^2 + 1)(2b^2 + 1)^2 \leq \frac{121}{4},$$

which is equivalent to $g(a) \leq 0$, where

$$g(a) = \ln(2a^2 + 1) + 2\ln(2b^2 + 1) - \ln 121 + \ln 4, \quad b = \frac{3-a}{2}, \quad a \in (0, 1].$$

Since $b'(a) = -1/2$, we have

$$\begin{aligned} g'(a) &= \frac{4a}{2a^2 + 1} - \frac{4b}{2b^2 + 1} = \frac{4(a-b)(1-2ab)}{(2a^2 + 1)(2b^2 + 1)} \\ &= \frac{6(a-1)(a^2 - 3a + 1)}{(2a^2 + 1)(2b^2 + 1)} \\ &= \frac{3(1-a)(3 + \sqrt{5} - 2a)(2a - 3 + \sqrt{5})}{2(2a^2 + 1)(2b^2 + 1)}, \end{aligned}$$

hence $g'\left(\frac{3-\sqrt{5}}{2}\right) = 0$, $g'(a) < 0$ for $a \in \left[0, \frac{3-\sqrt{5}}{2}\right)$, $g'(a) > 0$ for $a \in \left(\frac{3-\sqrt{5}}{2}, 1\right]$.

Therefore, g is strictly decreasing on $\left[0, \frac{3-\sqrt{5}}{2}\right]$ and strictly increasing on $\left[\frac{3-\sqrt{5}}{2}, 1\right]$. Since $g(0) = 0$, it suffices to show that $g(1) \leq 0$, which reduces to $27 \cdot 4 \leq 121$.

The equality holds for $a = 0$ and $b = c = 3/2$ (or any cyclic permutation).

□

P 1.89. If a, b, c are nonnegative real numbers so that $a + b + c \geq k_0$, where

$$k_0 = \frac{3}{8}\sqrt{66 + 10\sqrt{105}} \approx 4.867,$$

then

$$\sqrt[3]{(a^2 + 1)(b^2 + 1)(c^2 + 1)} \leq \left(\frac{a + b + c}{3}\right)^2 + 1.$$

(Vasile C., 2018)

Solution. Consider first the case $a + b + c = k_0$, and write the inequality as

$$f(a) + f(b) + f(c) \geq 3f(s), \quad s = \frac{a + b + c}{3} = \frac{k_0}{3},$$

where

$$f(u) = -\ln(u^2 + 1), \quad u \in [0, k_0].$$

For $u \in [s, k_0]$, $f(u)$ is convex because

$$f''(u) = \frac{6(3u^2 - 1)}{(3u^2 + 1)^2} > 0.$$

By the RHCF-Theorem, we only need to show that

$$f(x) + 2f(y) \geq 3f(s)$$

for $0 \leq x \leq s \leq y$ so that $x + 2y = 3s$; that is, to show that $g(x) \geq 0$ for $x \in [0, s]$, where

$$g(x) = f(x) + 2f(y) - 3f(s), \quad y = \frac{k_0 - x}{2}.$$

Since $y'(x) = -1/3$, we have

$$\begin{aligned} g'(x) &= f'(x) + 2y'f'(y) = \frac{-2x}{x^2 + 1} + \frac{2y}{y^2 + 1} \\ &= \frac{2(x - y)(xy - 1)}{(x^2 + 1)(y^2 + 1)} = \frac{3(s - x)(x^2 - k_0x + 2)}{2(x^2 + 1)(y^2 + 1)}. \end{aligned}$$

Since g is increasing on $[0, s_1]$ and decreasing on $[s_1, s]$, where $s_1 = \frac{k_0 - \sqrt{k_0^2 - 8}}{2}$, it suffices to show that $g(0) \geq 0$ and $g(s) \geq 0$. These inequalities are true because $g(0) = 0$ and $g(s) = 0$. The equality $g(0) = 0$ is equivalent to

$$\sqrt[3]{(y^2 + 1)^2} = \left(\frac{2y}{3}\right)^2 + 1,$$

where $y = \frac{k_0}{2}$.

According to RHCF-Theorem, if the inequality

$$f(a) + f(b) + f(c) \geq 3f\left(\frac{a + b + c}{3}\right)$$

holds for $a + b + c = k_0$, then it holds for $a + b + c > k_0$, too.

The equality holds for $a = b = c$. In addition, for $a + b + c = k_0$, the equality occurs again for $a = 0$ and $b = c = k_0/2$ (or any cyclic permutation).

□

P 1.90. If a, b, c, d are nonnegative real numbers so that $a + b + c + d = 4$, then

$$(a^2 + 3)(b^2 + 3)(c^2 + 3)(d^2 + 3) \leq 513.$$

(Vasile C., 2006)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) + f(d) \leq \ln 513,$$

where

$$f(u) = \ln(u^2 + 3), \quad u \in [0, 4].$$

From

$$f''(u) = \frac{2(3 - u^2)}{(u^2 + 3)^2},$$

it follows that f is strictly convex on $[0, \sqrt{3}]$ and strictly concave on $[\sqrt{3}, 4]$. By LCRCF-Theorem, it suffices to consider the cases $a = 0$ and $0 < a \leq b = c$.

Case 1: $a = 0$. We need to show that $b + c + d = 4$ involves

$$(b^2 + 3)(c^2 + 3)(d^2 + 3) \leq 171.$$

Substituting b, c, d by $4b/3, 4c/3, 4d/3$, respectively, we need to show that $b + c + d = 3$ involves

$$(b^2 + k)(c^2 + k)(d^2 + k) \leq k^2(k + 9),$$

where $k = 27/16$. According to Remark from the proof of P 1.86, this inequality holds for all $k \geq 9/8$.

Case 2: $0 < a \leq b = c = d$. We need to show that $a + 3b = 4$ ($0 < a \leq 1$) involves

$$(a^2 + 3)(b^2 + 3)^3 \leq 513,$$

which is equivalent to $g(a) \leq 0$, where

$$g(a) = \ln(a^2 + 3) + 3\ln(b^2 + 3) - \ln 513, \quad b = \frac{4 - a}{3}, \quad a \in (0, 1].$$

Since $b'(a) = -1/3$, we have

$$\begin{aligned} g'(a) &= \frac{2a}{a^2 + 3} - \frac{2b}{b^2 + 3} = \frac{2(a - b)(3 - ab)}{(a^2 + 3)(b^2 + 3)} \\ &= \frac{8(a - 1)(a^2 - 4a + 9)}{9(a^2 + 3)(b^2 + 3)}. \end{aligned}$$

Because

$$a^2 - 4a + 9 = (a - 2)^2 + 5 > 0,$$

we have $g'(a) > 0$ for $a \in [0, 1)$, g is strictly decreasing on $[0, 1]$, hence it suffices to show that $g(0) \leq 0$. This reduces to show that the original inequality holds for $a = 0$ and $b = c = d = 4/3$, which follows immediately from the case 1.

The equality holds for $a = b = c = 0$ and $d = 4$ (or any cyclic permutation).

□

P 1.91. If a, b, c, d are nonnegative real numbers so that $a + b + c + d = 4$, then

$$(a^2 + 2)(b^2 + 2)(c^2 + 2)(d^2 + 2) \leq 144.$$

(Vasile C., 2006)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) + f(d) \leq \ln 144,$$

where

$$f(u) = \ln(u^2 + 2), \quad u \in [0, 4].$$

From

$$f''(u) = \frac{2(2 - u^2)}{(u^2 + 2)^2},$$

it follows that f is strictly convex on $[0, \sqrt{2}]$ and strictly concave on $[\sqrt{2}, 4]$. By LCRCF-Theorem, it suffices to consider the cases $a = 0$ and $0 < a \leq b = c$.

Case 1: $a = 0$. We need to show that $b + c + d = 4$ involves

$$(b^2 + 2)(c^2 + 2)(d^2 + 2) \leq 72.$$

Substituting b, c, d by $4b/3, 4c/3, 4d/3$, respectively, we need to show that $b + c + d = 3$ involves

$$(8b^2 + 9)(8c^2 + 9)(8d^2 + 9) \leq 9^4.$$

This is true according to Remark from the proof of P 1.86.

Case 2: $0 < a \leq b = c = d$. We need to show that $a + 3b = 4$ ($0 < a \leq 1$) involves

$$(a^2 + 2)(b^2 + 2)^3 \leq 144,$$

which is equivalent to $g(a) \leq 0$, where

$$g(a) = \ln(a^2 + 2) + 3\ln(b^2 + 2) - \ln 144, \quad b = \frac{4 - a}{3}, \quad a \in (0, 1].$$

Since $b'(a) = -1/3$, we have

$$\begin{aligned} g'(a) &= \frac{2a}{a^2 + 2} - \frac{2b}{b^2 + 2} = \frac{2(a - b)(2 - ab)}{(a^2 + 2)(b^2 + 2)} \\ &= \frac{8(a - 1)(a^2 - 4a + 6)}{9(a^2 + 2)(b^2 + 2)}. \end{aligned}$$

Because

$$a^2 - 4a + 6 = (a - 2)^2 + 2 > 0,$$

we have $g'(a) > 0$ for $a \in [0, 1)$, g is strictly decreasing on $[0, 1]$, hence it suffices to show that $g(0) \leq 0$. This reduces to show that the original inequality holds for $a = 0$ and $b = c = d = 4/3$, which follows immediately from the case 1.

The equality holds for $a = b = c = 0$ and $d = 4$ (or any cyclic permutation), and also for $a = b = 0$ and $c = d = 2$ (or any permutation).

□

P 1.92. If a, b, c, d are nonnegative real numbers such that

$$a + b + c + d = 4,$$

then

$$\frac{a}{3a^3 + 2} + \frac{b}{3b^3 + 2} + \frac{c}{3c^3 + 2} + \frac{d}{3d^3 + 2} \leq \frac{4}{5}.$$

(Vasile Cîrtoaje, 2019)

Solution. Consider the function

$$f(u) = \frac{-u}{3u^3 + 2} : \mathbb{I} = [0, 4].$$

Since

$$f''(u) = \frac{18u^2(4 - 3u^3)}{(3u^3 + 2)^3}$$

is positive for $u \in [0, 1]$, f is left convex on $\mathbb{I}_{\leq 1}$. According to LHCF-Theorem and Note 1, it is enough to show that $h(x, y) \geq 0$ for $x, y \in [0, 4]$ such that $x + 3y = 4$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{3u^2 + 3u - 2}{3u^3 + 2},$$

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{2F(x, y)}{(3x^3 + 2)(3y^3 + 2)},$$

where

$$F(x, y) = 2(x^2 + xy + y^2) + 2(x + y) + 2 - 3x^2y^2 - 3xy(x + y).$$

From

$$4 = x + 3y \geq 2\sqrt{3xy},$$

we get $3xy \leq 4$. Thus, we have

$$F(x, y) \geq 2(x^2 + xy + y^2) + 2(x + y) + 2 - 4xy - 4(x + y) = 26(y - 1)^2 \geq 0.$$

The proof is completed. The equality occurs for $a = b = c = d = 1$.

□

P 1.93. If a, b, c, d are positive real numbers such that

$$\frac{1}{9a + 2} + \frac{1}{9b + 2} + \frac{1}{9c + 2} + \frac{1}{9d + 2} = \frac{4}{11},$$

then

$$3(a + b + c + d) + 20 \geq 8 \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right).$$

(V. Cîrtoaje, 2021)

Solution. Using the substitution

$$\frac{1}{9a+2} = \frac{a_1}{11}, \quad \frac{1}{9b+2} = \frac{a_2}{11}, \quad \dots,$$

that is

$$a = \frac{1}{9} \left(\frac{11}{a_1} - 2 \right), \quad b = \frac{1}{9} \left(\frac{11}{a_2} - 2 \right), \quad \dots,$$

the required inequality can be written as

$$f(a_1) + f(a_2) + f(a_3) + f(a_4) \geq 4f(s), \quad s = \frac{a_1 + a_2 + a_3 + a_4}{4} = 1,$$

where

$$f(u) = \frac{11}{u} - \frac{216u}{11-2u}, \quad u \in \left(0, \frac{11}{2}\right),$$

with

$$\frac{1}{22} f''(u) = \frac{1}{u^3} - \frac{432}{(11-2u)^3}.$$

For $u \in (0, 1]$, we have

$$\frac{1}{u^3} - \frac{432}{(11-2u)^3} \geq 1 - \frac{432}{9^3} > 0,$$

therefore f is convex on $(0, s]$. By the LHCF-Theorem, it suffices to show that

$$f(x) + 3f(y) \geq 4f(1),$$

where $x + 3y = 4$. According to Note 1, this is true if $h(x, y) \geq 0$ for $x + 3y = 4$. We have

$$\begin{aligned} g(u) &= \frac{f(u) - f(1)}{u - 1} = 11 \left(\frac{-1}{u} - \frac{24}{11-2u} \right), \\ h(x, y) &= \frac{g(x) - g(y)}{x - y} = 11 \left[\frac{1}{xy} - \frac{48}{(11-2x)(11-2y)} \right] \\ &= \frac{121[11 - 21(x+y) - 4xy]}{xy(11-2x)(11-2y)} = \frac{363(2y-1)^2}{xy(11-2x)(11-2y)} \geq 0. \end{aligned}$$

We have $f(a_1) + f(a_2) + f(a_3) + f(a_4) = 4f(1)$ for $a_1 = a_2 = a_3 = a_4 = 1$, and for $a_1 = \frac{5}{2}$ and $a_2 = a_3 = a_4 = \frac{1}{2}$ (or any cyclic permutation). Therefore, the original inequality is an equality for $a = b = c = d = 1$, and also for $a = \frac{4}{15}$ and $b = c = d = \frac{20}{9}$ (or any cyclic permutation). $\square$

P 1.94. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers such that $a_1 + a_2 + \dots + a_n = 1$, then

$$a_1^3 + a_2^3 + \dots + a_n^3 \leq \frac{1}{8} + a_1^4 + a_2^4 + \dots + a_n^4.$$

(Vasile C., 2018)

Solution. We use the induction method. For $n = 2$, denoting

$$a_1 a_2 = p, \quad p \leq 1/4,$$

we have

$$a_1^3 + a_2^3 = (a_1 + a_2)^3 - 3a_1 a_2 (a_1 + a_2) = 1 - 3p,$$

$$a_1^4 + a_2^4 = (a_1^2 + a_2^2)^2 - 2a_1^2 a_2^2 = 2p^2 - 4p + 1,$$

and the inequality is equivalent to

$$(4p - 1)^2 \geq 0.$$

Consider further that $n \geq 3$, $a_1 \leq a_2 \leq \dots \leq a_n$, and write the inequality as

$$f(a_1) + f(a_2) + \dots + f(a_n) \leq \frac{1}{8},$$

where

$$f(u) = u^3 - u^4, \quad u \in [0, 1].$$

From

$$f''(u) = 6u(1 - 2u),$$

it follows that f is strictly convex on $[0, 1/2]$ and strictly concave on $[1/2, 1]$. By LCRCF-Theorem, it suffices to consider the cases $a_1 = 0$ and $0 < a_1 \leq a_2 = \dots = a_n$.

Case 1: $a_1 = 0$. The inequality follows by the induction hypothesis.

Case 2: $0 < a_1 \leq a_2 = \dots = a_n$. We only need to prove the homogeneous inequality

$$8(a_1^4 + a_2^4 + \dots + a_n^4) + (a_1 + a_2 + \dots + a_n)^4 \geq 8(a_1 + a_2 + \dots + a_n)(a_1^3 + a_2^3 + \dots + a_n^3)$$

for $a_1 = x$ and $a_2 = \dots = a_{n-1} = 1$, that is

$$8(x^4 + n - 1) + (x + n - 1)^4 \geq 8(x + n - 1)(x^3 + n - 1),$$

$$x^4 - 4(n - 1)x^3 + 6(n - 1)^2 x^2 + 4(n - 1)(n^2 - 2n - 1)x + (n - 3)(n - 1)(n^2 - 5) \geq 0,$$

$$x^2(x - 2n + 2)^2 + 2(n - 1)^2 x^2 + 4(n - 1)(n^2 - 2n - 1)x + (n - 3)(n - 1)(n^2 - 5) \geq 0.$$

The equality holds for $a_1 = \dots = a_{n-2} = 0$ and $a_{n-1} = a_n = 1/2$ (or any permutation).

□

P 1.95. If $a_1, a_2, \dots, a_n$ ($n \geq 4$) are nonnegative real numbers such that

$$a_1 + a_2 + \dots + a_n = n,$$

then

$$n(n - 1)(2n - 1) \sum_{i=1}^n \frac{1}{a_i + n - 1} \geq a_1^2 + a_2^2 + \dots + a_n^2 + n^2(2n - 3).$$

(Vasile C., 2018)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{n(n-1)(2n-1)}{u+n-1} - u^2, \quad u \in [0, n].$$

For $u \leq 1$, we have

$$\frac{1}{2}f''(u) = \frac{n(n-1)(2n-1)}{(u+n-1)^3} - 1 \geq \frac{(n-1)(2n-1)}{n^2} = \frac{n^2 - 3n + 1}{n^2} > 0,$$

therefore f is convex on $[0, s]$. By the LHCF-Theorem, it suffices to show that

$$f(x) + (n-1)f(y) \geq nf(1),$$

where $x \geq 1 \geq y$ and $x + (n-1)y = n$. Write this inequality as follows:

$$n(n-1)(2n-1) \left(\frac{1}{x+n-1} + \frac{n-1}{y+n-1} - 1 \right) - [x^2 + (n-1)y^2 - n] \geq 0,$$

$$\frac{n(n-1)^2(2n-1)(y-1)^2}{[2n-1-(n-1)y](y+n-1)} - n(n-1)(y-1)^2 \geq 0,$$

$$n(n-1)y(y-1)^2[(n-1)y + n^2 - 4n + 2] \geq 0.$$

Clearly, the last inequality is true for $n \geq 4$.

The equality occurs for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = a_2 = \cdots = a_{n-1} = 0, \quad a_n = n$$

(or any cyclic permutation).

Remark. The inequality is equivalent to

$$(2n-1) \sum_{i=1}^n \frac{1}{a_i + n-1} + \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} a_i a_j \geq 2n.$$

□

P 1.96. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers such that

$$a_1 + a_2 + \cdots + a_n = n,$$

then

$$n(n^2 - n + 1) \sum_{i=1}^n \frac{1}{a_i + n-1} \leq a_1^2 + a_2^2 + \cdots + a_n^2 + n^2(n-1).$$

(Vasile C. and L. Giugiuc, 2021)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = u^2 - \frac{n(n^2 - n + 1)}{u + n - 1}, \quad u \in [0, n].$$

For $u \geq 1$, we have

$$\frac{1}{2}f''(u) = 1 - \frac{n(n^2 - n + 1)}{(u + n - 1)^3} \geq 1 - \frac{n^2 - n + 1}{n^2} = \frac{n - 1}{n^2} > 0,$$

therefore f is convex on $[s, n]$. By the RHCF-Theorem, it suffices to show that

$$f(x) + (n - 1)f(y) \geq nf(1),$$

where

$$x + (n - 1)y = n, \quad x \leq 1 \leq y \leq \frac{n}{n - 1}.$$

Write this inequality as follows:

$$x^2 + (n - 1)y^2 - n - n(n^2 - n + 1) \left(\frac{1}{x + n - 1} + \frac{n - 1}{y + n - 1} - 1 \right) \geq 0,$$

$$n(n - 1)(y - 1)^2 - \frac{n(n - 1)(n^2 - n + 1)(y - 1)^2}{[2n - 1 - (n - 1)y](y + n - 1)} \geq 0,$$

$$n(n - 1)(y - 1)^2[n - (n - 1)y](y + n - 2) \geq 0.$$

Clearly, the last inequality is true.

The equality occurs for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = 0$ and $a_2 = a_3 = \cdots = a_n = \frac{n}{n - 1}$ (or any cyclic permutation).

Remark. The inequality is equivalent to

$$(n^2 - n + 1) \sum_{i=1}^n \frac{1}{a_i + n - 1} + \frac{2}{n} \sum_{1 \leq i < j \leq n} a_i a_j \leq n^2.$$

□

P 1.97. If $a_1, a_2, \dots, a_n$ ($n \geq 4$) are nonnegative real numbers such that

$$\frac{1}{a_1 + n - 1} + \frac{1}{a_2 + n - 1} + \cdots + \frac{1}{a_n + n - 1} \geq 1,$$

then

$$a_1^2 + a_2^2 + \cdots + a_n^2 - n \geq \frac{n^2 - n - 1}{n - 2}(a_1 + a_2 + \cdots + a_n - n).$$

(L. Giugiuc and V. Cîrtoaje, 2021)

Solution. Using the substitution

$$\frac{1}{a_i + n - 1} = \frac{x_i}{n}, \quad i = 1, 2, \dots, n,$$

that is

$$a_i = \frac{n}{x_i} - n + 1, \quad 0 < x_i \leq \frac{n}{n-1}, \quad i = 1, 2, \dots, n,$$

the required inequality can be written as

$$f(x_1) + f(x_2) + \dots + f(x_n) \geq nf(s), \quad s = \frac{x_1 + x_2 + \dots + x_n}{n} = 1,$$

where

$$f(u) = \frac{n}{u^2} - \frac{3n^2 - 7n + 3}{(n-2)u}, \quad u \in \left(0, \frac{n}{n-1}\right],$$

with

$$\frac{1}{2}f''(u) = \frac{1}{u^4} \left(3n - \frac{3n^2 - 7n + 3}{n-2}u\right).$$

For $u \in (0, 1]$, we have

$$3n - \frac{3n^2 - 7n + 3}{n-2}u \geq 3n - \frac{3n^2 - 7n + 3}{n-2} = \frac{n-3}{n-2} > 0,$$

therefore f is convex on $(0, s]$. By the LHCF-Theorem, it suffices to show that

$$f(x) + (n-1)f(y) \geq nf(1),$$

where

$$x + (n-1)y = n, \quad y = \frac{n-x}{n-1}, \quad \frac{n}{n-1} \geq x \geq 1 \geq y > 0.$$

Write this inequality as follows:

$$\begin{aligned} n \left(\frac{1}{x^2} + \frac{n-1}{y^2} - n \right) - \frac{3n^2 - 7n + 3}{n-2} \left(\frac{1}{x} + \frac{n-1}{y} - n \right) &\geq 0, \\ \frac{n^2(x-1)^2[n+2(n-1)x-x^2]}{x^2(n-x)^2} - \frac{n(3n^2-7n+3)(x-1)^2}{(n-2)x(n-x)} &\geq 0. \end{aligned}$$

The inequality is true if

$$\frac{n[n+2(n-1)x-x^2]}{x(n-x)} - \frac{3n^2-7n+3}{n-2} \geq 0,$$

which is equivalent to

$$[n - (n-1)x][n(n-2) - (2n-3)x] \geq 0.$$

It is true for $n \geq 4$ since

$$n(n-2) - (2n-3)x \geq n(n-2) - \frac{n(2n-3)}{n-1} = \frac{n^2-5n+5}{n-1} > 0.$$

The proof is completed. The equality occurs for $a_1 = a_2 = \dots = a_n = 1$, and also for $a_1 = a_2 = \dots = a_{n-1} = \frac{n-1}{n-2}$ and $a_n = 0$ (or any cyclic permutation). □

P 1.98. Let a, b, c be nonnegative real numbers, no two of which are zero, and

$$x = \frac{2a}{b+c}, \quad y = \frac{2b}{c+a}, \quad z = \frac{2c}{a+b}.$$

Prove that

$$\frac{1}{\sqrt{5x+4}} + \frac{1}{\sqrt{5y+4}} + \frac{1}{\sqrt{5z+4}} \geq 1.$$

(V. Cîrtoaje, 2021)

Solution. Due to homogeneity, we may assume $a+b+c=3$, when

$$x = \frac{2a}{3-a}, \quad y = \frac{2b}{3-b}, \quad z = \frac{2c}{3-c}.$$

Thus, we need to show that

$$f(a) + f(b) + f(c) \geq 3f(s),$$

where

$$s = \frac{a+b+c}{3} = 1$$

and

$$f(u) = \sqrt{\frac{3-u}{2+u}}, \quad u \in [0, 3).$$

From

$$f''(u) = \frac{5(7-4u)}{4\sqrt{(2+u)^5(3-u)^3}},$$

it follows that f is convex on $[0, s]$. By the LHCF-Theorem, it suffices to consider the case where two of a, b, c are equal. Due to homogeneity, we may set $b=c=1$, when

$$x = a, \quad y = z = \frac{2}{a+1}.$$

So, we need to show that

$$\frac{1}{\sqrt{5a+4}} + 2\sqrt{\frac{a+1}{4a+14}} \geq 1.$$

Using the substitution

$$5a+4 = 9t^2, \quad t \geq \frac{2}{3},$$

the inequality becomes

$$\begin{aligned} \frac{1}{3t} + \frac{1}{3}\sqrt{\frac{2(9t^2+1)}{2t^2+3}} &\geq 1, \\ \sqrt{\frac{2(9t^2+1)}{2t^2+3}} &\geq \frac{3t-1}{t}. \end{aligned}$$

By squaring, the inequality becomes

$$\begin{aligned} 4t^3 - 9t^2 + 6t - 1 &\geq 0, \\ (t-1)^2(4t-1) &\geq 0. \end{aligned}$$

The equality occurs for $a=b=c$.

□

P 1.99. If a, b, c, d are positive real numbers such that $abcd = 1$, then

$$a^3 + b^3 + c^3 + d^3 + 4 \geq 2(a^2 + b^2 + c^2 + d^2).$$

(V. Cîrtoaje, 2021)

Solution. Using the substitutions

$$a = e^x, \quad b = e^y, \quad c = e^z, \quad d = e^w,$$

we need to show that

$$f(x) + f(y) + f(z) + f(w) \geq 4f(s), \quad s = \frac{x + y + z + w}{4} = 0,$$

where

$$f(u) = e^{3u} - 2e^{2u}, \quad u \in \mathbb{R}.$$

For $u \geq 0$, we have

$$f''(u) = e^{2u}(9e^u - 8) > 0,$$

hence f is convex for $u \geq 0$. By the RHCF-Theorem and Note 2, it suffices to show that $H(x, y) \geq 0$ for $x, y \in \mathbb{R}$ so that $x + 3y = 0$, where

$$H(x, y) = \frac{f'(x) - f'(y)}{x - y}.$$

From

$$f'(u) = 3e^{3u} - 4e^{2u},$$

we get

$$H(x, y) = \frac{e^x - e^y}{x - y} \cdot [3(e^{2x} + e^{x+y} + e^{2y}) - 4(e^x + e^y)].$$

Since $(e^x - e^y)/(x - y) > 0$, we need to prove that

$$3(e^{2x} + e^{x+y} + e^{2y}) - 4(e^x + e^y) \geq 0$$

for $x + 3y = 0$. This is true if

$$3(a^2 + ab + b^2) - 4(a + b) \geq 0$$

for $ab^3 = 1$, $a, b > 0$. The inequality is equivalent to

$$3(b^8 + b^4 + 1) - 4b^3(b^4 + 1) \geq 0,$$

$$3b^8 \geq (b^4 + 1)(4b^3 - 3).$$

Since

$$4b^3 \leq 2b^4 + 2b^2,$$

it suffices to show that

$$3b^8 \geq (b^4 + 1)(2b^4 + 2b^2 - 3).$$

Denoting $b^2 = x$, the inequality becomes

$$3x^4 \geq (x^2 + 1)(2x^2 + 2x - 3),$$

$$x^4 - 2x^3 + x^2 - 2x + 3 \geq 0,$$

$$x^2(x - 1)^2 - 2x + 3 \geq 0.$$

For $x \leq 1$, we have

$$x^2(x - 1)^2 - 2x + 3 \geq -2x + 3 > 0,$$

and for $x \geq 1$, we have

$$x^2(x - 1)^2 - 2x + 3 \geq (x - 1)^2 - 2x + 3 = (x - 2)^2 \geq 0.$$

The equality occurs for $a = b = c = d = 1$.

□

P 1.100. If $a, b, c, d \in [-1, 1]$ such that $a + b + c + d = 1$, then

$$\frac{-1}{9} \leq a^3 + b^3 + c^3 + d^3 \leq \frac{7}{4}.$$

(V. Cîrtoaje, 2021)

Solution. We will apply LCRCF-Theorem to $f(x) = -x^3$, which is convex on $[-1, 0]$ and concave on $[0, 1]$. In addition, assume that $a \leq b \leq c \leq d$.

I. Write the right inequality as $f(a) + f(b) + f(c) + f(d) \geq \frac{-7}{4}$. According to Note 7, it suffices to consider the cases $d = 1$ and $a = b = c \leq d$.

Case 1: $d = 1$. We need to show that $a^3 + b^3 + c^3 \leq \frac{3}{4}$ for $a + b + c = 0$. Since

$$a + b = -c \in [-1, 0],$$

we have

$$\frac{3}{4} - a^3 - b^3 - c^3 = \frac{3}{4} - a^3 - b^3 + (a + b)^3 = \frac{3}{4} + 3ab(a + b) \geq \frac{3}{4} + \frac{3}{4}(a + b)^3 \geq \frac{3}{4} - \frac{3}{4} = 0.$$

Case 2: $a = b = c \leq d$. We need to show that $3a^3 + d^3 \leq \frac{7}{4}$ for $3a + d = 1$. We have $3a = 1 - d \geq 0$, hence

$$\frac{7}{4} - 3a^3 - d^3 = \frac{7}{4} - 3a^3 - (1 - 3a)^3 = 24a^3 - 27a^2 + 9a + \frac{3}{4} > 3a(8a^2 - 9a + 3) \geq 0.$$

The equality occurs for $a = b = \frac{-1}{2}$ and $c = d = 1$ (or any permutation).

II. Write the left inequality as $f(a) + f(b) + f(c) + f(d) \leq \frac{1}{9}$. According to LCRCF-Theorem, it suffices to consider the cases $a = -1$ and $a \leq b = c = d$.

Case 1: $a = -1$. We need to show that $b^3 + c^3 + d^3 \geq \frac{8}{9}$ for $b + c + d = 2$. According to LCRCF-Theorem, it suffices to consider the cases $b = -1$ and $b \leq c = d$. For $b = -1$, we get $c + d = 3$, which is not possible. For $b \leq c = d$, we need to show that $b^3 + 2c^3 \geq \frac{8}{9}$ for $b + 2c = 2$. We have $b = 2(1 - c)$, hence

$$b^3 + 2c^3 - \frac{8}{9} = 8(1 - c)^3 + 2c^3 - \frac{8}{9} = \frac{2}{9}(32 - 108c + 108c^2 - 27c^3) = \frac{2}{9}(2 - 3c)^2(8 - 3c) \geq 0.$$

Case 2: $a \leq b = c = d$. We need to show that $a^3 + 3b^3 + \frac{1}{9} \geq 0$ for $a + 3b = 1$. From $1 = a + 3b \leq 4b$ and $3b = 1 - a \leq 2$, we have $\frac{1}{4} \leq b \leq \frac{2}{3}$, hence

$$\begin{aligned} a^3 + 3b^3 + \frac{1}{9} &= (1 - 3b)^3 + 3b^3 + \frac{1}{9} = \frac{1}{9}(10 - 81b + 243b^2 - 216b^3) = \frac{1}{9}(2 - 3b)(5 - 33b + 72b^2) \\ &\geq \frac{1}{9}(2 - 3b)(4 - 34b + 72b^2) = \frac{2}{9}(2 - 3b)(4b - 1)(9b - 2) \geq 0. \end{aligned}$$

The equality occurs for $a = -1$ and $b = c = d = \frac{2}{3}$ (or any permutation). □

P 1.101. Prove that $\frac{19}{4}$ is the minimum positive value of the constant k such that

$$\sqrt[3]{\frac{b+c}{ka+b+c}} + \sqrt[3]{\frac{c+a}{kb+c+a}} + \sqrt[3]{\frac{a+b}{kc+a+b}} \geq 3\sqrt[3]{\frac{2}{k+2}}$$

holds for any nonnegative real numbers a, b, c with $a + b + c > 0$.

(V. Cîrtoaje, 2024)

Solution. For $b = c = 0$, the inequality becomes $2 \geq 3\sqrt[3]{\frac{2}{k+2}}$, which is equivalent to $k \geq \frac{19}{4}$.

To show that $\frac{19}{4}$ is the minimum value of the constant k , we need to prove the inequality

$$\sqrt[3]{\frac{b+c}{19a+4b+4c}} + \sqrt[3]{\frac{c+a}{19b+4c+4a}} + \sqrt[3]{\frac{a+b}{19c+4a+4b}} \geq \sqrt[3]{2}.$$

Due to homogeneity, we may assume that $a + b + c = 3$. Thus, we need to show that

$$f(a) + f(b) + f(c) \geq 3f(s),$$

where

$$s = \frac{a+b+c}{3} = 1$$

and

$$f(u) = \sqrt[3]{\frac{3-u}{4+5u}}, \quad u \in [0, 3].$$

From

$$f''(u) = \frac{38(26-15u)}{3(4+5u)^{7/3}(3-u)^{5/3}},$$

it follows that f is convex on $[0, s]$. By the LHCF-Theorem, it suffices to consider the case where $a \geq b = c$. Due to homogeneity, we may set $a = 1$, hence $b = c := x \leq 1$. So, we need to show that

$$\sqrt[3]{\frac{2x}{8x+19}} + 2\sqrt[3]{\frac{x+1}{23x+4}} \geq \sqrt[3]{2}.$$

Denoting

$$t = \sqrt[3]{\frac{27x}{8x+19}} \leq 1,$$

the inequality becomes

$$t + \sqrt[3]{\frac{4(11t^3+27)}{15t^3+4}} \geq 3,$$

$$\frac{4(11t^3+27)}{15t^3+4} \geq (3-t)^3,$$

$$t(5t^5 - 45t^4 + 135t^3 - 119t^2 - 12t + 36) \geq 0,$$

$$t(t-1)^2(5t^3 - 35t^2 + 60t + 36) \geq 0.$$

It is true because

$$5t^3 - 35t^2 + 60t + 36 > 35t(1-t) \geq 0.$$

The equality occurs for $a = b = c$, and also for $b = c = 0$, or $c = a = 0$, or $a = b = 0$.

□

Chapter 2

Half Convex Function Method for Ordered Variables

2.1 Theoretical Basis

The following statement is known as the Right Half Convex Function Theorem for Ordered Variables (RHCF-OV Theorem).

RHCF-OV Theorem (*Vasile Cîrtoaje, 2008*). *Let f be a real function defined on an interval $\mathbb{I}$ and convex on $\mathbb{I}_{\geq s}$, where $s \in \text{int}(\mathbb{I})$. The inequality*

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \dots, a_n \in \mathbb{I}$ satisfying

$$a_1 + a_2 + \cdots + a_n = ns$$

and

$$a_1 \leq a_2 \leq \cdots \leq a_m \leq s, \quad m \in \{1, 2, \dots, n-1\},$$

if and only if

$$f(x) + (n-m)f(y) \geq (1+n-m)f(s)$$

for all $x, y \in \mathbb{I}$ so that

$$x \leq s \leq y, \quad x + (n-m)y = (1+n-m)s.$$

Proof. For

$$a_1 = x, \quad a_2 = \cdots = a_m = s, \quad a_{m+1} = \cdots = a_n = y,$$

the inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s)$$

becomes

$$f(x) + (n-m)f(y) \geq (1+n-m)f(s);$$

thus, the necessity is proved. To prove the sufficiency, we assume that

$$a_1 \leq a_2 \leq \cdots \leq a_n.$$

From $a_1 \leq a_2 \leq \dots \leq a_m \leq s$, it follows that there is an integer

$$k \in \{m, m+1, \dots, n-1\}$$

so that

$$a_1 \leq \dots \leq a_k \leq s \leq a_{k+1} \leq \dots \leq a_n.$$

Since f is convex on $\mathbb{I}_{\geq s}$, we may apply Jensen's inequality to get

$$f(a_{k+1}) + \dots + f(a_n) \geq (n-k)f(z),$$

where

$$z = \frac{a_{k+1} + \dots + a_n}{n-k}, \quad z \in \mathbb{I}.$$

Therefore, to prove the desired inequality

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq f(s),$$

it suffices to show that

$$f(a_1) + \dots + f(a_k) + (n-k)f(z) \geq nf(s). \quad (*)$$

Let $b_1, \dots, b_k$ be defined by

$$a_i + (n-m)b_i = (1+n-m)s, \quad i = 1, \dots, k.$$

We claim that

$$z \geq b_1 \geq \dots \geq b_k \geq s, \quad b_1, \dots, b_k \in \mathbb{I}.$$

Indeed, we have

$$\begin{aligned} b_1 &\geq \dots \geq b_k, \\ b_k - s &= \frac{s - a_k}{n-m} \geq 0, \end{aligned}$$

and

$$z \geq b_1$$

because

$$\begin{aligned} (n-m)b_1 &= (1+n-m)s - a_1 \\ &= -(m-1)s + (a_2 + \dots + a_k) + (a_{k+1} + \dots + a_n) \\ &\leq -(m-1)s + (k-1)s + (a_{k+1} + \dots + a_n) = \\ &= (k-m)s + (n-k)z \leq (n-m)z. \end{aligned}$$

Since $b_1, \dots, b_k \in \mathbb{I}_{\geq s}$, by hypothesis we have

$$f(a_1) + (n-m)f(b_1) \geq (1+n-m)f(s),$$

...

$$f(a_k) + (n-m)f(b_k) \geq (1+n-m)f(s),$$

hence

$$\begin{aligned} f(a_1) + \cdots + f(a_k) + (n-m)[f(b_1) + \cdots + f(b_k)] &\geq k(1+n-m)f(s), \\ f(a_1) + \cdots + f(a_k) &\geq k(1+n-m)f(s) - (n-m)[f(b_1) + \cdots + f(b_k)]. \end{aligned}$$

According to this result, the inequality (*) is true if

$$k(1+n-m)f(s) - (n-m)[f(b_1) + \cdots + f(b_k)] + (n-k)f(z) \geq nf(s),$$

which is equivalent to

$$pf(z) + (k-p)f(s) \geq f(b_1) + \cdots + f(b_k), \quad p = \frac{n-k}{n-m} \leq 1.$$

By Jensen's inequality, we have

$$pf(z) + (1-p)f(s) \geq f(w), \quad w = pz + (1-p)s \geq s.$$

Thus, we only need to show that

$$f(w) + (k-1)f(s) \geq f(b_1) + \cdots + f(b_k).$$

Since the decreasingly ordered vector $\vec{A}_k = (w, s, \dots, s)$ majorizes the decreasingly ordered vector $\vec{B}_k = (b_1, b_2, \dots, b_k)$, this inequality follows from Karamata's inequality for convex functions.

Similarly, we can prove the Left Half Convex Function Theorem for Ordered Variables (LHCF-OV Theorem).

LHCF-OV Theorem. *Let f be a real function defined on an interval $\mathbb{I}$ and convex on $\mathbb{I}_{\leq s}$, where $s \in \text{int}(\mathbb{I})$. The inequality*

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \dots, a_n \in \mathbb{I}$ satisfying

$$a_1 + a_2 + \cdots + a_n = ns$$

and

$$a_1 \geq a_2 \geq \cdots \geq a_m \geq s, \quad m \in \{1, 2, \dots, n-1\},$$

if and only if

$$f(x) + (n-m)f(y) \geq (1+n-m)f(s)$$

for all $x, y \in \mathbb{I}$ so that

$$x \geq s \geq y, \quad x + (n-m)y = (1+n-m)s.$$

From the RHCF-OV Theorem and the LHCF-OV Theorem, we find the HCF-OV Theorem (Half Convex Function Theorem for Ordered Variables).

HCF-OV Theorem. *Let f be a real function defined on an interval $\mathbb{I}$ and convex on $\mathbb{I}_{\geq s}$ (or $\mathbb{I}_{\leq s}$), where $s \in \text{int}(\mathbb{I})$. The inequality*

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \dots, a_n \in \mathbb{I}$ so that

$$a_1 + a_2 + \dots + a_n = ns$$

and at least m of $a_1, a_2, \dots, a_n$ are smaller (greater) than s , where $m \in \{1, 2, \dots, n-1\}$, if and only if

$$f(x) + (n-m)f(y) \geq (1+n-m)f(s)$$

for all $x, y \in \mathbb{I}$ satisfying $x + (n-m)y = (1+n-m)s$.

The RHCF-OV Theorem, the LHCF-OV Theorem and the HCF-OV Theorem are respectively generalizations of the RHCF-Theorem, the LHCF Theorem and the HCF-Theorem, because the last theorems can be obtained from the first theorems for $m = 1$.

Note 1. Let us denote

$$g(u) = \frac{f(u) - f(s)}{u - s}, \quad h(x, y) = \frac{g(x) - g(y)}{x - y}.$$

In many applications, it is useful to replace the hypothesis

$$f(x) + (n-m)f(y) \geq (1+n-m)f(s)$$

in the RHCF-OV Theorem and the LHCF-OV Theorem by the equivalent condition

$$h(x, y) \geq 0 \quad \text{for all } x, y \in \mathbb{I} \quad \text{so that } x + (n-m)y = (1+n-m)s.$$

This equivalence is true since

$$\begin{aligned} f(x) + (n-m)f(y) - (1+n-m)f(s) &= [f(x) - f(s)] + (n-m)[f(y) - f(s)] \\ &= (x-s)g(x) + (n-m)(y-s)g(y) \\ &= \frac{n-m}{1+n-m}(x-y)[g(x) - g(y)] \\ &= \frac{n-m}{1+n-m}(x-y)^2 h(x, y). \end{aligned}$$

Note 2. Assume that f is differentiable on $\mathbb{I}$, and let

$$H(x, y) = \frac{f'(x) - f'(y)}{x - y}.$$

The desired inequality of Jensen's type in the RHCF-OV Theorem and the LHCF-OV Theorem holds true by replacing the hypothesis

$$f(x) + (n-m)f(y) \geq (1+n-m)f(s)$$

with the more restrictive condition

$$H(x, y) \geq 0 \quad \text{for all } x, y \in \mathbb{I} \quad \text{so that } x + (n-m)y = (1+n-m)s.$$

To prove this, we will show that the new condition implies

$$f(x) + (n-m)f(y) \geq (1+n-m)f(s)$$

for all $x, y \in \mathbb{I}$ so that $x + (n - m)y = (1 + n - m)s$. Write this inequality as

$$f_1(x) \geq (1 + n - m)f(s),$$

where

$$f_1(x) = f(x) + (n - m)f\left(\frac{(1 + n - m)s - x}{n - m}\right).$$

From

$$\begin{aligned} f'_1(x) &= f'(x) - f'\left(\frac{(1 + n - m)s - x}{n - m}\right) \\ &= f'(x) - f'(y) \\ &= \frac{1 + n - m}{n - m}(x - s)H(x, y), \end{aligned}$$

it follows that f_1 is decreasing on $\mathbb{I}_{\leq s}$ and increasing on $\mathbb{I}_{\geq s}$; therefore,

$$f_1(x) \geq f_1(s) = (1 + n - m)f(s).$$

Note 3. The RHCF-OV Theorem and the LHCF-OV Theorem are also valid in the case when f is defined on $\mathbb{I} \setminus \{u_0\}$, where $u_0 \in \mathbb{I}_{< s}$ for the RHCF-OV Theorem, and $u_0 \in \mathbb{I}_{> s}$ for the LHCF-OV Theorem.

Note 4. The desired inequalities in the RHCF-OV Theorem and the LHCF-OV Theorem become equalities for

$$a_1 = a_2 = \cdots = a_n = s.$$

In addition, if there exist $x, y \in \mathbb{I}$ so that

$$x + (n - m)y = (1 + n - m)s, \quad f(x) + (n - m)f(y) = (1 + n - m)f(s), \quad x \neq y,$$

then the equality holds also for

$$a_1 = x, \quad a_2 = \cdots = a_m = s, \quad a_{m+1} = \cdots = a_n = y$$

Notice that these equality conditions are equivalent to

$$x + (n - m)y = (1 + n - m)s, \quad h(x, y) = 0$$

($x < y$ for the RHCF-OV Theorem, and $x > y$ for the LHCF-OV Theorem).

Note 5. The WRHCF-OV Theorem and the WLHCF-OV Theorem are extensions of the *weighted* Jensen's inequality to right and left half convex functions with ordered variables (Vasile Cîrtoaje, 2008).

WRHCF-OV Theorem. Let $p_1, p_2, \dots, p_n$ be positive real numbers so that

$$p_1 + p_2 + \cdots + p_n = 1,$$

and let f be a real function defined on an interval $\mathbb{I}$ and convex on $\mathbb{I}_{\geq s}$, where $s \in \text{int}(\mathbb{I})$. The inequality

$$p_1 f(x_1) + p_2 f(x_2) + \cdots + p_n f(x_n) \geq f(p_1 x_1 + p_2 x_2 + \cdots + p_n x_n)$$

holds for all $x_1, x_2, \dots, x_n \in \mathbb{I}$ so that $p_1x_1 + p_2x_2 + \dots + p_nx_n = s$ and

$$x_1 \leq x_2 \leq \dots \leq x_n, \quad x_m \leq s, \quad m \in \{1, 2, \dots, n-1\},$$

if and only if

$$f(x) + kf(y) \geq (1+k)f(s)$$

for all $x, y \in \mathbb{I}$ satisfying

$$x \leq s \leq y, \quad x + ky = (1+k)s,$$

where

$$k = \frac{p_{m+1} + p_{m+2} + \dots + p_n}{p_1}.$$

WLHCF-OV Theorem. Let $p_1, p_2, \dots, p_n$ be positive real numbers so that

$$p_1 + p_2 + \dots + p_n = 1,$$

and let f be a real function defined on an interval $\mathbb{I}$ and convex on $\mathbb{I}_{\leq s}$, where $s \in \text{int}(\mathbb{I})$. The inequality

$$p_1f(x_1) + p_2f(x_2) + \dots + p_nf(x_n) \geq f(p_1x_1 + p_2x_2 + \dots + p_nx_n)$$

holds for all $x_1, x_2, \dots, x_n \in \mathbb{I}$ so that $p_1x_1 + p_2x_2 + \dots + p_nx_n = s$ and

$$x_1 \geq x_2 \geq \dots \geq x_n, \quad x_m \geq s, \quad m \in \{1, 2, \dots, n-1\},$$

if and only if

$$f(x) + kf(y) \geq (1+k)f(s)$$

for all $x, y \in \mathbb{I}$ satisfying

$$x \geq s \geq y, \quad x + ky = (1+k)s,$$

where

$$k = \frac{p_{m+1} + p_{m+2} + \dots + p_n}{p_1}.$$

2.2 Applications

2.1. Find the least value $k_0 \in [1, \infty)$ of the constant k such that

$$\frac{1}{ka^2 + b + c} + \frac{1}{kb^2 + c + a} + \frac{1}{kc^2 + a + b} \geq \frac{3}{k+2}$$

for any nonnegative real numbers a, b, c with at most one of them larger than 1 and $a + b + c = 3$.

2.2. If a, b, c, d are real numbers so that

$$a \leq b \leq 1 \leq c \leq d, \quad a + b + c + d = 4,$$

then

$$(3a^2 - 2)(a - 1)^2 + (3b^2 - 2)(b - 1)^2 + (3c^2 - 2)(c - 1)^2 + (3d^2 - 2)(d - 1)^2 \geq 0.$$

2.3. If a, b, c, d are nonnegative real numbers so that

$$a \geq b \geq 1 \geq c \geq d, \quad a + b + c + d = 4,$$

then

$$\frac{1}{2a^3 + 5} + \frac{1}{2b^3 + 5} + \frac{1}{2c^3 + 5} + \frac{1}{2d^3 + 5} \leq \frac{4}{7}.$$

2.4. If

$$\frac{-2n-1}{n-1} \leq a_1 \leq \cdots \leq a_n \leq 1 \leq a_{n+1} \leq \cdots \leq a_{2n}, \quad a_1 + a_2 + \cdots + a_{2n} = 2n,$$

then

$$a_1^3 + a_2^3 + \cdots + a_{2n}^3 \geq 2n.$$

2.5. Let $a_1, a_2, \dots, a_n$ ($n \geq 3$) be real numbers so that $a_1 + a_2 + \cdots + a_n = n$. Prove that

(a) if $-3 \leq a_1 \leq \cdots \leq a_{n-2} \leq 1 \leq a_{n-1} \leq a_n$, then

$$a_1^3 + a_2^3 + \cdots + a_n^3 \geq a_1^2 + a_2^2 + \cdots + a_n^2;$$

(b) if $-\frac{n-1}{n-3} \leq a_1 \leq a_2 \leq 1 \leq \cdots \leq a_n$, then

$$a_1^3 + a_2^3 + \cdots + a_n^3 + n \geq 2(a_1^2 + a_2^2 + \cdots + a_n^2).$$

2.6. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$ and let $m \in \{1, 2, \dots, n-1\}$. Prove that

(a) if $a_1 \leq a_2 \leq \dots \leq a_m \leq 1$, then

$$(n-m)(a_1^3 + a_2^3 + \dots + a_n^3 - n) \geq (2n-2m+1)(a_1^2 + a_2^2 + \dots + a_n^2 - n);$$

(b) if $a_1 \geq a_2 \geq \dots \geq a_m \geq 1$, then

$$a_1^3 + a_2^3 + \dots + a_n^3 - n \leq (n-m+2)(a_1^2 + a_2^2 + \dots + a_n^2 - n).$$

2.7. Let $a_1, a_2, \dots, a_n$ ($n \geq 3$) be real numbers so that $a_1 + a_2 + \dots + a_n = n$. Prove that

(a) if $a_1 \leq \dots \leq a_{n-1} \leq 1 \leq a_n$, then

$$a_1^4 + a_2^4 + \dots + a_n^4 - n \geq 6(a_1^2 + a_2^2 + \dots + a_n^2 - n);$$

(b) if $a_1 \leq \dots \leq a_{n-2} \leq 1 \leq a_{n-1} \leq a_n$, then

$$a_1^4 + a_2^4 + \dots + a_n^4 - n \geq \frac{14}{3}(a_1^2 + a_2^2 + \dots + a_n^2 - n);$$

(c) if $a_1 \leq a_2 \leq 1 \leq a_3 \leq \dots \leq a_n$, then

$$a_1^4 + a_2^4 + \dots + a_n^4 - n \geq \frac{2(n^2 - 3n + 3)}{n^2 - 5n + 7} (a_1^2 + a_2^2 + \dots + a_n^2 - n).$$

2.8. Let a, b, c, d, e be nonnegative real numbers so that $a + b + c + d + e = 5$. Prove that

(a) if $a \geq b \geq 1 \geq c \geq d \geq e$, then

$$21(a^2 + b^2 + c^2 + d^2 + e^2) \geq a^4 + b^4 + c^4 + d^4 + e^4 + 100;$$

(b) if $a \geq b \geq c \geq 1 \geq d \geq e$, then

$$13(a^2 + b^2 + c^2 + d^2 + e^2) \geq a^4 + b^4 + c^4 + d^4 + e^4 + 60.$$

2.9. Let $a_1, a_2, \dots, a_n$ ($n \geq 3$) be nonnegative numbers so that $a_1 + a_2 + \dots + a_n = n$. Prove that

(a) if $a_1 \geq \dots \geq a_{n-1} \geq 1 \geq a_n$, then

$$7(a_1^3 + a_2^3 + \dots + a_n^3) \geq 3(a_1^4 + a_2^4 + \dots + a_n^4) + 4n;$$

(b) if $a_1 \geq \dots \geq a_{n-2} \geq 1 \geq a_{n-1} \geq a_n$, then

$$13(a_1^3 + a_2^3 + \dots + a_n^3) \geq 4(a_1^4 + a_2^4 + \dots + a_n^4) + 9n.$$

2.10. If $a_1, a_2, \dots, a_n$ are positive real numbers so that $a_1 + a_2 + \dots + a_n = n$ and

$$a_1 \geq \dots \geq a_m \geq 1 \geq a_{m+1} \geq \dots \geq a_n, \quad m \in \{1, 2, \dots, n-1\},$$

then

$$(n-m+1)^2 \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} - n \right) \geq 4(n-m)(a_1^2 + a_2^2 + \dots + a_n^2 - n).$$

2.11. If $a_1, a_2, \dots, a_n$ are positive real numbers so that $\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} = n$ and

$$a_1 \leq \dots \leq a_m \leq 1 \leq a_{m+1} \leq \dots \leq a_n, \quad m \in \{1, 2, \dots, n-1\},$$

then

$$a_1^2 + a_2^2 + \dots + a_n^2 - n \geq 2 \left(1 + \frac{\sqrt{n-m}}{n-m+1} \right) (a_1 + a_2 + \dots + a_n - n).$$

2.12. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers such that

$$a_1 \geq a_2 \geq \dots \geq a_k \geq 1 \geq a_{k+1} \geq \dots \geq a_n, \quad a_1 + a_2 + \dots + a_n = n.$$

If $n \geq 3$ and $1 \leq k \leq n-1$, then

$$\frac{1}{ka_1^2 + k + 1} + \frac{1}{ka_2^2 + k + 1} + \dots + \frac{1}{ka_n^2 + k + 1} \geq \frac{n}{2k + 1}.$$

2.13. If $a_1, a_2, \dots, a_{2n}$ are nonnegative real numbers so that

$$a_1 \geq \dots \geq a_n \geq 1 \geq a_{n+1} \geq \dots \geq a_{2n}, \quad a_1 + a_2 + \dots + a_{2n} = 2n,$$

then

$$\frac{1}{na_1^2 + n^2 + n + 1} + \frac{1}{na_2^2 + n^2 + n + 1} + \dots + \frac{1}{na_{2n}^2 + n^2 + n + 1} \leq \frac{2n}{(n+1)^2}.$$

2.14. If a, b, c, d, e, f are nonnegative real numbers so that

$$a \geq b \geq c \geq 1 \geq d \geq e \geq f, \quad a + b + c + d + e + f = 6,$$

then

$$\frac{3a+4}{3a^2+4} + \frac{3b+4}{3b^2+4} + \frac{3c+4}{3c^2+4} + \frac{3d+4}{3d^2+4} + \frac{3e+4}{3e^2+4} + \frac{3f+4}{3f^2+4} \leq 6.$$

2.15. If a, b, c, d, e, f are nonnegative real numbers so that

$$a \geq b \geq 1 \geq c \geq d \geq e \geq f, \quad a + b + c + d + e + f = 6,$$

then

$$\frac{a^2 - 1}{(2a + 7)^2} + \frac{b^2 - 1}{(2b + 7)^2} + \frac{c^2 - 1}{(2c + 7)^2} + \frac{d^2 - 1}{(2d + 7)^2} + \frac{e^2 - 1}{(2e + 7)^2} + \frac{f^2 - 1}{(2f + 7)^2} \geq 0.$$

2.16. If a, b, c, d, e, f are nonnegative real numbers so that

$$a \leq b \leq 1 \leq c \leq d \leq e \leq f, \quad a + b + c + d + e + f = 6,$$

then

$$\frac{a^2 - 1}{(2a + 5)^2} + \frac{b^2 - 1}{(2b + 5)^2} + \frac{c^2 - 1}{(2c + 5)^2} + \frac{d^2 - 1}{(2d + 5)^2} + \frac{e^2 - 1}{(2e + 5)^2} + \frac{f^2 - 1}{(2f + 5)^2} \leq 0.$$

2.17. If a, b, c are nonnegative real numbers so that

$$a \leq b \leq 1 \leq c, \quad a + b + c = 3,$$

then

$$\sqrt{\frac{2a}{b+c}} + \sqrt{\frac{2b}{c+a}} + \sqrt{\frac{2c}{a+b}} \geq 3.$$

2.18. If $a_1, a_2, \dots, a_8$ are nonnegative real numbers so that

$$a_1 \geq a_2 \geq a_3 \geq a_4 \geq 1 \geq a_5 \geq a_6 \geq a_7 \geq a_8, \quad a_1 + a_2 + \dots + a_8 = 8,$$

then

$$(a_1^2 + 1)(a_2^2 + 1) \cdots (a_8^2 + 1) \geq (a_1 + 1)(a_2 + 1) \cdots (a_8 + 1).$$

2.19. If a, b, c, d are real numbers so that

$$\frac{-1}{2} \leq a \leq b \leq 1 \leq c \leq d, \quad a + b + c + d = 4,$$

then

$$7 \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} \right) + 3 \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right) \geq 40.$$

2.20. Let a, b, c, d be real numbers such that $a + b + c + d = 4$. Prove that

(a) if $-1 \leq a \leq b \leq c \leq 1 \leq d$, then

$$3 \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} \right) \geq 8 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d};$$

(b) if $-1 \leq a \leq b \leq 1 \leq c \leq d$, then

$$2 \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} \right) \geq 4 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}.$$

2.21. If a, b, c, d are positive real numbers so that

$$a \geq b \geq 1 \geq c \geq d, \quad abcd = 1,$$

then

$$a^2 + b^2 + c^2 + d^2 - 4 \geq 18 \left(a + b + c + d - \frac{1}{a} - \frac{1}{b} - \frac{1}{c} - \frac{1}{d} \right).$$

2.22. If a, b, c, d are positive real numbers so that

$$a \leq b \leq 1 \leq c \leq d, \quad abcd = 1,$$

then

$$\sqrt{a^2 - a + 1} + \sqrt{b^2 - b + 1} + \sqrt{c^2 - c + 1} + \sqrt{d^2 - d + 1} \geq a + b + c + d.$$

2.23. If a, b, c, d are positive real numbers so that

$$a \leq b \leq c \leq 1 \leq d, \quad abcd = 1,$$

then

$$\frac{1}{a^3 + 3a + 2} + \frac{1}{b^3 + 3b + 2} + \frac{1}{c^3 + 3c + 2} + \frac{1}{d^3 + 3d + 2} \geq \frac{2}{3}.$$

2.24. If $a_1, a_2, \dots, a_n$ are positive real numbers so that

$$a_1 \geq \dots \geq a_{n-1} \geq 1 \geq a_n, \quad a_1 a_2 \dots a_n = 1,$$

then

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \geq a_1 + a_2 + \dots + a_n.$$

2.25. Let $a_1, a_2, \dots, a_n$ be positive real numbers so that

$$a_1 \leq \dots \leq a_{n-1} \leq 1 \leq a_n, \quad a_1 a_2 \dots a_n = 1.$$

If $k \geq 1$, then

$$\frac{1}{1 + ka_1} + \frac{1}{1 + ka_2} + \dots + \frac{1}{1 + ka_n} \geq \frac{n}{1 + k}.$$

2.26. If $a_1, a_2, \dots, a_9$ are positive real numbers so that

$$a_1 \leq \dots \leq a_8 \leq 1 \leq a_9, \quad a_1 a_2 \dots a_9 = 1,$$

then

$$\frac{1}{(a_1 + 2)^2} + \frac{1}{(a_2 + 2)^2} + \dots + \frac{1}{(a_9 + 2)^2} \geq 1.$$

2.27. Let $a_1, a_2, \dots, a_n$ be positive real numbers so that

$$a_1 \leq \dots \leq a_{n-1} \leq 1 \leq a_n, \quad a_1 a_2 \cdots a_n = 1.$$

If $p, q \geq 0$ so that

$$p + q \geq 1 + \frac{2pq}{p + 4q},$$

then

$$\frac{1}{1 + pa_1 + qa_1^2} + \frac{1}{1 + pa_2 + qa_2^2} + \dots + \frac{1}{1 + pa_n + qa_n^2} \geq \frac{n}{1 + p + q}.$$

2.28. Let $a_1, a_2, \dots, a_n$ be positive real numbers so that

$$a_1 \leq \dots \leq a_{n-1} \leq 1 \leq a_n, \quad a_1 a_2 \cdots a_n = 1.$$

If $m \geq 1$ and $0 < k \leq m$, then

$$\frac{1}{(a_1 + k)^m} + \frac{1}{(a_2 + k)^m} + \dots + \frac{1}{(a_n + k)^m} \geq \frac{n}{(1 + k)^m}.$$

2.29. If $a_1, a_2, \dots, a_n$ are positive real numbers so that

$$a_1 \leq \dots \leq a_{n-1} \leq 1 \leq a_n, \quad a_1 a_2 \cdots a_n = 1,$$

then

$$\frac{1}{\sqrt{1 + 3a_1}} + \frac{1}{\sqrt{1 + 3a_2}} + \dots + \frac{1}{\sqrt{1 + 3a_n}} \geq \frac{n}{2}.$$

2.30. Let $a_1, a_2, \dots, a_n$ be positive real numbers so that

$$a_1 \leq \dots \leq a_{n-1} \leq 1 \leq a_n, \quad a_1 a_2 \cdots a_n = 1.$$

If $0 < m < 1$ and $0 < k \leq \frac{1}{2^{1/m} - 1}$, then

$$\frac{1}{(a_1 + k)^m} + \frac{1}{(a_2 + k)^m} + \dots + \frac{1}{(a_n + k)^m} \geq \frac{n}{(1 + k)^m}.$$

2.31. Let $a_1, a_2, \dots, a_n$ be positive real numbers so that

$$a_1 \geq a_2 \geq \dots \geq a_k \geq 1 \geq a_{k+1} \geq \dots \geq a_n > 0, \quad a_1 a_2 \cdots a_n = 1.$$

If $n \geq 3$ and $1 \leq k \leq n - 1$, then

$$\frac{1}{ka_1 + 1} + \frac{1}{ka_2 + 1} + \dots + \frac{1}{ka_n + 1} \geq \frac{n}{k + 1}.$$

2.32. If $a_1, a_2, \dots, a_n$ ($n \geq 4$) are positive real numbers so that

$$a_1 \geq a_2 \geq a_3 \geq 1 \geq a_4 \geq \dots \geq a_n, \quad a_1 a_2 \cdots a_n = 1,$$

then

$$\frac{1}{(a_1 + 1)^2} + \frac{1}{(a_2 + 1)^2} + \dots + \frac{1}{(a_n + 1)^2} \geq \frac{n}{4}.$$

2.33. If $a_1, a_2, \dots, a_n$ are positive real numbers so that

$$a_1 \geq \dots \geq a_{n-1} \geq 1 \geq a_n, \quad a_1 a_2 \cdots a_n = 1,$$

then

$$\frac{1}{(a_1 + 3)^2} + \frac{1}{(a_2 + 3)^2} + \dots + \frac{1}{(a_n + 3)^2} \leq \frac{n}{16}.$$

2.34. Let $a_1, a_2, \dots, a_n$ be positive real numbers so that

$$a_1 \geq \dots \geq a_{n-1} \geq 1 \geq a_n, \quad a_1 a_2 \cdots a_n = 1.$$

If $p, q \geq 0$ so that $p + q \leq 1$, then

$$\frac{1}{1 + pa_1 + qa_1^2} + \frac{1}{1 + pa_2 + qa_2^2} + \dots + \frac{1}{1 + pa_n + qa_n^2} \leq \frac{n}{1 + p + q}.$$

2.35. Let $a_1, a_2, \dots, a_n$ be positive real numbers so that

$$a_1 \geq \dots \geq a_{n-1} \geq 1 \geq a_n, \quad a_1 a_2 \cdots a_n = 1.$$

If $m > 1$ and $k \geq \frac{1}{2^{1/m} - 1}$, then

$$\frac{1}{(a_1 + k)^m} + \frac{1}{(a_2 + k)^m} + \dots + \frac{1}{(a_n + k)^m} \leq \frac{n}{(1 + k)^m}.$$

2.36. If $a_1, a_2, \dots, a_n$ are positive real numbers so that

$$a_1 \geq \dots \geq a_{n-1} \geq 1 \geq a_n, \quad a_1 a_2 \cdots a_n = 1,$$

then

$$\frac{1}{\sqrt{1 + 2a_1}} + \frac{1}{\sqrt{1 + 2a_2}} + \dots + \frac{1}{\sqrt{1 + 2a_n}} \leq \frac{n}{\sqrt{3}}.$$

2.37. Let $a_1, a_2, \dots, a_n$ be positive real numbers so that

$$a_1 \geq \dots \geq a_{n-1} \geq 1 \geq a_n, \quad a_1 a_2 \cdots a_n = 1.$$

If $0 < m < 1$ and $k \geq m$, then

$$\frac{1}{(a_1 + k)^m} + \frac{1}{(a_2 + k)^m} + \dots + \frac{1}{(a_n + k)^m} \leq \frac{n}{(1 + k)^m}.$$

2.38. If $a_1, a_2, \dots, a_n$ ($n \geq 3$) are positive real numbers so that

$$a_1 \geq \dots \geq a_{n-2} \geq 1 \geq a_{n-1} \geq a_n, \quad a_1 a_2 \cdots a_n = 1,$$

then

$$\frac{1}{(a_1 + 5)^2} + \frac{1}{(a_2 + 5)^2} + \dots + \frac{1}{(a_n + 5)^2} \leq \frac{n}{36}.$$

2.39. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that

$$a_1 \geq \dots \geq a_{n-1} \geq 1 \geq a_n, \quad a_1^2 + a_2^2 + \dots + a_n^2 = n,$$

then

$$\frac{1}{3 - a_1} + \frac{1}{3 - a_2} + \dots + \frac{1}{3 - a_n} \leq \frac{n}{2}.$$

2.40. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that

$$a_1 \leq \dots \leq a_{n-1} \leq 1 \leq a_n, \quad a_1 + a_2 + \dots + a_n = n.$$

Prove that

$$a_1^3 + a_2^3 + \dots + a_n^3 - n \geq (n - 1)^2 \left[\left(\frac{n - a_1}{n - 1} \right)^3 + \left(\frac{n - a_2}{n - 1} \right)^3 + \dots + \left(\frac{n - a_n}{n - 1} \right)^3 - n \right].$$

2.3 Solutions

P 2.1. Find the least value $k_0 \in [1, \infty)$ of the constant k such that

$$\frac{1}{ka^2 + b + c} + \frac{1}{kb^2 + c + a} + \frac{1}{kc^2 + a + b} \geq \frac{3}{k + 2}$$

for any nonnegative real numbers a, b, c with at most one of them larger than 1 and $a + b + c = 3$.

(Vasile C., 2007)

Solution. For $a = 0$, $b = 1$ and $c = 2$, the constraint is satisfied and the inequality leads to $2k^2 - 6k + 1 \geq 0$, hence

$$k_0 = \frac{3 + \sqrt{7}}{2} \approx 2.823.$$

To prove that k_0 is the least value of k , we need to prove the inequality for $k = k_0$, hence for

$$2k^2 - 6k + 1 = 0.$$

Write the inequality as

$$f(a) + f(b) + f(c) \geq 3f(s), \quad s = \frac{a + b + c}{3} = 1,$$

where

$$f(u) = \frac{1}{ku^2 - u + 3}, \quad u \in \mathbb{I} = [0, 3].$$

Without loss of generality, assume that

$$a \leq b \leq 1 \leq c.$$

We have

$$f''(u) = \frac{2g(u)}{(ku^2 - u + 3)^3},$$

where $g(u) = 3k^2u^2 - 3ku - 3k + 1$. For $u \geq 1$, we have

$$g(u) = 3(ku - 1/2)^2 - 3k + 1/4 \geq 3(k - 1/2)^2 - 3k + 1/4 = 3k^2 - 6k + 1 = k^2 > 0,$$

hence f is convex on $\mathbb{I}_{\geq s}$. By applying the RHCF-OV Theorem for $n = 3$ and $m = 2$, it suffices to show that $f(x) + f(y) \geq 2f(1)$ for $0 \leq x \leq y$ and $x + y = 2$. Denoting $p = xy$, we have $p \leq 1$, while the inequality is equivalent to

$$\frac{2k + 2 - kp}{k^2p^2 - (8k - 1)p + 12k + 3} \geq \frac{1}{k + 2}, \quad 1 - 6k + 2k^2 - (k^2 - 6k + 1)p - k^2p^2 \geq 0,$$

$$(1 - p)(1 - 6k + 2k^2 + k^2p^2) \geq 0, \quad (1 - p)p^2 \geq 0.$$

So, the proof is finished. For $k = k_0$, the equality occurs when $a = b = c = 1$, and also when $a = 0$, $b = 1$ and $c = 2$ (or any permutation).

□

P 2.2. If a, b, c, d are real numbers so that

$$a \leq b \leq 1 \leq c \leq d, \quad a + b + c + d = 4,$$

then

$$(3a^2 - 2)(a - 1)^2 + (3b^2 - 2)(b - 1)^2 + (3c^2 - 2)(c - 1)^2 + (3d^2 - 2)(d - 1)^2 \geq 0.$$

(Vasile C., 2007)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) + f(d) \geq 4f(s), \quad s = \frac{a + b + c + d}{4} = 1,$$

where

$$f(u) = (3u^2 - 2)(u - 1)^2, \quad u \in \mathbb{I} = \mathbb{R}.$$

From

$$f''(u) = 2(18u^2 - 18u + 1),$$

it follows that $f''(u) > 0$ for $u \geq 1$, hence f is convex on $\mathbb{I}_{\geq s}$. Therefore, we may apply the RHCF-OV Theorem for $n = 4$ and $m = 2$. Thus, it suffices to show that $f(x) + 2f(y) \geq 3f(1)$ for all real x, y so that $x + 2y = 3$. Using Note 1, we only need to show that $h(x, y) \geq 0$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

We have

$$g(u) = 3(u^3 + u^2 + u + 1) - 6(u^2 + u + 1) + u + 1 = 3u^3 - 3u^2 - 2u - 2,$$

$$h(x, y) = 3(x^2 + xy + y^2) - 3(x + y) - 2 = (3y - 4)^2 \geq 0.$$

From $x + 2y = 3$ and $h(x, y) = 0$, we get $x = 1/3$, $y = 4/3$. Therefore, in accordance with Note 4, the equality holds for $a = b = c = d = 1$, and also for

$$a = \frac{1}{3}, \quad b = 1, \quad c = d = \frac{4}{3}.$$

Remark. Similarly, we can prove the following generalization:

- Let $a_1, a_2, \dots, a_{2n}$ be real numbers so that

$$a_1 \leq \dots \leq a_n \leq 1 \leq a_{n+1} \leq \dots \leq a_{2n}, \quad a_1 + a_2 + \dots + a_{2n} = 2n.$$

If $k = \frac{n}{n^2 - n + 1}$, then

$$(a_1^2 - k)(a_1 - 1)^2 + (a_2^2 - k)(a_2 - 1)^2 + \dots + (a_{2n}^2 - k)(a_{2n} - 1)^2 \geq 0,$$

with equality for $a_1 = a_2 = \dots = a_{2n} = 1$, and also for

$$a_1 = \frac{1}{n^2 - n + 1}, \quad a_2 = \dots = a_n = 1, \quad a_{n+1} = \dots = a_{2n} = \frac{n^2}{n^2 - n + 1}.$$

□

P 2.3. If a, b, c, d are nonnegative real numbers so that

$$a \geq b \geq 1 \geq c \geq d, \quad a + b + c + d = 4,$$

then

$$\frac{1}{2a^3 + 5} + \frac{1}{2b^3 + 5} + \frac{1}{2c^3 + 5} + \frac{1}{2d^3 + 5} \leq \frac{4}{7}.$$

(Vasile C., 2009)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) + f(d) \geq 4f(s), \quad s = \frac{a + b + c + d}{4} = 1,$$

where

$$f(u) = \frac{-1}{2u^3 + 5}, \quad u \geq 0.$$

From

$$f''(u) = \frac{12u(5 - 4u^3)}{(2u^3 + 5)^3},$$

it follows that $f''(u) \geq 0$ for $u \in [0, 1]$, hence f is convex on $[0, s]$. Therefore, we may apply the LHCF-OV Theorem for $n = 4$ and $m = 2$. Using Note 1, we only need to show that $h(x, y) \geq 0$ for $x, y \geq 0$ so that $x + 2y = 3$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{2(u^2 + u + 1)}{7(2u^3 + 5)},$$

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{2E}{7(2x^3 + 5)(2y^3 + 5)},$$

where

$$E = -2x^2y^2 - 2xy(x + y) - 2(x^2 + xy + y^2) + 5(x + y) + 5.$$

Since

$$E = (1 - 2y)^2(2 + 3y - 2y^2) = (1 - 2y)^2(2 + xy) \geq 0,$$

the proof is completed. From $x + 2y = 3$ and $h(x, y) = 0$, we get $x = 2$, $y = 1/2$. Therefore, in accordance with Note 4, the equality holds for $a = b = c = d = 1$, and also for

$$a = 2, \quad b = 1, \quad c = d = \frac{1}{2}.$$

Remark. Similarly, we can prove the following generalization.

- If $a_1, a_2, \dots, a_{2n}$ are nonnegative real numbers so that

$$a_1 \geq \dots \geq a_n \geq 1 \geq a_{n+1} \geq \dots \geq a_{2n}, \quad a_1 + a_2 + \dots + a_{2n} = 2n.$$

then

$$\frac{1}{a_1^3 + n + \frac{1}{n}} + \frac{1}{a_2^3 + n + \frac{1}{n}} + \dots + \frac{1}{a_{2n}^3 + n + \frac{1}{n}} \geq \frac{2n^2}{n^2 + n + 1},$$

with equality for $a_1 = a_2 = \dots = a_{2n} = 1$, and also for

$$a_1 = n, \quad a_2 = \dots = a_n = 1, \quad a_{n+1} = \dots = a_{2n} = \frac{1}{n}.$$

□

P 2.4. If

$$\frac{-2n-1}{n-1} \leq a_1 \leq \cdots \leq a_n \leq 1 \leq a_{n+1} \leq \cdots \leq a_{2n}, \quad a_1 + a_2 + \cdots + a_{2n} = 2n,$$

then

$$a_1^3 + a_2^3 + \cdots + a_{2n}^3 \geq 2n.$$

(Vasile C., 2007)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_{2n}) \geq 2nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_{2n}}{2n} = 1,$$

where

$$f(u) = u^3, \quad u \geq \frac{-2n-1}{n-1}.$$

From $f''(u) = 6u$, it follows that $f(u)$ is convex for $u \geq s$. Therefore, we may apply the RHCF-OV Theorem for $2n$ numbers and $m = n$. By Note 1, it suffices to show that $h(x, y) \geq 0$ for all $x, y \geq \frac{-2n-1}{n-1}$ so that $x + ny = 1 + n$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = u^2 + u + 1,$$

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = x + y + 1 = \frac{(n-1)x + 2n + 1}{n-1} \geq 0.$$

From $x + ny = 1 + n$ and $h(x, y) = 0$, we get

$$x = \frac{-2n-1}{n-1}, \quad y = \frac{n+2}{n-1}.$$

In accordance with Note 4, the equality holds for $a_1 = a_2 = \cdots = a_{2n} = 1$, and also for

$$a_1 = \frac{-2n-1}{n-1}, \quad a_2 = \cdots = a_n = 1, \quad a_{n+1} = \cdots = a_{2n} = \frac{n+2}{n-1}.$$

□

P 2.5. Let $a_1, a_2, \dots, a_n$ ($n \geq 3$) be real numbers so that $a_1 + a_2 + \cdots + a_n = n$. Prove that

(a) if $-3 \leq a_1 \leq \cdots \leq a_{n-2} \leq 1 \leq a_{n-1} \leq a_n$, then

$$a_1^3 + a_2^3 + \cdots + a_n^3 \geq a_1^2 + a_2^2 + \cdots + a_n^2;$$

(b) if $-\frac{n-1}{n-3} \leq a_1 \leq a_2 \leq 1 \leq \cdots \leq a_n$, then

$$a_1^3 + a_2^3 + \cdots + a_n^3 + n \geq 2(a_1^2 + a_2^2 + \cdots + a_n^2).$$

(Vasile C., 2007)

Solution. (a) Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = u^3 - u^2, \quad u \geq -3.$$

For $u \geq 1$, we have

$$f''(u) = 6u - 2 > 0,$$

hence $f(u)$ is convex for $u \geq s$. Thus, we may apply the RHCF-OV Theorem for $m = n - 2$. According to this theorem, it suffices to show that

$$f(x) + 2f(y) \geq 3f(1)$$

for $-3 \leq x \leq y$ satisfying $x + 2y = 3$. Using Note 1, we only need to show that $h(x, y) \geq 0$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

We have

$$g(u) = u^2,$$

$$h(x, y) = x + y = \frac{x + 3}{2} \geq 0.$$

From $x + 2y = 3$ and $h(x, y) = 0$, we get $x = -3$ and $y = 3$. Therefore, in accordance with Note 4, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = -3, \quad a_2 = \cdots = a_{n-2} = 1, \quad a_{n-1} = a_n = 3.$$

(b) Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = u^3 - 2u^2, \quad u \geq -\frac{n-1}{n-3}.$$

For $u \geq 1$, we have

$$f''(u) = 6u - 4 > 0,$$

hence $f(u)$ is convex for $u \geq s$. Thus, we may apply the RHCF-OV Theorem for $m = 2$. According to this theorem, it suffices to show that

$$f(x) + (n-2)f(y) \geq (n-1)f(1)$$

for $-\frac{n-1}{n-3} \leq x \leq y$ satisfying $x + (n-2)y = n-1$. Using Note 1, we only need to show that $h(x, y) \geq 0$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

We have

$$g(u) = u^2 - u - 1,$$

$$h(x, y) = x + y - 1 = \frac{(n-3)x + n-1}{n-1} \geq 0.$$

From $x + (n-2)y = n-1$ and $h(x, y) = 0$, we get $x = -\frac{n-1}{n-3}$ and $y = \frac{n-1}{n-3}$. Therefore, in accordance with Note 4, the equality holds for $a_1 = a_2 = \dots = a_n = 1$. If $n \geq 4$, then the equality holds also for

$$a_1 = -\frac{n-1}{n-3}, \quad a_2 = 1, \quad a_3 = \dots = a_n = \frac{n-1}{n-3}.$$

□

P 2.6. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$ and let $m \in \{1, 2, \dots, n-1\}$. Prove that

(a) if $a_1 \leq a_2 \leq \dots \leq a_m \leq 1$, then

$$(n-m)(a_1^3 + a_2^3 + \dots + a_n^3 - n) \geq (2n-2m+1)(a_1^2 + a_2^2 + \dots + a_n^2 - n);$$

(b) if $a_1 \geq a_2 \geq \dots \geq a_m \geq 1$, then

$$a_1^3 + a_2^3 + \dots + a_n^3 - n \leq (n-m+2)(a_1^2 + a_2^2 + \dots + a_n^2 - n).$$

(Vasile C., 2007)

Solution. (a) Write the inequality as

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \dots + a_n}{n} = 1,$$

where

$$f(u) = (n-m)u^3 - (2n-2m+1)u^2, \quad u \in \mathbb{I} = [0, n].$$

For $u \geq 1$, we have

$$\begin{aligned} f''(u) &= 6(n-m)u - 2(2n-2m+1) \\ &\geq 6(n-m) - 2(2n-2m+1) = 2(n-m-1) \geq 0, \end{aligned}$$

hence f is convex on $\mathbb{I}_{\geq s}$. Thus, by the RHCF-OV Theorem and Note 1, we only need to show that $h(x, y) \geq 0$ for all nonnegative numbers x, y so that $x + (n-m)y = n-m+1$. We have

$$\begin{aligned} g(u) &= \frac{f(u) - f(1)}{u-1} = (n-m)(u^2 + u + 1) - (2n-2m+1)(u+1) \\ &= (n-m)u^2 - (n-m+1)u - n + m - 1, \end{aligned}$$

$$h(x, y) = \frac{g(x) - g(y)}{x-y} = (n-m)(x+y) - n + m - 1 = (n-m-1)x \geq 0.$$

From $x + (n - m)y = 1 + n - m$ and $h(x, y) = 0$, we get $x = 0$, $y = (n - m + 1)/(n - m)$. Therefore, in accordance with Note 4, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = 0, \quad a_2 = \cdots = a_m = 1, \quad a_{m+1} = \cdots = a_n = 1 + \frac{1}{n - m}.$$

(b) Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = (n - m + 2)u^2 - u^3, \quad u \in \mathbb{I} = [0, n].$$

For $u \leq 1$, we have

$$f''(u) = 2(n - m + 2 - 3u) \geq 2(n - m + 2 - 3) = 2(n - m - 1) \geq 0,$$

hence f is convex on $\mathbb{I}_{\leq s}$. By the LHCF-OV Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for all $x, y \geq 0$ so that $x + (n - m)y = 1 + n - m$. We have

$$\begin{aligned} g(u) &= \frac{f(u) - f(1)}{u - 1} = (n - m + 2)(u + 1) - (u^2 + u + 1) \\ &= -u^2 + (n - m + 1)u + n - m + 1, \\ h(x, y) &= \frac{g(x) - g(y)}{x - y} = -(x + y) + n - m + 1 = (n - m - 1)y \geq 0. \end{aligned}$$

From $x + (n - m)y = 1 + n - m$ and $h(x, y) = 0$, we get $x = n - m + 1$, $y = 0$. Therefore, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = n - m + 1, \quad a_2 = \cdots = a_m = 1, \quad a_{m+1} = \cdots = a_n = 0.$$

Remark 1. For $m = 1$, we get the following results:

- If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then

$$(n - 1)(a_1^3 + a_2^3 + \cdots + a_n^3 - n) \geq (2n - 1)(a_1^2 + a_2^2 + \cdots + a_n^2 - n),$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = 0, \quad a_2 = a_3 = \cdots = a_n = \frac{n}{n - 1}$$

(or any cyclic permutation).

- If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then

$$a_1^3 + a_2^3 + \cdots + a_n^3 - n \leq (n + 1)(a_1^2 + a_2^2 + \cdots + a_n^2 - n),$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = n, \quad a_2 = a_3 = \cdots = a_n = 0$$

(or any cyclic permutation).

Remark 2. For $m = n - 1$, we get the following statements:

- If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that

$$a_1 \leq \dots \leq a_{n-1} \leq 1 \leq a_n, \quad a_1 + a_2 + \dots + a_n = n,$$

then

$$a_1^3 + a_2^3 + \dots + a_n^3 + 2n \geq 3(a_1^2 + a_2^2 + \dots + a_n^2),$$

with equality for $a_1 = a_2 = \dots = a_n = 1$, and also for

$$a_1 = 0, \quad a_2 = \dots = a_{n-1} = 1, \quad a_n = 2.$$

- If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that

$$a_1 \geq \dots \geq a_{n-1} \geq 1 \geq a_n, \quad a_1 + a_2 + \dots + a_n = n,$$

then

$$a_1^3 + a_2^3 + \dots + a_n^3 + 2n \leq 3(a_1^2 + a_2^2 + \dots + a_n^2),$$

with equality for $a_1 = a_2 = \dots = a_n = 1$, and also for

$$a_1 = 2, \quad a_2 = \dots = a_{n-1} = 1, \quad a_n = 0.$$

Remark 3. Replacing n with $2n$ and choosing then $m = n$, we get the following results:

- If $a_1, a_2, \dots, a_{2n}$ are nonnegative real numbers so that

$$a_1 \leq \dots \leq a_n \leq 1 \leq a_{n+1} \leq \dots \leq a_{2n}, \quad a_1 + a_2 + \dots + a_{2n} = 2n,$$

then

$$n(a_1^3 + a_2^3 + \dots + a_{2n}^3 - 2n) \geq (2n + 1)(a_1^2 + a_2^2 + \dots + a_{2n}^2 - 2n),$$

with equality for $a_1 = a_2 = \dots = a_{2n} = 1$, and also for

$$a_1 = 0, \quad a_2 = \dots = a_n = 1, \quad a_{n+1} = \dots = a_{2n} = 1 + \frac{1}{n}.$$

- If $a_1, a_2, \dots, a_{2n}$ are nonnegative real numbers so that

$$a_1 \geq \dots \geq a_n \geq 1 \geq a_{n+1} \geq \dots \geq a_{2n}, \quad a_1 + a_2 + \dots + a_{2n} = 2n,$$

then

$$a_1^3 + a_2^3 + \dots + a_{2n}^3 - 2n \leq (n + 2)(a_1^2 + a_2^2 + \dots + a_{2n}^2 - 2n),$$

with equality for $a_1 = a_2 = \dots = a_{2n} = 1$, and also for

$$a_1 = n + 1, \quad a_2 = \dots = a_n = 1, \quad a_{n+1} = \dots = a_{2n} = 0.$$

□

P 2.7. Let $a_1, a_2, \dots, a_n$ ($n \geq 3$) be real numbers so that $a_1 + a_2 + \dots + a_n = n$. Prove that

(a) if $a_1 \leq \dots \leq a_{n-1} \leq 1 \leq a_n$, then

$$a_1^4 + a_2^4 + \dots + a_n^4 - n \geq 6(a_1^2 + a_2^2 + \dots + a_n^2 - n);$$

(b) if $a_1 \leq \dots \leq a_{n-2} \leq 1 \leq a_{n-1} \leq a_n$, then

$$a_1^4 + a_2^4 + \dots + a_n^4 - n \geq \frac{14}{3}(a_1^2 + a_2^2 + \dots + a_n^2 - n);$$

(c) if $a_1 \leq a_2 \leq 1 \leq a_3 \leq \dots \leq a_n$, then

$$a_1^4 + a_2^4 + \dots + a_n^4 - n \geq \frac{2(n^2 - 3n + 3)}{n^2 - 5n + 7} (a_1^2 + a_2^2 + \dots + a_n^2 - n).$$

(Vasile C., 2009)

Solution. Consider the inequality

$$a_1^4 + a_2^4 + \dots + a_n^4 - n \geq k(a_1^2 + a_2^2 + \dots + a_n^2 - n), \quad k \leq 6,$$

and write it as

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \dots + a_n}{n} = 1,$$

where

$$f(u) = u^4 - ku^2, \quad u \in \mathbb{R}.$$

From $f''(u) = 2(6u^2 - k)$, it follows that f is convex for $u \geq 1$. Therefore, we may apply the RHCF-OV Theorem for $m = n - 1$, $m = n - 2$ and $m = 2$, respectively. By Note 1, it suffices to show that $h(x, y) \geq 0$ for all real x, y so that $x + (n - m)y = 1 + n - m$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = u^3 + u^2 + u + 1 - k(u + 1),$$

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = x^2 + xy + y^2 + x + y + 1 - k.$$

(a) We need to show that $h(x, y) \geq 0$ for $k = 6$, $m = n - 1$, $x + y = 2$. Indeed, we have

$$h(x, y) = 1 - xy = \frac{1}{4}(x - y)^2 \geq 0.$$

From $x + y = 2$ and $h(x, y) = 0$, we get $x = y = 1$. Therefore, in accordance with Note 4, the equality holds for $a_1 = a_2 = \dots = a_n = 1$.

(b) For $k = 14/3$, $m = n - 2$ and $x + 2y = 3$, we have

$$h(x, y) = \frac{1}{3}(3y - 5)^2 \geq 0.$$

From $x + 2y = 3$ and $h(x, y) = 0$, we get $x = -1/3$ and $y = 5/3$. Therefore, the equality holds for $a_1 = a_2 = \dots = a_n = 1$, and also for

$$a_1 = \frac{-1}{3}, \quad a_2 = \dots = a_{n-2} = 1, \quad a_{n-1} = a_n = \frac{5}{3}.$$

(c) We have $k = \frac{2(n^2 - 3n + 3)}{n^2 - 5n + 7}$, $m = 2$ and $x + (n - 2)y = n - 1$, which involve

$$h(x, y) = \frac{[(n^2 - 5n + 7)y - n^2 + 3n - 1]^2}{n^2 - 5n + 7} \geq 0.$$

From $x + (n - 2)y = n - 1$ and $h(x, y) = 0$, we get

$$x = \frac{-n^2 + 5n - 5}{n^2 - 5n + 7}, \quad y = \frac{n^2 - 3n + 1}{n^2 - 5n + 7}.$$

Therefore, the equality holds for $a_1 = a_2 = \dots = a_n = 1$, and also for

$$a_1 = \frac{-n^2 + 5n - 5}{n^2 - 5n + 7}, \quad a_2 = 1, \quad a_3 = \dots = a_n = \frac{n^2 - 3n + 1}{n^2 - 5n + 7}.$$

□

P 2.8. Let a, b, c, d, e be nonnegative real numbers so that $a + b + c + d + e = 5$. Prove that

(a) if $a \geq b \geq 1 \geq c \geq d \geq e$, then

$$21(a^2 + b^2 + c^2 + d^2 + e^2) \geq a^4 + b^4 + c^4 + d^4 + e^4 + 100;$$

(b) if $a \geq b \geq c \geq 1 \geq d \geq e$, then

$$13(a^2 + b^2 + c^2 + d^2 + e^2) \geq a^4 + b^4 + c^4 + d^4 + e^4 + 60.$$

(Vasile C., 2009)

Solution. Consider the inequality

$$k(a^2 + b^2 + c^2 + d^2 + e^2 - 5) \geq a^4 + b^4 + c^4 + d^4 + e^4 - 5, \quad k \geq 6,$$

and write it as

$$f(a) + f(b) + f(c) + f(d) + f(e) \geq 5f(s), \quad s = \frac{a + b + c + d + e}{5} = 1,$$

where

$$f(u) = ku^2 - u^4, \quad u \geq 0.$$

From $f''(u) = 2(k - 6u^2)$, it follows that f is convex on $[0, 1]$. Therefore, we may apply the LHCF-OV Theorem for $m = 2$ and $m = 3$, respectively. By Note 1, it suffices to show that $h(x, y) \geq 0$ for all $x, y \geq 0$ so that $x + (5 - m)y = 6 - m$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = k(u + 1) - (u^3 + u^2 + u + 1),$$

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = k - (x^2 + xy + y^2 + x + y + 1).$$

(a) We need to show that $h(x, y) \geq 0$ for $k = 21$, $n = 5$, $m = 2$ and $x + 3y = 4$; indeed, we have

$$h(x, y) = 21 - (x^2 + xy + y^2 + x + y + 1) = y(22 - 7y) = y(10 + 3x + 2y) \geq 0.$$

From $x + 3y = 4$ and $h(x, y) = 0$, we get $x = 4$ and $y = 0$. Therefore, in accordance with Note 4, the equality holds for $a = b = c = d = e = 1$, and also for

$$a = 4, \quad b = 1, \quad c = d = e = 0.$$

(b) We have $k = 13$, $n = 5$, $m = 3$ and $x + 2y = 3$, which involve

$$h(x, y) = 13 - (x^2 + xy + y^2 + x + y + 1) = y(10 - 3y) = y(4 + 2x + y) \geq 0.$$

From $x + 2y = 3$ and $h(x, y) = 0$, we get $x = 3$ and $y = 0$. Therefore, the equality holds for $a = b = c = d = e = 1$, and also for

$$a = 3, \quad b = c = 1, \quad d = e = 0.$$

□

P 2.9. Let $a_1, a_2, \dots, a_n$ ($n \geq 3$) be nonnegative numbers so that $a_1 + a_2 + \dots + a_n = n$. Prove that

(a) if $a_1 \geq \dots \geq a_{n-1} \geq 1 \geq a_n$, then

$$7(a_1^3 + a_2^3 + \dots + a_n^3) \geq 3(a_1^4 + a_2^4 + \dots + a_n^4) + 4n;$$

(b) if $a_1 \geq \dots \geq a_{n-2} \geq 1 \geq a_{n-1} \geq a_n$, then

$$13(a_1^3 + a_2^3 + \dots + a_n^3) \geq 4(a_1^4 + a_2^4 + \dots + a_n^4) + 9n.$$

(Vasile C., 2009)

Solution. Consider the inequality

$$k(a_1^3 + a_2^3 + \dots + a_n^3 - n) \geq a_1^4 + a_2^4 + \dots + a_n^4 - n, \quad k \geq 2,$$

and write it as

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \dots + a_n}{n} = 1,$$

where

$$f(u) = ku^3 - u^4, \quad u \geq 0.$$

From $f''(u) = 6u(k - 2u^2)$, it follows that f is convex on $[0, 1]$. Therefore, we may apply the LHCF-OV Theorem for $m = n - 1$ and $m = n - 2$, respectively. By Note 1, it suffices to show that $h(x, y) \geq 0$ for $x \geq y \geq 0$ so that $x + my = 1 + m$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = k(u^2 + u + 1) - (u^3 + u^2 + u + 1),$$

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = -(x^2 + xy + y^2) + (k - 1)(x + y + 1).$$

(a) We need to show that $h(x, y) \geq 0$ for $k = 7/3$, $m = n - 1$, $x + y = 2$. Indeed,

$$h(x, y) = xy \geq 0.$$

From $x > y$, $x + y = 2$ and $h(x, y) = 0$, we get $x = 2$ and $y = 0$. Therefore, in accordance with Note 4, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = 2, \quad a_2 = \cdots = a_{n-1} = 1, \quad a_n = 0.$$

(b) We have $k = 13/4$, $m = n - 2$, $x + 2y = 3$, which involve

$$h(x, y) = 3y(9 - 4y) = 3y(3 + 2x) \geq 0.$$

From $x + 2y = 3$ and $h(x, y) = 0$, we get $x = 3$ and $y = 0$. Therefore, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = 3, \quad a_2 = \cdots = a_{n-2} = 1, \quad a_{n-1} = a_n = 0.$$

□

P 2.10. If $a_1, a_2, \dots, a_n$ are positive real numbers so that $a_1 + a_2 + \cdots + a_n = n$ and

$$a_1 \geq \cdots \geq a_m \geq 1 \geq a_{m+1} \geq \cdots \geq a_n, \quad m \in \{1, 2, \dots, n-1\},$$

then

$$(n - m + 1)^2 \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n \right) \geq 4(n - m)(a_1^2 + a_2^2 + \cdots + a_n^2 - n).$$

(Vasile C., 2007)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{(n - m + 1)^2}{u} - 4(n - m)u^2, \quad u > 0.$$

For $u \in (0, 1]$, we have

$$\begin{aligned} f''(u) &= \frac{2(n-m+1)^2}{u^3} - 8(n-m) \\ &\geq 2(n-m+1)^2 - 8(n-m) = 2(n-m-1)^2 \geq 0. \end{aligned}$$

Since f is convex on $(0, s]$, we may apply the LHCF-OV Theorem. By Note 1, it suffices to show that $h(x, y) \geq 0$ for all $x, y > 0$ so that $x + (n-m)y = 1 + n - m$. We have

$$\begin{aligned} g(u) &= \frac{f(u) - f(1)}{u - 1} = \frac{-(n-m+1)^2}{u} - 4(n-m)(u+1), \\ h(x, y) &= \frac{(n-m+1)^2}{xy} - 4(n-m) = \frac{[n-m+1-2(n-m)y]^2}{xy} \geq 0. \end{aligned}$$

From $x + (n-m)y = 1 + n - m$ and $h(x, y) = 0$, we get

$$x = \frac{n-m+1}{2}, \quad y = \frac{n-m+1}{2(n-m)}.$$

Therefore, in accordance with Note 4, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = \frac{n-m+1}{2}, \quad a_2 = a_3 = \cdots = a_m = 1, \quad a_{m+1} = \cdots = a_n = \frac{n-m+1}{2(n-m)}.$$

Remark 1. For $m = n - 1$, we get the following elegant statement:

- If $a_1, a_2, \dots, a_n$ are positive real numbers so that

$$a_1 \geq \cdots \geq a_{n-1} \geq 1 \geq a_n, \quad a_1 + a_2 + \cdots + a_n = n,$$

then

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \geq a_1^2 + a_2^2 + \cdots + a_n^2,$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$

Remark 2. Replacing n with $2n$ and choosing then $m = n$, we get the following statement:

- If $a_1, a_2, \dots, a_{2n}$ are positive real numbers so that

$$a_1 \geq \cdots \geq a_n \geq 1 \geq a_{n+1} \geq \cdots \geq a_{2n}, \quad a_1 + a_2 + \cdots + a_{2n} = 2n,$$

then

$$(n+1)^2 \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{2n}} - 2n \right) \geq 4n(a_1^2 + a_2^2 + \cdots + a_{2n}^2 - 2n),$$

with equality for $a_1 = a_2 = \cdots = a_{2n} = 1$, and also for

$$a_1 = \frac{n+1}{2}, \quad a_2 = a_3 = \cdots = a_n = 1, \quad a_{n+1} = \cdots = a_{2n} = \frac{n+1}{2n}.$$

□

P 2.11. If $a_1, a_2, \dots, a_n$ are positive real numbers so that $\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} = n$ and

$$a_1 \leq \dots \leq a_m \leq 1 \leq a_{m+1} \leq \dots \leq a_n, \quad m \in \{1, 2, \dots, n-1\},$$

then

$$a_1^2 + a_2^2 + \dots + a_n^2 - n \geq 2 \left(1 + \frac{\sqrt{n-m}}{n-m+1} \right) (a_1 + a_2 + \dots + a_n - n).$$

(Vasile C., 2007)

Solution. Replacing each a_i by $1/a_i$, we need to prove that

$$a_1 \geq \dots \geq a_m \geq 1 \geq a_{m+1} \geq \dots \geq a_n, \quad a_1 + a_2 + \dots + a_n = n$$

involves

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \dots + a_n}{n} = 1,$$

where

$$f(u) = \frac{1}{u^2} - \frac{2k}{u}, \quad k = 1 + \frac{\sqrt{m-n}}{n-m+1}, \quad u > 0.$$

For $u \in (0, 1]$, we have

$$f''(u) = \frac{6-4ku}{u^4} \geq \frac{6-4k}{u^4} = \frac{2(\sqrt{n-m}-1)^2}{(n-m+1)u^4} \geq 0.$$

Thus, f is convex on $(0, 1]$. By the LHCF-OV Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for $x, y > 0$ so that $x + (n-m)y = 1 + n - m$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

We have

$$g(u) = \frac{-1}{u^2} + \frac{2k-1}{u}$$

and

$$h(x, y) = \frac{1}{xy} \left(\frac{1}{x} + \frac{1}{y} + 1 - 2k \right).$$

We only need to show that

$$\frac{1}{x} + \frac{1}{y} \geq 1 + \frac{2\sqrt{n-m}}{n-m+1}.$$

Indeed, using the Cauchy-Schwarz inequality, we get

$$\frac{1}{x} + \frac{1}{y} \geq \frac{(1 + \sqrt{n-m})^2}{x + (n-m)y} = \frac{(1 + \sqrt{n-m})^2}{n-m+1} = 1 + \frac{2\sqrt{n-m}}{n-m+1}.$$

From $x + (n-m)y = 1 + n - m$ and $h(x, y) = 0$, we get

$$x = \frac{n-m+1}{1 + \sqrt{n-m}}, \quad y = \frac{n-m+1}{n-m + \sqrt{n-m}}.$$

By Note 4, we have

$$f(a_1) + f(a_2) + \cdots + f(a_n) = nf(1)$$

for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = \frac{n-m+1}{1+\sqrt{n-m}}, \quad a_2 = a_3 = \cdots = a_m = 1, \quad a_{m+1} = \cdots = a_n = \frac{n-m+1}{n-m+\sqrt{n-m}}.$$

Therefore, the original inequality becomes an equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = \frac{1+\sqrt{n-m}}{n-m+1}, \quad a_2 = a_3 = \cdots = a_m = 1, \quad a_{m+1} = \cdots = a_n = \frac{n-m+\sqrt{n-m}}{n-m+1}.$$

Remark. Replacing n with $2n$ and choosing then $m = n$, we get the statement below.

- If $a_1, a_2, \dots, a_{2n}$ are positive real numbers so that

$$a_1 \leq \cdots \leq a_n \leq 1 \leq a_{n+1} \leq \cdots \leq a_{2n}, \quad \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{2n}} = 2n,$$

then

$$a_1^2 + a_2^2 + \cdots + a_{2n}^2 - 2n \geq 2 \left(1 + \frac{\sqrt{n}}{n+1} \right) (a_1 + a_2 + \cdots + a_{2n} - 2n).$$

with equality for $a_1 = a_2 = \cdots = a_{2n} = 1$, and also for

$$a_1 = \frac{1+\sqrt{n}}{n+1}, \quad a_2 = a_3 = \cdots = a_n = 1, \quad a_{n+1} = \cdots = a_{2n} = \frac{n+\sqrt{n}}{n+1}.$$

□

P 2.12. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers such that

$$a_1 \geq a_2 \geq \cdots \geq a_k \geq 1 \geq a_{k+1} \geq \cdots \geq a_n, \quad a_1 + a_2 + \cdots + a_n = n.$$

If $n \geq 3$ and $1 \leq k \leq n-1$, then

$$\frac{1}{ka_1^2 + k + 1} + \frac{1}{ka_2^2 + k + 1} + \cdots + \frac{1}{ka_n^2 + k + 1} \geq \frac{n}{2k+1}.$$

(Vasile C., 2007)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

and

$$f(u) = \frac{1}{ku^2 + k + 1}, \quad u \geq 0.$$

For $u \geq 1$, we have

$$f''(u) = \frac{2k(3ku^2 - k - 1)}{(ku^2 + k + 1)^3} \geq \frac{2k(2k - 1)}{(ku^2 + k + 1)^3} > 0,$$

hence $f(u)$ is convex for $u \geq s$. Therefore, we may apply the RHCF-OV Theorem for $m = n - k$. By Note 1, it suffices to show that $h(x, y) \geq 0$ for all $x, y \geq 0$ so that $x + ky = 1 + k$. Since

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{-k(u + 1)}{(2k + 1)(ku^2 + k + 1)},$$

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{k}{2k + 1} \cdot \frac{k(xy + x + y - 1) - 1}{(kx^2 + k + 1)(ky^2 + k + 1)},$$

we only need to show that

$$k(xy + x + y - 1) - 1 \geq 0.$$

Indeed,

$$\begin{aligned} k(xy + x + y - 1) - 1 &= ky(1 + x) + kx - k - 1 = (1 + k - x)(1 + x) + kx - k - 1 \\ &= x(2k - x) = x(k - 1 + ky) \geq x(k - 1) \geq 0. \end{aligned}$$

The proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = a_2 = \cdots = a_k = 1 + \frac{1}{k}, \quad a_{k+1} = \cdots = a_{n-1} = 1, \quad a_n = 0.$$

□

Solution. Consider the inequality

$$\frac{1}{a_1^2 + k} + \frac{1}{a_2^2 + k} + \cdots + \frac{1}{a_n^2 + k} \geq \frac{n}{1 + k}, \quad k \in [0, 3];$$

and write it as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

and

$$f(u) = \frac{1}{u^2 + k}, \quad u \geq 0.$$

For $u \geq 1$, we have

$$f''(u) = \frac{2(3u^2 - k)}{(u^2 + k)^3} \geq \frac{2(3 - k)}{(u^2 + k)^3} \geq 0,$$

hence $f(u)$ is convex for $u \geq s$. Therefore, we may apply the RHCF-OV Theorem for $m = n - 1$ and $m = n - 2$, respectively. By Note 1, it suffices to show that $h(x, y) \geq 0$ for all $x, y \geq 0$ so that $x + (n - m)y = 1 + n - m$. Since

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{-u - 1}{(1 + k)(u^2 + k)},$$

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{xy + x + y - k}{(1 + k)(x^2 + k)(y^2 + k)},$$

we only need to show that

$$xy + x + y - k \geq 0.$$

(a) We need to show that $xy + x + y - k \geq 0$ for $k = 2$, $m = n - 1$, $x + y = 2$; indeed, we have

$$xy + x + y - k = xy \geq 0.$$

From $x < y$, $x + y = 2$ and $xy + x + y - k = 0$, we get $x = 0$ and $y = 2$. Therefore, by Note 4, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = 0, \quad a_2 = \cdots = a_{n-1} = 1, \quad a_n = 2.$$

(b) We have $k = 3/2$, $m = n - 2$, $x + 2y = 3$, hence

$$xy + x + y - k = \frac{x(4-x)}{2} = \frac{x(1+2y)}{2} \geq 0.$$

From $x + 2y = 3$ and $xy + x + y - k = 0$, we get $x = 0$ and $y = 3/2$. Therefore, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = 0, \quad a_2 = \cdots = a_{n-2} = 1, \quad a_{n-1} = a_n = \frac{3}{2}.$$

□

P 2.13. If $a_1, a_2, \dots, a_{2n}$ are nonnegative real numbers so that

$$a_1 \geq \cdots \geq a_n \geq 1 \geq a_{n+1} \geq \cdots \geq a_{2n}, \quad a_1 + a_2 + \cdots + a_{2n} = 2n,$$

then

$$\frac{1}{na_1^2 + n^2 + n + 1} + \frac{1}{na_2^2 + n^2 + n + 1} + \cdots + \frac{1}{na_{2n}^2 + n^2 + n + 1} \leq \frac{2n}{(n+1)^2}.$$

(Vasile C., 2007)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_{2n}) \geq 2nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_{2n}}{2n} = 1,$$

where

$$f(u) = \frac{-1}{nu^2 + n^2 + n + 1}, \quad u \geq 0.$$

For $u \in [0, 1]$, we have

$$f''(u) = \frac{2nu(n^2 + n + 1 - 3nu^2)}{(nu^2 + n^2 + n + 1)^3} \geq \frac{2nu(n^2 + n + 1 - 3n)}{(nu^2 + n^2 + n + 1)^3} \geq 0,$$

hence f is convex on $[0, s]$. Therefore, we may apply the LHCF-OV Theorem for $2n$ numbers and $m = n$. By Note 1, it suffices to show that $h(x, y) \geq 0$ for all $x, y \geq 0$ so that $x + ny = 1 + n$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{n(u+1)}{(n+1)^2(nu^2 + n^2 + n + 1)},$$

$$\begin{aligned}
h(x, y) &= \frac{g(x) - g(y)}{x - y} \\
&= \frac{n(n^2 + n + 1 - nx - ny - nxy)}{(n+1)^2(nx^2 + n^2 + n + 1)(ny^2 + n^2 + n + 1)} \\
&= \frac{n(ny - 1)^2}{(n+1)^2(nx^2 + n^2 + n + 1)(ny^2 + n^2 + n + 1)} \geq 0.
\end{aligned}$$

From $x + ny = 1 + n$ and $h(x, y) = 0$, we get $x = n$ and $y = 1/n$. Therefore, the equality holds for $a_1 = a_2 = \cdots = a_{2n} = 1$, and also for

$$a_1 = n, \quad a_2 = \cdots = a_n = 1, \quad a_{n+1} = \cdots = a_{2n} = \frac{1}{n}.$$

□

P 2.14. If a, b, c, d, e, f are nonnegative real numbers so that

$$a \geq b \geq c \geq 1 \geq d \geq e \geq f, \quad a + b + c + d + e + f = 6,$$

then

$$\frac{3a+4}{3a^2+4} + \frac{3b+4}{3b^2+4} + \frac{3c+4}{3c^2+4} + \frac{3d+4}{3d^2+4} + \frac{3e+4}{3e^2+4} + \frac{3f+4}{3f^2+4} \leq 6.$$

(Vasile C., 2009)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) + f(d) + f(e) + f(f) \geq 6f(s), \quad s = \frac{a+b+c+d+e+f}{6} = 1,$$

where

$$f(u) = \frac{-3u-4}{3u^2+4}, \quad u \geq 0.$$

For $u \in [0, 1]$, we have

$$f''(u) = \frac{6(16 - 9u^3) + 216u(1 - u)}{(3u^2 + 4)^3} > 0,$$

hence f is convex on $[0, s]$. Therefore, we may apply the LHCF-OV Theorem for $n = 6$ and $m = 3$. By Note 1, it suffices to show that $h(x, y) \geq 0$ for all $x, y \geq 0$ so that $x + 3y = 4$. We have

$$\begin{aligned}
g(u) &= \frac{f(u) - f(1)}{u - 1} = \frac{3u}{3u^2 + 4}, \\
h(x, y) &= \frac{g(x) - g(y)}{x - y} = \frac{3(4 - 3xy)}{(3x^2 + 4)(3y^2 + 4)} \\
&= \frac{3(x - 2)^2}{(3x^2 + 4)(3y^2 + 4)} \geq 0.
\end{aligned}$$

From $x + 3y = 4$ and $h(x, y) = 0$, we get $x = 2$ and $y = 2/3$. Therefore, in accordance with Note 4, the equality holds for $a = b = c = d = e = f = 1$, and also for

$$a = 2, \quad b = c = 1, \quad d = e = f = \frac{2}{3}.$$

□

P 2.15. If a, b, c, d, e, f are nonnegative real numbers so that

$$a \geq b \geq 1 \geq c \geq d \geq e \geq f, \quad a + b + c + d + e + f = 6,$$

then

$$\frac{a^2 - 1}{(2a + 7)^2} + \frac{b^2 - 1}{(2b + 7)^2} + \frac{c^2 - 1}{(2c + 7)^2} + \frac{d^2 - 1}{(2d + 7)^2} + \frac{e^2 - 1}{(2e + 7)^2} + \frac{f^2 - 1}{(2f + 7)^2} \geq 0.$$

(Vasile C., 2009)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) + f(d) + f(e) + f(f) \geq 6f(s), \quad s = \frac{a + b + c + d + e + f}{6} = 1,$$

where

$$f(u) = \frac{u^2 - 1}{(2u + 7)^2}, \quad u \geq 0.$$

For $u \in [0, 1]$, we have

$$f''(u) = \frac{2(37 - 28u)}{(2u + 7)^4} > 0,$$

hence f is convex on $[0, s]$. Therefore, we may apply the LHCF-OV Theorem for $n = 6$ and $m = 2$. By Note 1, it suffices to show that $h(x, y) \geq 0$ for all $x, y \geq 0$ so that $x + 4y = 5$. We have

$$\begin{aligned} g(u) &= \frac{f(u) - f(1)}{u - 1} = \frac{u + 1}{(2u + 7)^2}, \\ h(x, y) &= \frac{g(x) - g(y)}{x - y} = \frac{21 - 4x - 4y - 4xy}{(2x + 7)^2(2y + 7)^2} \\ &= \frac{(x - 4)^2}{(2x + 7)^2(2y + 7)^2} \geq 0. \end{aligned}$$

From $x + 4y = 5$ and $h(x, y) = 0$, we get $x = 4$ and $y = 1/4$. Therefore, the equality holds only for $a = b = c = d = e = f = 1$, and also for

$$a = 4, \quad b = 1, \quad c = d = e = f = \frac{1}{4}.$$

□

P 2.16. If a, b, c, d, e, f are nonnegative real numbers so that

$$a \leq b \leq 1 \leq c \leq d \leq e \leq f, \quad a + b + c + d + e + f = 6,$$

then

$$\frac{a^2 - 1}{(2a + 5)^2} + \frac{b^2 - 1}{(2b + 5)^2} + \frac{c^2 - 1}{(2c + 5)^2} + \frac{d^2 - 1}{(2d + 5)^2} + \frac{e^2 - 1}{(2e + 5)^2} + \frac{f^2 - 1}{(2f + 5)^2} \leq 0.$$

(Vasile C., 2009)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) + f(d) + f(e) + f(f) \geq 6f(s), \quad s = \frac{a + b + c + d + e + f}{6} = 1,$$

where

$$f(u) = \frac{1 - u^2}{(2u + 5)^2}, \quad u \geq 0.$$

For $u \geq 1$, we have

$$f''(u) = \frac{2(20u - 13)}{(2u + 5)^4} > 0,$$

hence $f(u)$ is convex for $u \geq s$. Therefore, we may apply the RHCF-OV Theorem for $n = 6$ and $m = 2$. By Note 1, it suffices to show that $h(x, y) \geq 0$ for all $x, y \geq 0$ so that $x + 4y = 5$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{-u - 1}{(2u + 5)^2},$$

$$\begin{aligned} h(x, y) &= \frac{g(x) - g(y)}{x - y} \\ &= \frac{4xy + 4x + 4y - 5}{(2x + 5)^2(2y + 5)^2} \\ &= \frac{4xy + 3x}{(2x + 5)^2(2y + 5)^2} \geq 0. \end{aligned}$$

From $x + 4y = 5$ and $h(x, y) = 0$, we get $x = 0$ and $y = 5/4$. Therefore, in accordance with Note 4, the equality holds only for $a = b = c = d = e = f = 1$, and also for

$$a = 0, \quad b = 1, \quad c = d = e = f = \frac{5}{4}.$$

□

P 2.17. If a, b, c are nonnegative real numbers so that

$$a \leq b \leq 1 \leq c, \quad a + b + c = 3,$$

then

$$\sqrt{\frac{2a}{b+c}} + \sqrt{\frac{2b}{c+a}} + \sqrt{\frac{2c}{a+b}} \geq 3.$$

(Vasile C., 2008)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) \geq 3f(s), \quad s = \frac{a + b + c}{3} = 1,$$

where

$$f(u) = \sqrt{\frac{u}{3-u}}, \quad u \in [0, 3).$$

From

$$f''(u) = \frac{3(4u-3)}{4u^{3/2}(3-u)^{5/2}},$$

it follows that $f(u)$ is convex for $u \geq s$. Therefore, we may apply the RHCF-OV Theorem for $n = 3$ and $m = 2$. So, it suffices to show that

$$f(x) + f(y) \geq 2f(1)$$

for $x + y = 2$, $0 \leq x \leq 1 \leq y$. This inequality is true if $g(x) \geq 0$, where

$$g(x) = f(x) + f(y) - 2f(1), \quad y = 2 - x, \quad x \in [0, 1].$$

Since $y' = -1$, we have

$$g'(x) = f'(x) - f'(y) = \frac{3}{2} \left[\frac{1}{\sqrt{x(3-x)^3}} - \frac{1}{\sqrt{y(3-y)^3}} \right].$$

The derivative $f'(x)$ has the same sign as $h(x)$, where

$$\begin{aligned} h(x) &= y(3-y)^3 - x(3-x)^3 = (2-x)(1+x)^3 - x(3-x)^3 \\ &= 2(1-11x+15x^2-5x^3) = 2(1-x)(1-10x+5x^2). \end{aligned}$$

Let

$$x_1 = 1 - \frac{2}{\sqrt{5}}.$$

Since $h(x_1) = 0$, $h(x) > 0$ for $x \in [0, x_1)$ and $h(x) < 0$ for $x \in (x_1, 1)$, it follows that g is increasing on $[0, x_1]$ and decreasing on $[x_1, 1]$. From

$$g(0) = f(0) + f(2) - 2f(1) = 0,$$

$$g(1) = f(1) + f(1) - 2f(1) = 0,$$

it follows that $g(x) \geq 0$ for $x \in [0, 1]$.

The equality holds for $a = b = c = 1$, and also for $a = 0$, $b = 1$ and $c = 2$.

□

P 2.18. If $a_1, a_2, \dots, a_8$ are nonnegative real numbers so that

$$a_1 \geq a_2 \geq a_3 \geq a_4 \geq 1 \geq a_5 \geq a_6 \geq a_7 \geq a_8, \quad a_1 + a_2 + \dots + a_8 = 8,$$

then

$$(a_1^2 + 1)(a_2^2 + 1) \cdots (a_8^2 + 1) \geq (a_1 + 1)(a_2 + 1) \cdots (a_8 + 1).$$

(Vasile C., 2008)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_8) \geq 8f(s), \quad s = \frac{a_1 + a_2 + \cdots + a_8}{8} = 1,$$

where

$$f(u) = \ln(u^2 + 1) - \ln(u + 1), \quad u \geq 0.$$

For $u \in [0, 1]$, we have

$$f''(u) = \frac{2(1 - u^2)}{(u^2 + 1)^2} + \frac{1}{(u + 1)^2} = \frac{(u^2 - u^4) + 4u(1 - u^2) + u^2 + 3}{(u^2 + 1)^2(u + 1)^2} > 0.$$

Therefore, f is convex on $[0, s]$. According to the LHCF-OV Theorem applied for $n = 8$ and $m = 4$, it suffices to show that $f(x) + 4f(y) \geq 5f(1)$ for $x, y \geq 0$ so that $x + 4y = 5$. Using Note 2, we only need to show that $H(x, y) \geq 0$ for $x, y \geq 0$ so that $x + 4y = 5$, where

$$H(x, y) = \frac{f'(x) - f'(y)}{x - y} = \frac{2(1 - xy)}{(x^2 + 1)(y^2 + 1)} + \frac{1}{(x + 1)(y + 1)}.$$

The inequality $H(x, y) \geq 0$ is equivalent to

$$2(1 - xy)(x + 1)(y + 1) + (x^2 + 1)(y^2 + 1) \geq 0.$$

Since $2(x^2 + 1) \geq (x + 1)^2$ and $2(y^2 + 1) \geq (y + 1)^2$, it suffices to prove that

$$8(1 - xy) + (x + 1)(y + 1) \geq 0.$$

Indeed,

$$8(1 - xy) + (x + 1)(y + 1) = 8x^2 - 38x + 14 = 28(x - 19/28)^2 + 31/28 > 0.$$

The proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_8$.

□

P 2.19. If a, b, c, d are real numbers so that

$$\frac{-1}{2} \leq a \leq b \leq 1 \leq c \leq d, \quad a + b + c + d = 4,$$

then

$$7 \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} \right) + 3 \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right) \geq 40.$$

(Vasile C., 2011)

Solution. We have

$$d = 4 - a - b - c \leq 4 + \frac{1}{2} + \frac{1}{2} - 1 = 4.$$

Write the inequality as

$$f(a) + f(b) + f(c) + f(d) \geq 4f(s), \quad s = \frac{a + b + c + d}{4} = 1,$$

where

$$f(u) = \frac{7}{u^2} + \frac{3}{u}, \quad u \in \mathbb{I} = \left[\frac{-1}{2}, 4 \right] \setminus \{0\}.$$

Clearly, $f(u)$ is convex for $u \geq 1$ (because $\frac{7}{u^2}$ and $\frac{3}{u}$ are convex). According to Note 3, we may apply the RHCF-OV Theorem for $n = 4$ and $m = 2$. By Note 1, we only need to show that $h(x, y) \geq 0$ for $x, y \in \mathbb{I}$ so that $x + 2y = 3$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

We have

$$g(u) = -\frac{7}{u^2} - \frac{10}{u},$$

$$h(x, y) = \frac{7(x + y) + 10xy}{x^2y^2} = \frac{(2x + 1)(-5x + 21)}{2x^2y^2} \geq 0.$$

From $x + 2y = 3$ and $h(x, y) = 0$, we get $x = -1/2$, $y = 7/3$. Therefore, in accordance with Note 4, the equality holds for $a = b = c = d = 1$, and also for

$$a = \frac{-1}{2}, \quad b = 1, \quad c = d = \frac{7}{4}.$$

□

P 2.20. Let a, b, c, d be real numbers such that $a + b + c + d = 4$. Prove that

(a) if $-1 \leq a \leq b \leq c \leq 1 \leq d$, then

$$3 \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} \right) \geq 8 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d};$$

(b) if $-1 \leq a \leq b \leq 1 \leq c \leq d$, then

$$2 \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} \right) \geq 4 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}.$$

(Vasile C., 2011)

Solution. (a) We have

$$d = 4 - a - b - c \leq 4 + 1 + 1 + 1 = 7.$$

Write the desired inequality as

$$f(a) + f(b) + f(c) + f(d) \geq 4f(s), \quad s = \frac{a + b + c + d}{4} = 1,$$

where

$$f(u) = \frac{3}{u^2} - \frac{1}{u}, \quad u \in \mathbb{I} = [-1, 7] \setminus \{0\}.$$

From

$$f''(u) = \frac{2(9-u)}{u^4} > 0,$$

it follows that f is convex on $\mathbb{I}_{\geq s}$. According to Note 3, we may apply the RHCF-OV Theorem for $n = 4$ and $m = 3$. By Note 1, it suffices to show that $h(x, y) \geq 0$ for all $x, y \in \mathbb{I}$ so that $x + y = 2$. We have

$$\begin{aligned} g(u) &= \frac{f(u) - f(1)}{u - 1} = -\frac{2}{u} - \frac{3}{u^2}, \\ h(x, y) &= \frac{g(x) - g(y)}{x - y} = \frac{3(x + y) + 2xy}{x^2y^2} \\ &= \frac{2(x + 1)(3 - x)}{x^2y^2} = \frac{2(x + 1)(y + 1)}{x^2y^2} \geq 0. \end{aligned}$$

From $x < y$, $x + y = 2$ and $h(x, y) = 0$, we get $x = -1$ and $y = 3$. Therefore, in accordance with Note 4, the equality holds for $a = b = c = d = 1$, and also for

$$a = -1, \quad b = c = 1, \quad d = 3.$$

(b) We have

$$d = 4 - a - b - c \leq 4 + 1 + 1 - 1 = 5.$$

Write the desired inequality as

$$f(a) + f(b) + f(c) + f(d) \geq 4f(s), \quad s = \frac{a + b + c + d}{4} = 1,$$

where

$$f(u) = \frac{2}{u^2} - \frac{1}{u}, \quad u \in \mathbb{I} = [-1, 5] \setminus \{0\}.$$

From

$$f''(u) = \frac{2(6-u)}{u^4} > 0,$$

it follows that f is convex on $\mathbb{I}_{\geq s}$. According to Note 3, we may apply the RHCF-OV Theorem for $n = 4$ and $m = 2$. By Note 1, it suffices to show that $h(x, y) \geq 0$ for all $x, y \in \mathbb{I}$ so that $x + 2y = 3$. We have

$$\begin{aligned} g(u) &= \frac{f(u) - f(1)}{u - 1} = -\frac{1}{u} - \frac{2}{u^2}, \\ h(x, y) &= \frac{g(x) - g(y)}{x - y} = \frac{2(x + y) + xy}{x^2y^2} \\ &= \frac{(x + 1)(6 - x)}{2x^2y^2} \geq 0. \end{aligned}$$

From $x + 2y = 3$ and $h(x, y) = 0$, we get $x = -1$ and $y = 2$. Therefore, the equality holds for $a = b = c = d = 1$, and also for

$$a = -1, \quad b = 1, \quad c = d = 2.$$

□

P 2.21. If a, b, c, d are positive real numbers so that

$$a \geq b \geq 1 \geq c \geq d, \quad abcd = 1,$$

then

$$a^2 + b^2 + c^2 + d^2 - 4 \geq 18 \left(a + b + c + d - \frac{1}{a} - \frac{1}{b} - \frac{1}{c} - \frac{1}{d} \right).$$

(Vasile C., 2008)

Solution. Using the substitution

$$a = e^x, \quad b = e^y, \quad c = e^z, \quad d = e^w,$$

we need to show that

$$f(x) + f(y) + f(z) + f(w) \geq 4f(s),$$

where

$$x \geq y \geq 0 \geq z \geq w, \quad s = \frac{x + y + z + w}{4} = 0,$$

$$f(u) = e^{2u} - 1 - 18(e^u - e^{-u}), \quad u \in \mathbb{R}.$$

For $u \leq 0$, we have

$$f''(u) = 4e^{2u} + 18(e^{-u} - e^u) > 0,$$

hence f is convex on $(-\infty, s]$. By the LHCF-OV Theorem applied for $n = 4$ and $m = 2$, it suffices to show that $f(x) + 2f(y) \geq 3f(0)$ for all real x, y so that $x + 2y = 0$; that is, to show that

$$a^2 + 2b^2 - 3 - 18 \left(a + 2b - \frac{1}{a} - \frac{2}{b} \right) \geq 0$$

for all $a, b > 0$ so that $ab^2 = 1$. This inequality is equivalent to

$$\frac{(b^2 - 1)^2(2b^2 + 1)}{b^4} + \frac{18(b - 1)^3(b + 1)}{b^2} \geq 0,$$

$$\frac{(b - 1)^2(2b - 1)^2(b + 1)(5b + 1)}{b^4} \geq 0.$$

The proof is completed. The equality holds for $a = b = c = d = 1$, and also for

$$a = 4, \quad b = 1, \quad c = d = 1/2.$$

□

P 2.22. If a, b, c, d are positive real numbers so that

$$a \leq b \leq 1 \leq c \leq d, \quad abcd = 1,$$

then

$$\sqrt{a^2 - a + 1} + \sqrt{b^2 - b + 1} + \sqrt{c^2 - c + 1} + \sqrt{d^2 - d + 1} \geq a + b + c + d.$$

(Vasile C., 2008)

Solution. Using the substitution

$$a = e^x, \quad b = e^y, \quad c = e^z, \quad d = e^w,$$

we need to show that

$$f(x) + f(y) + f(z) + f(w) \geq 4f(s),$$

where

$$x \leq y \leq 0 \leq z \leq w, \quad s = \frac{x + y + z + w}{4} = 0,$$

$$f(u) = \sqrt{e^{2u} - e^u + 1} - e^u, \quad u \in \mathbb{R}.$$

We claim that f is convex for $u \geq 0$. Since

$$e^{-u} f''(u) = \frac{4e^{3u} - 6e^{2u} + 9e^u - 2}{4(e^{2u} - e^u + 1)^{3/2}} - 1,$$

we need to show that

$$4t^3 - 6t^2 + 9t - 2 \geq 0$$

and

$$(4t^3 - 6t^2 + 9t - 2)^2 \geq 16(t^2 - t + 1)^3,$$

where $t = e^u \geq 1$. Indeed, we have

$$4t^3 - 6t^2 + 9t - 2 \geq 4t^3 - 6t^2 + 7t > 4t^3 - 6t^2 + 2t = 2t(t - 1)(2t - 1) \geq 0$$

and

$$(4t^3 - 6t^2 + 9t - 2)^2 - 16(t^2 - t + 1)^3 = 12t^3(t - 1) + 9t^2 + 12(t - 1) > 0.$$

By the RHCF-OV Theorem applied for $n = 4$ and $m = 2$, it suffices to show that $f(x) + 2f(y) \geq 3f(0)$ for all real x, y so that $x + 2y = 0$; that is, to show that

$$\sqrt{a^2 - a + 1} + 2\sqrt{b^2 - b + 1} \geq a + 2b$$

for all $a, b > 0$ so that $ab^2 = 1$. This inequality is equivalent to

$$\begin{aligned} \frac{\sqrt{b^4 - b^2 + 1}}{b^2} + 2\sqrt{b^2 - b + 1} &\geq \frac{1}{b^2} + 2b, \\ \frac{\sqrt{b^4 - b^2 + 1} - 1}{b^2} + 2(\sqrt{b^2 - b + 1} - 1) &\geq 0, \\ \frac{b^2 - 1}{\sqrt{b^4 - b^2 + 1} + 1} + \frac{2(1 - b)}{\sqrt{b^2 - b + 1} + b} &\geq 0. \end{aligned}$$

Since

$$\frac{b^2 - 1}{\sqrt{b^4 - b^2 + 1} + 1} \geq \frac{b^2 - 1}{b^2 + 1},$$

it suffices to show that

$$\frac{b^2 - 1}{b^2 + 1} + \frac{2(1 - b)}{\sqrt{b^2 - b + 1} + b} \geq 0,$$

which is equivalent to

$$\begin{aligned} (b-1) \left[\frac{b+1}{b^2+1} - \frac{2}{\sqrt{b^2-b+1}+b} \right] &\geq 0, \\ (b-1) \left[(b+1)\sqrt{b^2-b+1} - b^2+b-2 \right] &\geq 0, \\ \frac{(b-1)^2(3b^2-2b+3)}{(b+1)\sqrt{b^2-b+1}+b^2-b+2} &\geq 0. \end{aligned}$$

The last inequality is clearly true. The equality holds for $a = b = c = d = 1$.

□

P 2.23. If a, b, c, d are positive real numbers so that

$$a \leq b \leq c \leq 1 \leq d, \quad abcd = 1,$$

then

$$\frac{1}{a^3+3a+2} + \frac{1}{b^3+3b+2} + \frac{1}{c^3+3c+2} + \frac{1}{d^3+3d+2} \geq \frac{2}{3}.$$

(Vasile C., 2007)

Solution. Using the substitution

$$a = e^x, \quad b = e^y, \quad c = e^z, \quad d = e^w,$$

we need to show that

$$f(x) + f(y) + f(z) + f(w) \geq 4f(s),$$

where

$$x \leq y \leq z \leq 0 \leq w, \quad s = \frac{x+y+z+w}{4} = 0,$$

$$f(u) = \frac{1}{e^{3u} + 3e^u + 2}, \quad u \in \mathbb{R}.$$

We claim that f is convex for $u \geq 0$. Indeed, denoting $t = e^u$, $t \geq 1$, we have

$$\begin{aligned} f''(u) &= \frac{3t(3t^5 + 2t^3 - 6t^2 + 3t - 2)}{(t^3 + 3t + 2)^3} \\ &= \frac{3t(t-1)(3t^4 + 3t^3 + 5t^2 - t + 2)}{(t^3 + 3t + 2)^3} \geq 0. \end{aligned}$$

By the RHCF-OV Theorem applied for $n = 4$ and $m = 3$, it suffices to show that $f(x) + f(y) \geq 2f(0)$ for all real x, y so that $x + y = 0$; that is, to show that

$$\frac{1}{a^3+3a+2} + \frac{1}{b^3+3b+2} \geq \frac{1}{3}$$

for all $a, b > 0$ so that $ab = 1$. This inequality is equivalent to

$$(a-1)^4(a^2+a+1) \geq 0.$$

The equality holds for $a = b = c = d = 1$.

□

P 2.24. If $a_1, a_2, \dots, a_n$ are positive real numbers so that

$$a_1 \geq \dots \geq a_{n-1} \geq 1 \geq a_n, \quad a_1 a_2 \cdots a_n = 1,$$

then

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \geq a_1 + a_2 + \dots + a_n.$$

(Vasile C., 2007)

Solution. Using the substitution

$$a_i = e^{x_i}, \quad i = 1, 2, \dots, n,$$

we need to show that

$$f(x_1) + f(x_2) + \dots + f(x_n) \geq n f(s),$$

where

$$x_1 \geq \dots \geq x_{n-1} \geq 0 \geq x_n, \quad s = \frac{x_1 + x_2 + \dots + x_n}{n} = 0,$$

$$f(u) = e^{-u} - e^u, \quad u \in \mathbb{R}.$$

For $u \leq 0$, we have

$$f''(u) = e^{-u} - e^u \geq 0,$$

therefore $f(u)$ is convex for $u \leq s$. By the LHCF-OV Theorem applied for $m = n - 1$, it suffices to show that $f(x) + f(y) \geq 2f(0)$ for all real x, y so that $x + y = 0$; that is, to show that

$$\frac{1}{a} - a + \frac{1}{b} - b \geq 0$$

for all $a, b > 0$ so that $ab = 1$. This is true since

$$\frac{1}{a} - a + \frac{1}{b} - b = \frac{1}{a} - a + a - \frac{1}{a} = 0.$$

The equality holds for

$$a_1 \geq 1, \quad a_2 = \dots = a_{n-1} = 1, \quad a_n = 1/a_1.$$

□

P 2.25. Let $a_1, a_2, \dots, a_n$ be positive real numbers so that

$$a_1 \leq \dots \leq a_{n-1} \leq 1 \leq a_n, \quad a_1 a_2 \cdots a_n = 1.$$

If $k \geq 1$, then

$$\frac{1}{1 + k a_1} + \frac{1}{1 + k a_2} + \dots + \frac{1}{1 + k a_n} \geq \frac{n}{1 + k}.$$

(Vasile C., 2007)

Solution. Using the substitution

$$a_i = e^{x_i}, \quad i = 1, 2, \dots, n,$$

we need to show that

$$f(x_1) + f(x_2) + \dots + f(x_n) \geq nf(s),$$

where

$$x_1 \leq \dots \leq x_{n-1} \leq 0 \leq x_n, \quad s = \frac{x_1 + x_2 + \dots + x_n}{n} = 0,$$

$$f(u) = \frac{1}{1 + ke^u}, \quad u \in \mathbb{R}.$$

For $u \geq 0$, we have

$$f''(u) = \frac{ke^u(ke^u - 1)}{(1 + ke^u)^3} \geq 0,$$

therefore $f(u)$ is convex for $u \geq s$. By the RHCF-OV Theorem applied for $m = n - 1$, it suffices to show that $f(x) + f(y) \geq 2f(0)$ for all real x, y so that $x + y = 0$; that is, to show that

$$\frac{1}{1 + ka} + \frac{1}{1 + kb} \geq \frac{2}{1 + k}$$

for all $a, b > 0$ so that $ab = 1$. This is true since

$$\frac{1}{1 + ka} + \frac{1}{1 + kb} - \frac{2}{1 + k} = \frac{k(k-1)(a-1)^2}{(1 + ka)(a + k)} \geq 0.$$

The equality holds for $a_1 = a_2 = \dots = a_n = 1$. If $k = 1$, then the equality holds for

$$a_1 \leq 1, \quad a_2 = \dots = a_{n-1} = 1, \quad a_n = 1/a_1.$$

□

P 2.26. If $a_1, a_2, \dots, a_9$ are positive real numbers so that

$$a_1 \leq \dots \leq a_8 \leq 1 \leq a_9, \quad a_1 a_2 \dots a_9 = 1,$$

then

$$\frac{1}{(a_1 + 2)^2} + \frac{1}{(a_2 + 2)^2} + \dots + \frac{1}{(a_9 + 2)^2} \geq 1.$$

(Vasile C., 2007)

Solution. Using the substitution

$$a_i = e^{x_i}, \quad i = 1, 2, \dots, 9,$$

we can write the inequality as

$$f(x_1) + f(x_2) + \dots + f(x_9) \geq 9f(s),$$

where

$$x_1 \leq \cdots \leq x_8 \leq 0 \leq x_9, \quad s = \frac{x_1 + x_2 + \cdots + x_9}{9} = 0,$$

$$f(u) = \frac{1}{(e^u + 2)^2}, \quad u \in \mathbb{R}.$$

For $u \in [0, \infty)$, we have

$$f''(u) = \frac{4e^u(e^u - 1)}{(e^u + 2)^4} \geq 0,$$

hence f is convex on $[s, \infty)$. According to the RHCF-OV Theorem (case $n = 9$ and $m = 8$), it suffices to show that $f(x) + f(y) \geq 2f(0)$ for all real x, y so that $x + y = 0$; that is, to show that

$$\frac{1}{(a+2)^2} + \frac{1}{(b+2)^2} \geq \frac{2}{9}$$

for all $a, b > 0$ so that $ab = 1$. Write this inequality as

$$\frac{b^2}{(2b+1)^2} + \frac{1}{(b+2)^2} \geq \frac{2}{9},$$

which is equivalent to the obvious inequality

$$(b-1)^4 \geq 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_9 = 1$.

□

P 2.27. Let $a_1, a_2, \dots, a_n$ be positive real numbers so that

$$a_1 \leq \cdots \leq a_{n-1} \leq 1 \leq a_n, \quad a_1 a_2 \cdots a_n = 1.$$

If $p, q \geq 0$ so that

$$p + q \geq 1 + \frac{2pq}{p + 4q},$$

then

$$\frac{1}{1 + pa_1 + qa_1^2} + \frac{1}{1 + pa_2 + qa_2^2} + \cdots + \frac{1}{1 + pa_n + qa_n^2} \geq \frac{n}{1 + p + q}.$$

(Vasile C., 2007)

Solution. Using the substitution

$$a_i = e^{x_i}, \quad i = 1, 2, \dots, n,$$

we can write the inequality as

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s),$$

where

$$x_1 \leq \cdots \leq x_{n-1} \leq 0 \leq x_n, \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

$$f(u) = \frac{1}{1 + pe^u + qe^{2u}}, \quad u \in \mathbb{R}.$$

We have

$$f''(u) = \frac{e^u f_1(u)}{(1 + pe^u + qe^{2u})^3},$$

where

$$f_1(u) = 4q^2 e^{3u} + 3p q e^{2u} + (p^2 - 4q) e^u - p.$$

The hypothesis $p + q \geq 1 + \frac{2pq}{p + 4q}$ is equivalent to

$$p^2 + 3pq + 4q^2 \geq p + 4q.$$

For $u \in [0, \infty)$, we have

$$f_1(u) \geq 4q^2 e^u + 3p q e^u + (p^2 - 4q) e^u - p \geq p(e^u - 1) \geq 0,$$

hence f is convex on $[s, \infty)$. According to the RHCF-OV Theorem (case $m = n - 1$), it suffices to show that $f(x) + f(y) \geq 2f(0)$ for all real x, y so that $x + y = 0$; that is, to show that

$$\frac{1}{1 + pa + qa^2} + \frac{1}{1 + pb + qb^2} \geq \frac{2}{1 + p + q}$$

for all $a, b > 0$ so that $ab = 1$. Write this inequality as

$$\frac{1}{1 + pa + qa^2} + \frac{a^2}{a^2 + pa + q} \geq \frac{2}{1 + p + q}$$

which is equivalent to

$$(a - 1)^2 h(a) \geq 0,$$

where

$$\begin{aligned} h(a) &= q(p + q - 1)(a^2 + 1) + (p^2 + pq + 2q^2 - p - 2q)a \\ &\geq 2q(p + q - 1)a + (p^2 + pq + 2q^2 - p - 2q)a \\ &= (p^2 + 3pq + 4q^2 - p - 4q)a \geq 0. \end{aligned}$$

The equality holds for $a_1 = a_2 = \dots = a_n = 1$.

Remark. For $p = 1$, $q = 1/4$ and $n = 9$, we get the preceding P 2.26. □

P 2.28. Let $a_1, a_2, \dots, a_n$ be positive real numbers so that

$$a_1 \leq \dots \leq a_{n-1} \leq 1 \leq a_n, \quad a_1 a_2 \dots a_n = 1.$$

If $m \geq 1$ and $0 < k \leq m$, then

$$\frac{1}{(a_1 + k)^m} + \frac{1}{(a_2 + k)^m} + \dots + \frac{1}{(a_n + k)^m} \geq \frac{n}{(1 + k)^m}.$$

(Vasile C., 2007)

Solution. Using the substitution

$$a_i = e^{x_i}, \quad i = 1, 2, \dots, n,$$

we can write the inequality as

$$f(x_1) + f(x_2) + \dots + f(x_n) \geq nf(s),$$

where

$$x_1 \leq \dots \leq x_{n-1} \leq 0 \leq x_n, \quad s = \frac{x_1 + x_2 + \dots + x_n}{n} = 0,$$

$$f(u) = \frac{1}{(e^u + k)^m}, \quad u \in \mathbb{R}.$$

For $u \in [0, \infty)$, we have

$$f''(u) = \frac{me^u(me^u - k)}{(e^u + k)^{m+2}} \geq 0,$$

hence f is convex on $[s, \infty)$. According to the RHCF-OV Theorem (case $m = n - 1$), it suffices to show that $f(x) + f(y) \geq 2f(0)$ for all real x, y so that $x \leq y$ and $x + y = 0$; that is, to show that

$$\frac{1}{(a + k)^m} + \frac{1}{(b + k)^m} \geq \frac{2}{(1 + k)^m}$$

for all $a, b > 0$ so that $a \in (0, 1]$ and $ab = 1$. Write this inequality as $g(a) \geq 0$, where

$$g(a) = \frac{1}{(a + k)^m} + \frac{a^m}{(ka + 1)^m} - \frac{2}{(1 + k)^m},$$

with

$$\frac{g'(a)}{m} = \frac{a^{m-1}(a + k)^{m+1} - (ka + 1)^{m+1}}{(a + k)^{m+1}(ka + 1)^{m+1}}.$$

If $g'(a) \leq 0$ for $a \in (0, 1]$, then g is decreasing, hence $g(a) \geq g(1) = 0$. Thus, it suffices to show that

$$a^{m-1} \leq \left(\frac{ka + 1}{a + k} \right)^{m+1}.$$

Since

$$\frac{ka + 1}{a + k} - \frac{ma + 1}{a + m} = \frac{(m - k)(1 - a^2)}{(a + k)(a + m)} \geq 0,$$

we only need to show that

$$a^{m-1} \leq \left(\frac{ma + 1}{a + m} \right)^{m+1},$$

which is equivalent to $h(a) \leq 0$ for $a \in (0, 1]$, where

$$h(a) = (m - 1) \ln a + (m + 1) \ln(a + m) - (m + 1) \ln(ma + 1),$$

with

$$h'(a) = \frac{m - 1}{a} + \frac{m + 1}{a + m} - \frac{m(m + 1)}{ma + 1} = \frac{m(m - 1)(a - 1)^2}{a(a + m)(ma + 1)}.$$

Since $h'(a) \geq 0$, $h(a)$ is increasing for $a \in (0, 1]$, therefore $h(a) \leq h(1) = 0$. The equality holds for $a_1 = a_2 = \dots = a_n = 1$.

Remark. For $k = m = 2$ and $n = 9$, we get the inequality in P 2.26.

□

P 2.29. If $a_1, a_2, \dots, a_n$ are positive real numbers so that

$$a_1 \leq \dots \leq a_{n-1} \leq 1 \leq a_n, \quad a_1 a_2 \dots a_n = 1,$$

then

$$\frac{1}{\sqrt{1+3a_1}} + \frac{1}{\sqrt{1+3a_2}} + \dots + \frac{1}{\sqrt{1+3a_n}} \geq \frac{n}{2}.$$

(Vasile C., 2007)

Solution. Using the substitution

$$a_i = e^{x_i}, \quad i = 1, 2, \dots, n,$$

we can write the inequality as

$$f(x_1) + f(x_2) + \dots + f(x_n) \geq n f(s),$$

where

$$x_1 \leq \dots \leq x_{n-1} \leq 0 \leq x_n, \quad s = \frac{x_1 + x_2 + \dots + x_n}{n} = 0,$$

$$f(u) = \frac{1}{\sqrt{1+3e^u}}, \quad u \in \mathbb{R}.$$

For $u \geq 0$, we have

$$f''(u) = \frac{3e^u(3e^u - 2)}{4(1+3e^u)^{5/2}} > 0,$$

hence f is convex on $[s, \infty)$. According to the RHCF-OV Theorem (case $m = n - 1$), it suffices to show that $f(x) + f(y) \geq 2f(0)$ for all real x, y so that $x + y = 0$; that is, to show that

$$\frac{1}{\sqrt{1+3a}} + \frac{1}{\sqrt{1+3b}} \geq 1$$

for all $a, b > 0$ so that $ab = 1$. Write this inequality as

$$\frac{1}{\sqrt{1+3a}} + \sqrt{\frac{a}{a+3}} \geq 1.$$

Substituting $\frac{1}{\sqrt{1+3a}} = t$, $0 < t < 1$, the inequality becomes

$$\sqrt{\frac{1-t^2}{8t^2+1}} \geq 1-t.$$

By squaring, we get

$$t(1-t)(2t-1)^2 \geq 0,$$

which is true. The equality holds for $a_1 = a_2 = \dots = a_n = 1$.

□

P 2.30. Let $a_1, a_2, \dots, a_n$ be positive real numbers so that

$$a_1 \leq \dots \leq a_{n-1} \leq 1 \leq a_n, \quad a_1 a_2 \dots a_n = 1.$$

If $0 < m < 1$ and $0 < k \leq \frac{1}{2^{1/m} - 1}$, then

$$\frac{1}{(a_1 + k)^m} + \frac{1}{(a_2 + k)^m} + \dots + \frac{1}{(a_n + k)^m} \geq \frac{n}{(1 + k)^m}.$$

(Vasile C., 2007)

Solution. By Bernoulli's inequality, we have

$$2^{1/m} > 1 + \frac{1}{m},$$

hence

$$k \leq \frac{1}{2^{1/m} - 1} < m < 1.$$

Using the substitution

$$a_i = e^{x_i}, \quad i = 1, 2, \dots, n,$$

we can write the inequality as

$$f(x_1) + f(x_2) + \dots + f(x_n) \geq n f(s),$$

where

$$x_1 \leq \dots \leq x_{n-1} \leq 0 \leq x_n, \quad s = \frac{x_1 + x_2 + \dots + x_n}{n} = 0,$$

$$f(u) = \frac{1}{(e^u + k)^m}, \quad u \in \mathbb{R}.$$

For $u \in [0, \infty)$, we have

$$f''(u) = \frac{me^u(me^u - k)}{(e^u + k)^{m+2}} \geq 0,$$

hence f is convex on $[s, \infty)$. According to the RHCF-OV Theorem (case $m = n - 1$), it suffices to show that $f(x) + f(y) \geq 2f(0)$ for all real x, y so that $x + y = 0$; that is, to show that

$$\frac{1}{(a + k)^m} + \frac{1}{(b + k)^m} \geq \frac{2}{(1 + k)^m}$$

for all $a, b > 0$ so that $ab = 1$. Write this inequality as $g(a) \geq 0$ for $a \geq 1$, where

$$g(a) = \frac{1}{(a + k)^m} + \frac{a^m}{(ka + 1)^m} - \frac{2}{(1 + k)^m}.$$

The derivative

$$\frac{g'(a)}{m} = \frac{a^{m-1}(a + k)^{m+1} - (ka + 1)^{m+1}}{(a + k)^{m+1}(ka + 1)^{m+1}}$$

has the same sign as the function

$$h(a) = (m - 1) \ln a + (m + 1) \ln(a + k) - (m + 1) \ln(ka + 1).$$

We have

$$h'(a) = \frac{m-1}{a} + (m+1) \left(\frac{1}{a+k} - \frac{k}{ka+1} \right) = \frac{kh_1(a)}{a(a+k)(ka+1)},$$

where

$$h_1(a) = (m-1)(a^2+1) - 2 \left(k - \frac{m}{k} \right) a.$$

The discriminant D of the quadratic function $h_1(a)$ is

$$\frac{D}{4} = \left(k - \frac{m}{k} \right)^2 - (m-1)^2 = (1-k^2) \left(\frac{m^2}{k^2} - 1 \right).$$

Since $D > 0$, the roots a_1 and a_2 of $h_1(a)$ are real and unequal. If $a_1 < a_2$, then $h_1(a) \geq 0$ for $a \in [a_1, a_2]$ and $h_1(a) \leq 0$ for $a \in (-\infty, a_1] \cup [a_2, \infty)$. Since

$$h_1(1) = \frac{2(k+1)(m-k)}{k} > 0,$$

it follows that $a_1 < 1 < a_2$, therefore $h_1(a)$ and $h'(a)$ are positive for $a \in [1, a_2]$ and negative for $a \in (a_2, \infty)$, h is increasing on $[1, a_2]$ and decreasing on $[a_2, \infty)$. From $h(1) = 0$ and

$$\lim_{a \rightarrow \infty} h(a) = -\infty,$$

it follows that there is $a_3 > a_2$ so that $h(a)$ and $g'(a)$ are positive for $a \in (1, a_3)$ and negative for $a \in (a_3, \infty)$. As a result, g is increasing on $[1, a_3]$ and decreasing on $[a_3, \infty)$. Since $g(1) = 0$ and

$$\lim_{a \rightarrow \infty} g(a) = \frac{1}{k^m} - \frac{2}{(1+k)^m} \geq 0,$$

it follows that $g(a) \geq 0$ for $a \geq 1$. This completes the proof. The equality holds for $a_1 = a_2 = \dots = a_n = 1$.

Remark. For $k = \frac{1}{3}$ and $m = \frac{1}{2}$, we get the preceding P 2.29.

□

P 2.31. Let $a_1, a_2, \dots, a_n$ be positive real numbers so that

$$a_1 \geq a_2 \geq \dots \geq a_k \geq 1 \geq a_{k+1} \geq \dots \geq a_n, \quad a_1 a_2 \dots a_n = 1.$$

If $n \geq 3$ and $1 \leq k \leq n-1$, then

$$\frac{1}{ka_1+1} + \frac{1}{ka_2+1} + \dots + \frac{1}{ka_n+1} \geq \frac{n}{k+1}.$$

(Vasile C., 2007)

Solution. Using the substitution

$$a_i = e^{x_i}, \quad i = 1, 2, \dots, n,$$

we can write the inequality as

$$f(x_1) + f(x_2) + \dots + f(x_n) \geq nf(s),$$

where

$$x_1 \geq x_2 \geq \dots \geq x_k \geq 1 \geq x_{k+1} \geq \dots \geq x_n, \quad s = \frac{x_1 + x_2 + \dots + x_n}{n} = 0,$$

$$f(u) = \frac{1}{ke^u + 1}, \quad u \in \mathbb{R}.$$

For $u \in [0, \infty)$, we have

$$f''(u) = \frac{ke^u(ke^u - 1)}{(ke^u + 1)^3} > 0,$$

hence f is convex on $[s, \infty)$. According to the RHCF-OV Theorem (case $m = n - k$), it suffices to show that $f(x) + kf(y) \geq (1+k)f(0)$ for all real x, y so that $x + ky = 0$; that is, to show that

$$\frac{1}{ka + 1} + \frac{k}{kb + 1} \geq 1$$

for all $a, b > 0$ so that $ab^k = 1$. The inequality is equivalent to

$$\frac{b^k}{b^k + k} + \frac{k}{kb + 1} \geq 1, \quad kg(b) \geq 0,$$

where

$$g(b) = b^k - kb + k - 1.$$

From $g'(b) = k(b^{k-1} - 1)$, it follows that $g(b)$ is decreasing on $(0, 1]$ and increasing on $[1, \infty)$, hence

$$g(b) \geq g(1) = 0.$$

The equality holds for $a_1 = a_2 = \dots = a_n = 1$.

□

P 2.32. If $a_1, a_2, \dots, a_n$ ($n \geq 4$) are positive real numbers so that

$$a_1 \geq a_2 \geq a_3 \geq 1 \geq a_4 \geq \dots \geq a_n, \quad a_1 a_2 \dots a_n = 1,$$

then

$$\frac{1}{(a_1 + 1)^2} + \frac{1}{(a_2 + 1)^2} + \dots + \frac{1}{(a_n + 1)^2} \geq \frac{n}{4}.$$

(Vasile C., 2007)

Solution. Using the substitution

$$a_i = e^{x_i}, \quad i = 1, 2, \dots, n,$$

we can write the inequality as

$$f(x_1) + f(x_2) + \dots + f(x_n) \geq nf(s),$$

where

$$x_1 \geq x_2 \geq x_3 \geq 0 \geq x_4 \geq \dots \geq x_n, \quad s = \frac{x_1 + x_2 + \dots + x_n}{n} = 0,$$

$$f(u) = \frac{1}{(e^u + 1)^2}, \quad u \in \mathbb{R}.$$

For $u \in [0, \infty)$, we have

$$f''(u) = \frac{2e^u(2e^u - 1)}{(e^u + 1)^4} > 0,$$

hence f is convex on $[s, \infty)$. According to the RHCF-OV Theorem (case $m = 3$), it suffices to show that $f(x) + 3f(y) \geq 4f(0)$ for all real x, y so that $x + 3y = 0$; that is, to show that

$$\frac{1}{(a+1)^2} + \frac{3}{(b+1)^2} \geq 1$$

for all $a, b > 0$ so that $ab^3 = 1$. The inequality is equivalent to

$$\frac{b^6}{(b^3+1)^2} + \frac{3}{(b+1)^2} \geq 1.$$

Using the Cauchy-Schwarz inequality, it suffices to show that

$$\frac{(b^3+3)^2}{(b^3+1)^2 + 3(b+1)^2} \geq 1,$$

which is equivalent to the obvious inequality

$$(b-1)^2(4b+5) \geq 0.$$

The equality holds for $a_1 = a_2 = \dots = a_n = 1$.

□

P 2.33. If $a_1, a_2, \dots, a_n$ are positive real numbers so that

$$a_1 \geq \dots \geq a_{n-1} \geq 1 \geq a_n, \quad a_1 a_2 \dots a_n = 1,$$

then

$$\frac{1}{(a_1+3)^2} + \frac{1}{(a_2+3)^2} + \dots + \frac{1}{(a_n+3)^2} \leq \frac{n}{16}.$$

(Vasile C., 2007)

Solution. Using the substitution

$$a_i = e^{x_i}, \quad i = 1, 2, \dots, n,$$

we can write the inequality as

$$f(x_1) + f(x_2) + \dots + f(x_n) \geq nf(s),$$

where

$$x_1 \geq \dots \geq x_{n-1} \geq 0 \geq x_n, \quad s = \frac{x_1 + x_2 + \dots + x_n}{n} = 0,$$

$$f(u) = \frac{-1}{(e^u + 3)^2}, \quad u \in \mathbb{R}.$$

For $u \in (-\infty, 0]$, we have

$$f''(u) = \frac{2e^u(3 - 2e^u)}{(e^u + 3)^4} > 0,$$

hence f is convex on $(-\infty, s]$. According to the LHCF-OV Theorem (case $m = n - 1$), it suffices to show that $f(x) + f(y) \geq 2f(0)$ for all real x, y so that $x + y = 0$; that is, to show that

$$\frac{1}{(a + 3)^2} + \frac{1}{(b + 3)^2} \leq \frac{1}{8}$$

for all $a, b > 0$ so that $ab = 1$. Write this inequality as

$$\frac{b^2}{(3b + 1)^2} + \frac{1}{(b + 3)^2} \leq \frac{1}{8},$$

which is equivalent to the obvious inequality

$$(b^2 - 1)^2 + 12b(b - 1)^2 \geq 0.$$

The equality holds for $a_1 = a_2 = \dots = a_n = 1$.

Remark. Similarly, we can prove the following generalization:

- Let $a_1, a_2, \dots, a_n$ be positive real numbers so that

$$a_1 \geq \dots \geq a_{n-1} \geq 1 \geq a_n, \quad a_1 a_2 \dots a_n = 1,$$

If $k \geq 1 + \sqrt{2}$, then

$$\frac{1}{(a_1 + k)^2} + \frac{1}{(a_2 + k)^2} + \dots + \frac{1}{(a_n + k)^2} \leq \frac{n}{(1 + k)^2},$$

with equality for $a_1 = a_2 = \dots = a_n = 1$.

□

P 2.34. Let $a_1, a_2, \dots, a_n$ be positive real numbers so that

$$a_1 \geq \dots \geq a_{n-1} \geq 1 \geq a_n, \quad a_1 a_2 \dots a_n = 1.$$

If $p, q \geq 0$ so that $p + q \leq 1$, then

$$\frac{1}{1 + pa_1 + qa_1^2} + \frac{1}{1 + pa_2 + qa_2^2} + \dots + \frac{1}{1 + pa_n + qa_n^2} \leq \frac{n}{1 + p + q}.$$

(Vasile C., 2007)

Solution. Using the substitution

$$a_i = e^{x_i}, \quad i = 1, 2, \dots, n,$$

we can write the inequality as

$$f(x_1) + f(x_2) + \dots + f(x_n) \geq nf(s),$$

where

$$x_1 \geq \dots \geq x_{n-1} \geq 0 \geq x_n, \quad s = \frac{x_1 + x_2 + \dots + x_n}{n} = 0,$$

$$f(u) = \frac{-1}{1 + pe^u + qe^{2u}}, \quad u \in \mathbb{R}.$$

For $u \leq 0$, we have

$$\begin{aligned} f''(u) &= \frac{e^u[-4q^2e^{3u} - 3pqe^{2u} + (4q - p^2)e^u + p]}{(1 + pe^u + qe^{2u})^3} \\ &\geq \frac{e^{2u}[-4q^2 - 3pq + (4q - p^2) + p]}{(1 + pe^u + qe^{2u})^3} \\ &= \frac{e^{2u}[(p + 4q)(1 - p - q) + 2pq]}{(1 + pe^u + qe^{2u})^3} \geq 0, \end{aligned}$$

therefore $f(u)$ is convex for $u \leq s$. According to the LHCF-OV Theorem (case $m = n - 1$), it suffices to show that $f(x) + f(y) \geq 2f(0)$ for all real x, y so that $x + y = 0$; that is, to show that

$$\frac{1}{1 + pa + qa^2} + \frac{1}{1 + pb + qb^2} \leq \frac{2}{1 + p + q}$$

for all $a, b > 0$ so that $ab = 1$. Write this inequality as

$$(a - 1)^2[q(1 - p - q)a^2 + (p + 2q - p^2 - pq - 2q^2)a + q(1 - p - q)] \geq 0,$$

which is true because

$$p + 2q - p^2 - pq - 2q^2 \geq (p + 2q)(p + q) - p^2 - pq - 2q^2 = 2pq \geq 0.$$

The equality holds for $a_1 = a_2 = \dots = a_n = 1$.

□

P 2.35. Let $a_1, a_2, \dots, a_n$ be positive real numbers so that

$$a_1 \geq \dots \geq a_{n-1} \geq 1 \geq a_n, \quad a_1 a_2 \dots a_n = 1.$$

If $m > 1$ and $k \geq \frac{1}{2^{1/m} - 1}$, then

$$\frac{1}{(a_1 + k)^m} + \frac{1}{(a_2 + k)^m} + \dots + \frac{1}{(a_n + k)^m} \leq \frac{n}{(1 + k)^m}.$$

(Vasile C., 2007)

Solution. By Bernoulli's inequality, we have

$$2^{1/m} < 1 + \frac{1}{m},$$

hence

$$k \geq \frac{1}{2^{1/m} - 1} > m > 1.$$

Using the substitution

$$a_i = e^{x_i}, \quad i = 1, 2, \dots, n,$$

we can write the inequality as

$$f(x_1) + f(x_2) + \dots + f(x_n) \geq n f(s),$$

where

$$x_1 \geq \dots \geq x_{n-1} \geq 0 \geq x_n, \quad s = \frac{x_1 + x_2 + \dots + x_n}{n} = 0,$$

$$f(u) = \frac{-1}{(e^u + k)^m}, \quad u \in \mathbb{R}.$$

For $u \leq 0$, we have

$$f''(u) = \frac{m e^u (k - m e^u)}{(e^u + k)^{m+2}} \geq 0,$$

hence f is convex $u \leq s$. By the LHCF-OV Theorem (case $m = n - 1$), it suffices to show that $f(x) + f(y) \geq 2f(0)$ for all real x, y so that $x + y = 0$; that is, to show that

$$\frac{1}{(a + k)^m} + \frac{1}{(b + k)^m} \leq \frac{2}{(1 + k)^m}$$

for all $a, b > 0$ so that $ab = 1$. Write this inequality as $g(a) \leq 0$ for $a \geq 1$, where

$$g(a) = \frac{1}{(a + k)^m} + \frac{a^m}{(ka + 1)^m} - \frac{2}{(1 + k)^m}.$$

The derivative

$$\frac{g'(a)}{m} = \frac{a^{m-1}(a + k)^{m+1} - (ka + 1)^{m+1}}{(a + k)^{m+1}(ka + 1)^{m+1}}$$

has the same sign as the function

$$h(a) = (m - 1) \ln a + (m + 1) \ln(a + k) - (m + 1) \ln(ka + 1).$$

We have

$$h'(a) = \frac{m-1}{a} + (m+1) \left(\frac{1}{a+k} - \frac{k}{ka+1} \right) = \frac{kh_1(a)}{a(a+k)(ka+1)},$$

where

$$h_1(a) = (m-1)(a^2+1) - 2 \left(k - \frac{m}{k} \right) a.$$

The discriminant D of the quadratic function $h_1(a)$ is

$$\frac{D}{4} = \left(k - \frac{m}{k} \right)^2 - (m-1)^2 = (k^2-1) \left(1 - \frac{m^2}{k^2} \right).$$

Since $D > 0$, the roots a_1 and a_2 of $h_1(a)$ are real and unequal. If $a_1 < a_2$, then $h_1(a) \leq 0$ for $a \in [a_1, a_2]$ and $h_1(a) \geq 0$ for $a \in (-\infty, a_1] \cup [a_2, \infty)$. Since

$$h_1(1) = \frac{2(k+1)(m-k)}{k} < 0,$$

it follows that $a_1 < 1 < a_2$, therefore $h_1(a)$ and $h'(a)$ are negative for $a \in [1, a_2)$ and positive for $a \in (a_2, \infty)$, $h(a)$ is decreasing for $a \in [1, a_2]$ and increasing for $a \in [a_2, \infty)$. From $h(1) = 0$ and

$$\lim_{a \rightarrow \infty} h(a) = \infty,$$

it follows that there is $a_3 > a_2$ so that $h(a)$ and $g'(a)$ are negative for $a \in (1, a_3)$ and positive for $a \in (a_3, \infty)$. As a result, g is decreasing on $[1, a_3]$ and increasing on $[a_3, \infty)$. Since $g(1) = 0$ and

$$\lim_{a \rightarrow \infty} g(a) = \frac{1}{k^m} - \frac{2}{(1+k)^m} \leq 0,$$

it follows that $g(a) \leq 0$ for $a \geq 1$. This completes the proof. The equality holds for $a_1 = a_2 = \dots = a_n = 1$. □

P 2.36. If $a_1, a_2, \dots, a_n$ are positive real numbers so that

$$a_1 \geq \dots \geq a_{n-1} \geq 1 \geq a_n, \quad a_1 a_2 \dots a_n = 1,$$

then

$$\frac{1}{\sqrt{1+2a_1}} + \frac{1}{\sqrt{1+2a_2}} + \dots + \frac{1}{\sqrt{1+2a_n}} \leq \frac{n}{\sqrt{3}}.$$

(Vasile C., 2007)

Solution. Using the substitution

$$a_i = e^{x_i}, \quad i = 1, 2, \dots, n,$$

we can write the inequality as

$$f(x_1) + f(x_2) + \dots + f(x_n) \geq nf(s),$$

where

$$x_1 \geq \cdots \geq x_{n-1} \geq 0 \geq x_n, \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

$$f(u) = \frac{-1}{\sqrt{1+2e^u}}, \quad u \in \mathbb{R}.$$

For $u \leq 0$, we have

$$f''(u) = \frac{e^u(1-e^u)}{(1+2e^u)^{5/2}} > 0,$$

hence f is convex on $(-\infty, s]$. According to the LHCF-OV Theorem (case $m = n-1$), it suffices to show that $f(x) + f(y) \geq 2f(0)$ for all real x, y so that $x + y = 0$; that is, to show that

$$\sqrt{\frac{3}{1+2a}} + \sqrt{\frac{3}{1+2b}} \leq 2$$

for all $a, b > 0$ so that $ab = 1$. By the Cauchy-Schwarz inequality, we get

$$\sqrt{\frac{3}{1+2a}} + \sqrt{\frac{3}{1+2b}} \leq \sqrt{\left(\frac{3}{1+2a} + 1\right) \left(1 + \frac{3}{1+2b}\right)} = 2.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 2.37. Let $a_1, a_2, \dots, a_n$ be positive real numbers so that

$$a_1 \geq \cdots \geq a_{n-1} \geq 1 \geq a_n, \quad a_1 a_2 \cdots a_n = 1.$$

If $0 < m < 1$ and $k \geq m$, then

$$\frac{1}{(a_1 + k)^m} + \frac{1}{(a_2 + k)^m} + \cdots + \frac{1}{(a_n + k)^m} \leq \frac{n}{(1 + k)^m}.$$

(Vasile C., 2007)

Solution. Using the substitution

$$a_i = e^{x_i}, \quad i = 1, 2, \dots, n,$$

we can write the inequality as

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s),$$

where

$$x_1 \geq \cdots \geq x_{n-1} \geq 0 \geq x_n, \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

$$f(u) = \frac{-1}{(e^u + k)^m}, \quad u \in \mathbb{R}.$$

For $u \leq 0$, we have

$$f''(u) = \frac{me^u(k - me^u)}{(e^u + k)^{m+2}} \geq 0,$$

hence f is convex on $(-\infty, s]$. According to the LHCF-OV Theorem (case $m = n - 1$), it suffices to show that $f(x) + f(y) \geq 2f(0)$ for all real x, y so that $x + y = 0$; that is, to show that

$$\frac{1}{(a+k)^m} + \frac{1}{(b+k)^m} \leq \frac{2}{(1+k)^m}$$

for all $a, b > 0$ so that $ab = 1$. Write this inequality as $g(a) \leq 0$ for $a \geq 1$, where

$$g(a) = \frac{1}{(a+k)^m} + \frac{a^m}{(ka+1)^m} - \frac{2}{(1+k)^m},$$

with

$$\frac{g'(a)}{m} = \frac{a^{m-1}(a+k)^{m+1} - (ka+1)^{m+1}}{(a+k)^{m+1}(ka+1)^{m+1}}.$$

If $g'(a) \leq 0$ for $a \geq 1$, then g is decreasing, hence $g(a) \leq g(1) = 0$. Thus, it suffices to show that

$$a^{m-1} \leq \left(\frac{ka+1}{a+k} \right)^{m+1}.$$

Since

$$\frac{ka+1}{a+k} - \frac{ma+1}{a+m} = \frac{(k-m)(a^2-1)}{(a+k)(a+m)} \geq 0,$$

we only need to show that

$$a^{m-1} \leq \left(\frac{ma+1}{a+m} \right)^{m+1},$$

which is equivalent to $h(a) \leq 0$ for $a \geq 1$, where

$$h(a) = (m-1) \ln a + (m+1) \ln(a+m) - (m+1) \ln(ma+1),$$

$$h'(a) = \frac{m-1}{a} + \frac{m+1}{a+m} - \frac{m(m+1)}{ma+1} = \frac{m(m-1)(a-1)^2}{a(a+m)(ma+1)}.$$

Since $h'(a) \leq 0$, $h(a)$ is decreasing for $a \geq 1$, hence

$$h(a) \leq h(1) = 0.$$

This completes the proof. The equality holds for $a_1 = a_2 = \dots = a_n = 1$.

Remark. For $k = \frac{1}{2}$ and $m = \frac{1}{2}$, we get the preceding P 2.36.

□

P 2.38. If $a_1, a_2, \dots, a_n$ ($n \geq 3$) are positive real numbers so that

$$a_1 \geq \dots \geq a_{n-2} \geq 1 \geq a_{n-1} \geq a_n, \quad a_1 a_2 \dots a_n = 1,$$

then

$$\frac{1}{(a_1+5)^2} + \frac{1}{(a_2+5)^2} + \dots + \frac{1}{(a_n+5)^2} \leq \frac{n}{36}.$$

(Vasile C., 2007)

Solution. Using the substitution

$$a_i = e^{x_i}, \quad i = 1, 2, \dots, n,$$

we can write the inequality as

$$f(x_1) + f(x_2) + \dots + f(x_n) \geq nf(s),$$

where

$$x_1 \geq \dots \geq x_{n-2} \geq 0 \geq x_{n-1} \geq x_n, \quad s = \frac{x_1 + x_2 + \dots + x_n}{n} = 0,$$

$$f(u) = \frac{-1}{(e^u + 5)^2}, \quad u \in \mathbb{R}.$$

For $u \in (-\infty, 0]$, we have

$$f''(u) = \frac{2e^u(5 - 2e^u)}{(e^u + 5)^4} > 0,$$

hence f is convex on $(-\infty, s]$. According to the LHCF-OV Theorem (case $m = n - 2$), it suffices to show that $f(x) + 2f(y) \geq 3f(0)$ for all real x, y so that $x + 2y = 0$; that is, to show that

$$\frac{1}{(a + 5)^2} + \frac{2}{(b + 5)^2} \leq \frac{1}{12}$$

for all $a, b > 0$ so that $ab^2 = 1$. Since

$$\frac{1}{(a + 5)^2} = \frac{b^4}{(5b^2 + 1)^2} \leq \frac{b^4}{(4b^2 + 2b)^2} = \frac{b^2}{4(2b + 1)^2},$$

it suffices to show that

$$\frac{b^2}{4(2b + 1)^2} + \frac{2}{(b + 5)^2} \leq \frac{1}{12},$$

which is equivalent to the obvious inequality

$$(b - 1)^2(b^2 + 16b + 1) \geq 0.$$

The equality holds for $a_1 = a_2 = \dots = a_n = 1$.

Remark. Similarly, we can prove the following refinement:

- Let $a_1, a_2, \dots, a_n$ be positive real numbers so that

$$a_1 \geq \dots \geq a_{n-2} \geq 1 \geq a_{n-1} \geq a_n, \quad a_1 a_2 \dots a_n = 1.$$

If $k \geq 2 + \sqrt{6}$, then

$$\frac{1}{(a_1 + k)^2} + \frac{1}{(a_2 + k)^2} + \dots + \frac{1}{(a_n + k)^2} \leq \frac{n}{(1 + k)^2},$$

with equality for $a_1 = a_2 = \dots = a_n = 1$.

□

P 2.39. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that

$$a_1 \geq \dots \geq a_{n-1} \geq 1 \geq a_n, \quad a_1^2 + a_2^2 + \dots + a_n^2 = n,$$

then

$$\frac{1}{3-a_1} + \frac{1}{3-a_2} + \dots + \frac{1}{3-a_n} \leq \frac{n}{2}.$$

(Vasile C., 2007)

Solution. From

$$n = a_1^2 + (a_2^2 + \dots + a_{n-1}^2) + a_n^2 \geq a_1^2 + (n-2) + 0,$$

we get

$$a_1 \leq \sqrt{2}.$$

Replacing $a_1, a_2, \dots, a_n$ by $\sqrt{a_1}, \sqrt{a_2}, \dots, \sqrt{a_n}$, we have to prove that

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq nf(s),$$

where

$$2 \geq a_1 \geq \dots \geq a_{n-1} \geq 1 \geq a_n, \quad s = \frac{a_1 + a_2 + \dots + a_n}{n} = 1,$$

$$f(u) = \frac{1}{\sqrt{u}-3}, \quad u \in [0, 2].$$

For $u \in [0, 1]$, we have

$$f''(u) = \frac{3(1-\sqrt{u})}{4u\sqrt{u}(3-\sqrt{u})^3} \geq 0.$$

Therefore, f is convex on $[0, s]$. According to the LHCF-OV Theorem and Note 1 (case $m = n-1$), it suffices to show that $h(x, y) \geq 0$ for $x, y \geq 0$ so that $x + y = 2$. Since

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{-1}{2(3-\sqrt{u})(1+\sqrt{u})}$$

and

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{2 - \sqrt{x} - \sqrt{y}}{2(\sqrt{x} + \sqrt{y})(1 + \sqrt{x})(1 + \sqrt{y})(3 - \sqrt{x})(3 - \sqrt{y})},$$

we need to show that

$$\sqrt{x} + \sqrt{y} \leq 2.$$

Indeed, we have

$$\sqrt{x} + \sqrt{y} \leq \sqrt{2(x+y)} = 2.$$

This completes the proof. The equality holds for $a_1 = a_2 = \dots = a_n = 1$.

□

P 2.40. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that

$$a_1 \leq \dots \leq a_{n-1} \leq 1 \leq a_n, \quad a_1 + a_2 + \dots + a_n = n.$$

Prove that

$$a_1^3 + a_2^3 + \dots + a_n^3 - n \geq (n-1)^2 \left[\left(\frac{n-a_1}{n-1} \right)^3 + \left(\frac{n-a_2}{n-1} \right)^3 + \dots + \left(\frac{n-a_n}{n-1} \right)^3 - n \right].$$

(Vasile C., 2010)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \dots + a_n}{n} = 1,$$

where

$$f(u) = u^3 - (n-1)^2 \left(\frac{n-u}{n-1} \right)^3, \quad u \geq 0.$$

For $u \geq 1$, we have

$$f''(u) = \frac{6n(u-1)}{n-1} \geq 0.$$

Therefore, $f(u)$ is convex for $u \geq s$. Thus, by the RHCF-OV Theorem (case $m = n-1$), it suffices to show that $f(x) + f(y) \geq 2f(1)$ for $x, y \geq 0$ so that $x + y = 2$. We have

$$\begin{aligned} f(x) + f(y) - 2f(1) &= x^3 + y^3 - 2 - (n-1)^2 \left[\left(\frac{n-x}{n-1} \right)^3 + \left(\frac{n-y}{n-1} \right)^3 - 2 \right] \\ &= 6(1-xy) - 6(n-1)^2 \left[1 - \frac{(n-x)(n-y)}{(n-1)^2} \right] = 0. \end{aligned}$$

This completes the proof. The equality holds for

$$a_1 \leq 1, \quad a_2 = \dots = a_{n-1} = 1, \quad a_n = 2 - a_1.$$

□

Chapter 3

Partially Convex Function Method

3.1 Theoretical Basis

The following statement is known as the Right Partially Convex Function Theorem (RPCF-Theorem).

Right Partially Convex Function Theorem (Vasile Cîrtoaje, 2012). *Let f be a real function defined on an interval $\mathbb{I}$ and convex on $[s, s_0]$, where $s, s_0 \in \mathbb{I}$, $s < s_0$. In addition, f is decreasing on $\mathbb{I}_{\leq s_0}$ and $f(u) \geq f(s_0)$ for $u \in \mathbb{I}$. The inequality*

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \dots, a_n \in \mathbb{I}$ satisfying

$$a_1 + a_2 + \cdots + a_n = ns$$

if and only if

$$f(x) + (n-1)f(y) \geq nf(s)$$

for all $x, y \in \mathbb{I}$ so that $x \leq s \leq y$ and $x + (n-1)y = ns$.

Proof. For

$$a_1 = x, \quad a_2 = a_3 = \cdots = a_n = y,$$

the inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq f(s)$$

becomes

$$f(x) + (n-1)f(y) \geq nf(s);$$

therefore, the necessity is obvious.

The proof of sufficiency is based on Lemma below. According to this lemma, it suffices to consider that $a_1, a_2, \dots, a_n \in \mathbb{J}$, where

$$\mathbb{J} = \mathbb{I}_{\leq s_0}.$$

Because $f(u)$ is convex on $\mathbb{J}_{\geq s}$, the desired inequality follows from the RHCF Theorem (see Chapter 1) applied to the interval $\mathbb{J}$.

Lemma. Let f be a real function defined on an interval $\mathbb{I}$. In addition, f is decreasing on $\mathbb{I}_{\leq s_0}$, and $f(u) \geq f(s_0)$ for $u \in \mathbb{I}$, where $s, s_0 \in \mathbb{I}$, $s < s_0$. If the inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s)$$

holds for all $a_1, a_2, \dots, a_n \in \mathbb{I}_{\leq s_0}$ so that $a_1 + a_2 + \cdots + a_n = ns$, then it holds for all $a_1, a_2, \dots, a_n \in \mathbb{I}$ so that $a_1 + a_2 + \cdots + a_n = ns$.

Proof. For $i = 1, 2, \dots, n$, define the numbers

$$b_i = \begin{cases} a_i, & a_i \leq s_0 \\ s_0, & a_i > s_0. \end{cases}$$

Clearly, $b_i \in \mathbb{I}_{\leq s_0}$ and $b_i \leq a_i$. Since $f(u) \geq f(s_0)$ for $u \in \mathbb{I}_{\geq s_0}$, it follows that $f(b_i) \leq f(a_i)$ for $i = 1, 2, \dots, n$. Therefore,

$$b_1 + b_2 + \cdots + b_n \leq a_1 + a_2 + \cdots + a_n = ns$$

and

$$f(b_1) + f(b_2) + \cdots + f(b_n) \leq f(a_1) + f(a_2) + \cdots + f(a_n).$$

Thus, it suffices to show that

$$f(b_1) + f(b_2) + \cdots + f(b_n) \geq nf(s)$$

for all $b_1, b_2, \dots, b_n \in \mathbb{I}_{\leq s_0}$ so that $b_1 + b_2 + \cdots + b_n \leq ns$. By hypothesis, this inequality is true for $b_1, b_2, \dots, b_n \in \mathbb{I}_{\leq s_0}$ and $b_1 + b_2 + \cdots + b_n = ns$. Since $f(u)$ is decreasing on $\mathbb{I}_{\leq s_0}$, the more we have $f(b_1) + f(b_2) + \cdots + f(b_n) \geq nf(s)$ for $b_1, b_2, \dots, b_n \in \mathbb{I}_{\leq s_0}$ and $b_1 + b_2 + \cdots + b_n \leq ns$.

Similarly, we can prove the Left Partially Convex Function Theorem (LPCF-Theorem).

Left Partially Convex Function Theorem (Vasile Cîrtoaje, 2012). Let f be a real function defined on an interval $\mathbb{I}$ and convex on $[s_0, s]$, where $s_0, s \in \mathbb{I}$, $s_0 < s$. In addition, f is increasing on $\mathbb{I}_{\geq s_0}$ and $f(u) \geq f(s_0)$ for $u \in \mathbb{I}$. The inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \dots, a_n \in \mathbb{I}$ satisfying

$$a_1 + a_2 + \cdots + a_n = ns$$

if and only if

$$f(x) + (n-1)f(y) \geq nf(s)$$

for all $x, y \in \mathbb{I}$ so that $x \geq s \geq y$ and $x + (n-1)y = ns$.

From the RPCF-Theorem and the LPCF-Theorem, we find the PCF-Theorem (Partially Convex Function Theorem).

Partially Convex Function Theorem (Vasile Cîrtoaje, 2012). Let f be a real function defined on an interval $\mathbb{I}$ and convex on $[s_0, s]$ or $[s, s_0]$, where $s_0, s \in \mathbb{I}$. In addition, f is decreasing on $\mathbb{I}_{\leq s_0}$ and increasing on $\mathbb{I}_{\geq s_0}$. The inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \dots, a_n \in \mathbb{I}$ satisfying

$$a_1 + a_2 + \dots + a_n = ns$$

if and only if

$$f(x) + (n-1)f(y) \geq nf(s)$$

for all $x, y \in \mathbb{I}$ so that $x + (n-1)y = ns$.

Note 1. Let us denote

$$g(u) = \frac{f(u) - f(s)}{u - s}, \quad h(x, y) = \frac{g(x) - g(y)}{x - y}.$$

As shown in Note 1 from Chapter 1, we may replace the hypothesis condition in the RPCF-Theorem and the LPCF-Theorem), namely

$$f(x) + (n-1)f(y) \geq nf(s),$$

by the condition

$$h(x, y) \geq 0 \quad \text{for all } x, y \in \mathbb{I} \quad \text{so that } x + (n-1)y = ns.$$

Note 2. Assume that f is differentiable on $\mathbb{I}$, and let

$$H(x, y) = \frac{f'(x) - f'(y)}{x - y}.$$

As shown in Note 2 from Chapter 1, the inequalities in the RPCF-Theorem and the LPCF-Theorem hold true by replacing the hypothesis

$$f(x) + (n-1)f(y) \geq nf(s)$$

with the more restrictive condition

$$H(x, y) \geq 0 \quad \text{for all } x, y \in \mathbb{I} \quad \text{so that } x + (n-1)y = ns.$$

Note 3. The desired inequalities in the RPCF-Theorem and the LPCF-Theorem become equalities for

$$a_1 = a_2 = \dots = a_n = s.$$

In addition, if there exist $x, y \in \mathbb{I}$ so that

$$x + (n-1)y = ns, \quad f(x) + (n-1)f(y) = nf(s), \quad x \neq y,$$

then the equality holds also for

$$a_1 = x, \quad a_2 = \dots = a_n = y$$

(or any cyclic permutation). Notice that these equality conditions are equivalent to

$$x + (n-1)y = ns, \quad h(x, y) = 0$$

($x < y$ for the RPCF-Theorem, and $x > y$ for the LPCF-Theorem).

Note 4. From the proof of the RPCF-Theorem, it follows that this theorem is also valid in the case when f is defined on $\mathbb{I} \setminus \{u_0\}$, where $u_0 \in \mathbb{I}_{>s_0}$. Similarly, the LPCF-Theorem is also valid in the case when f is defined on $\mathbb{I} \setminus \{u_0\}$, where $u_0 \in \mathbb{I}_{<s_0}$.

Note 5. The RPCF-Theorem holds true by replacing the condition

$$f \text{ is decreasing on } \mathbb{I}_{\leq s_0}$$

with

$$ns - (n-1)s_0 \leq \inf \mathbb{I}.$$

More precisely, the following theorem holds:

Theorem 1. *Let f be a function defined on a real interval $\mathbb{I}$, convex on $[s, s_0]$ and satisfying*

$$\min_{u \in \mathbb{I}_{\geq s}} f(u) = f(s_0),$$

where

$$s, s_0 \in \mathbb{I}, \quad s < s_0, \quad ns - (n-1)s_0 \leq \inf \mathbb{I}.$$

If

$$f(x) + (n-1)f(y) \geq nf(s)$$

for all $x, y \in \mathbb{I}$ so that $x \leq s \leq y$ and $x + (n-1)y = ns$, then

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)$$

for all $x_1, x_2, \dots, x_n \in \mathbb{I}$ satisfying $x_1 + x_2 + \cdots + x_n = ns$.

In order to prove Theorem 1, we define the function

$$f_0(u) = \begin{cases} f(u), & u \leq s_0, \quad u \in \mathbb{I} \\ f(s_0), & u \geq s_0, \quad u \in \mathbb{I}, \end{cases}$$

which is convex on $\mathbb{I}_{\geq s}$. Taking into account that $f_0(s) = f(s)$ and $f_0(u) \leq f(u)$ for all $u \in \mathbb{I}$, it suffices to prove that

$$f_0(x_1) + f_0(x_2) + \cdots + f_0(x_n) \geq nf_0(s)$$

for all $x_1, x_2, \dots, x_n \in \mathbb{I}$ satisfying $x_1 + x_2 + \cdots + x_n = ns$. According to the HCF-Theorem and Note 5 from Chapter 1, we only need to show that

$$f_0(x) + (n-1)f_0(y) \geq nf_0(s)$$

for all $x, y \in \mathbb{I}$ so that $x \leq s \leq y$ and $x + (n-1)y = ns$. Since

$$y - s_0 = \frac{ns - x}{n-1} - s_0 = \frac{ns - (n-1)s_0 - x}{n-1} \leq \frac{ns - (n-1)s_0 - \inf \mathbb{I}}{n-1} \leq 0,$$

the inequality $f_0(x) + (n-1)f_0(y) \geq nf_0(s)$ turns into $f(x) + (n-1)f(y) \geq nf(s)$, which holds (by hypothesis) for all $x, y \in \mathbb{I}$ so that $x \leq s \leq y$ and $x + (n-1)y = ns$.

Similarly, the LPCF-Theorem holds true by replacing the condition

$$f \text{ is increasing on } \mathbb{I}_{\geq s_0}$$

with

$$ns - (n - 1)s_0 \geq \sup \mathbb{I}.$$

More precisely, the following theorem holds:

Theorem 2. *Let f be a function defined on a real interval $\mathbb{I}$, convex on $[s_0, s]$ and satisfying*

$$\min_{u \in \mathbb{I}_{\leq s}} f(u) = f(s_0),$$

where

$$s, s_0 \in \mathbb{I}, \quad s > s_0, \quad ns - (n - 1)s_0 \geq \sup \mathbb{I}.$$

If

$$f(x) + (n - 1)f(y) \geq nf(s)$$

for all $x, y \in \mathbb{I}$ so that $x \geq s \geq y$ and $x + (n - 1)y = ns$, then

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)$$

for all $x_1, x_2, \dots, x_n \in \mathbb{I}$ satisfying $x_1 + x_2 + \cdots + x_n = ns$.

The proof of Theorem 2 is similar to the proof of Theorem 1.

Note 6. From the proof of Theorem 1, it follows that Theorem 1 is also valid in the case in which f is defined on $\mathbb{I} \setminus \{u_0\}$, where u_0 is an interior point of $\mathbb{I}$ so that $u_0 \notin [s, s_0]$. Similarly, Theorem 2 is also valid in the case in which f is defined on $\mathbb{I} \setminus \{u_0\}$, where u_0 is an interior point of $\mathbb{I}$ so that $u_0 \notin [s_0, s]$.

Note 7. In the same manner, we can extend *weighted* Jensen's inequality to right and left partially convex functions establishing the WRPCF-Theorem, the WLPCF-Theorem and the WPCF-Theorem (Vasile Cîrtoaje, 2014).

WRPCF-Theorem. *Let $p_1, p_2, \dots, p_n$ be positive real numbers so that*

$$p_1 + p_2 + \cdots + p_n = 1, \quad p = \min\{p_1, p_2, \dots, p_n\},$$

and let f be a real function defined on an interval $\mathbb{I}$ and convex on $[s, s_0]$, where $s, s_0 \in \mathbb{I}$, $s < s_0$. In addition, f is decreasing on $\mathbb{I}_{\leq s_0}$ and $f(u) \geq f(s_0)$ for $u \in \mathbb{I}$. The inequality

$$p_1 f(a_1) + p_2 f(a_2) + \cdots + p_n f(a_n) \geq f(p_1 a_1 + p_2 a_2 + \cdots + p_n a_n)$$

holds for all $a_1, a_2, \dots, a_n \in \mathbb{I}$ satisfying

$$p_1 a_1 + p_2 a_2 + \cdots + p_n a_n = s,$$

if and only if

$$pf(x) + (1 - p)f(y) \geq f(s)$$

for all $x, y \in \mathbb{I}$ so that $x \leq s \leq y$ and $px + (1 - p)y = s$.

WLPCF-Theorem. *Let $p_1, p_2, \dots, p_n$ be positive real numbers so that*

$$p_1 + p_2 + \dots + p_n = 1, \quad p = \min\{p_1, p_2, \dots, p_n\},$$

and let f be a real function defined on an interval $\mathbb{I}$ and convex on $[s_0, s]$, where $s_0, s \in \mathbb{I}$, $s_0 < s$. In addition, f is increasing on $\mathbb{I}_{\geq s_0}$ and $f(u) \geq f(s_0)$ for $u \in \mathbb{I}$. The inequality

$$p_1 f(a_1) + p_2 f(a_2) + \dots + p_n f(a_n) \geq f(p_1 a_1 + p_2 a_2 + \dots + p_n a_n)$$

holds for all $a_1, a_2, \dots, a_n \in \mathbb{I}$ satisfying

$$p_1 a_1 + p_2 a_2 + \dots + p_n a_n = s,$$

if and only if

$$p f(x) + (1 - p) f(y) \geq f(s)$$

for all $x, y \in \mathbb{I}$ so that $x \geq s \geq y$ and $px + (1 - p)y = s$.

3.2 Applications

3.1. If a, b, c are real numbers so that $a + b + c = 3$, then

$$\frac{16a - 5}{32a^2 + 1} + \frac{16b - 5}{32b^2 + 1} + \frac{16c - 5}{32c^2 + 1} \leq 1.$$

3.2. If a, b, c, d are real numbers so that $a + b + c + d = 4$, then

$$\frac{18a - 5}{12a^2 + 1} + \frac{18b - 5}{12b^2 + 1} + \frac{18c - 5}{12c^2 + 1} + \frac{18d - 5}{12d^2 + 1} \leq 4.$$

3.3. If a, b, c, d, e, f are real numbers so that $a + b + c + d + e + f = 6$, then

$$\frac{5a - 1}{5a^2 + 1} + \frac{5b - 1}{5b^2 + 1} + \frac{5c - 1}{5c^2 + 1} + \frac{5d - 1}{5d^2 + 1} + \frac{5e - 1}{5e^2 + 1} + \frac{5f - 1}{5f^2 + 1} \leq 4.$$

3.4. If $a_1, a_2, \dots, a_n$ ($n \geq 3$) are real numbers so that $a_1 + a_2 + \dots + a_n = n$, then

$$\frac{n(n+1) - 2a_1}{n^2 + (n-2)a_1^2} + \frac{n(n+1) - 2a_2}{n^2 + (n-2)a_2^2} + \dots + \frac{n(n+1) - 2a_n}{n^2 + (n-2)a_n^2} \leq n.$$

3.5. If a, b, c, d are real numbers so that $a + b + c + d = 4$, then

$$\frac{a(a-1)}{3a^2 + 4} + \frac{b(b-1)}{3b^2 + 4} + \frac{c(c-1)}{3c^2 + 4} + \frac{d(d-1)}{3d^2 + 4} \geq 0.$$

3.6. If a, b, c, d are real numbers so that $a + b + c + d = 4$, then

$$\frac{1}{3a^2 - 3a + 4} + \frac{1}{3b^2 - 3b + 4} + \frac{1}{3c^2 - 3c + 4} + \frac{1}{3d^2 - 3d + 4} \leq 1.$$

3.7. If a, b, c, d are real numbers so that $a + b + c + d = 4$, then

$$\frac{1}{4a^2 - 5a + 4} + \frac{1}{4b^2 - 5b + 4} + \frac{1}{4c^2 - 5c + 4} + \frac{1}{4d^2 - 5d + 4} \leq \frac{4}{3}.$$

3.8. If a, b, c, d are real numbers so that $a + b + c + d = 0$, then

$$\frac{a+1}{a^2+3} + \frac{b+1}{b^2+3} + \frac{c+1}{c^2+3} + \frac{d+1}{d^2+3} \leq \frac{4}{3}.$$

3.9. Let $a_1, a_2, \dots, a_n \neq -k$ be real numbers so that $a_1 + a_2 + \dots + a_n = n$, where

$$k \geq \frac{n}{2\sqrt{n-1}}.$$

Then,

$$\frac{a_1(a_1-1)}{(a_1+k)^2} + \frac{a_2(a_2-1)}{(a_2+k)^2} + \dots + \frac{a_n(a_n-1)}{(a_n+k)^2} \geq 0.$$

3.10. Let $a_1, a_2, \dots, a_n \neq -k$ be real numbers so that $a_1 + a_2 + \dots + a_n = n$. If

$$k \geq 1 + \frac{n}{\sqrt{n-1}},$$

then

$$\frac{a_1^2-1}{(a_1+k)^2} + \frac{a_2^2-1}{(a_2+k)^2} + \dots + \frac{a_n^2-1}{(a_n+k)^2} \geq 0.$$

3.11. Let a_1, a_2, a_3, a_4, a_5 be real numbers so that $a_1 + a_2 + a_3 + a_4 + a_5 \geq 5$. If

$$k \in \left[\frac{1}{6}, \frac{25}{14} \right],$$

then

$$\sum \frac{1}{ka_1^2 + a_2 + a_3 + a_4 + a_5} \leq \frac{5}{k+4}.$$

3.12. Let $a_1, a_2, \dots, a_5$ be nonnegative numbers so that $a_1 + a_2 + a_3 + a_4 + a_5 \geq 5$. If $k \in [k_1, k_2]$, where

$$k_1 = \frac{29 - \sqrt{761}}{10} \approx 0.1414, \quad k_2 = \frac{25}{14} \approx 1.7857,$$

then

$$\sum \frac{1}{ka_1^2 + a_2 + a_3 + a_4 + a_5} \leq \frac{5}{k+4}.$$

3.13. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n \geq n$. If $k > 1$, then

$$\frac{1}{a_1^k + a_2 + \dots + a_n} + \frac{1}{a_1 + a_2^k + \dots + a_n} + \dots + \frac{1}{a_1 + a_2 + \dots + a_n^k} \leq 1.$$

3.14. Let $a_1, a_2, \dots, a_5$ be nonnegative numbers so that $a_1 + a_2 + a_3 + a_4 + a_5 \geq 5$. If

$$k \in \left[\frac{4}{9}, \frac{61}{5} \right],$$

then

$$\sum \frac{a_1}{ka_1^2 + a_2 + a_3 + a_4 + a_5} \leq \frac{5}{k+4}.$$

3.15. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n \geq n$. If $k > 1$, then

$$\frac{a_1}{a_1^k + a_2 + \dots + a_n} + \frac{a_2}{a_1 + a_2^k + \dots + a_n} + \dots + \frac{a_n}{a_1 + a_2 + \dots + a_n^k} \leq 1.$$

3.16. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n \leq n$. If $k \geq 1 - \frac{1}{n}$, then

$$\frac{1 - a_1}{ka_1^2 + a_2 + \dots + a_n} + \frac{1 - a_2}{a_1 + ka_2^2 + \dots + a_n} + \dots + \frac{1 - a_n}{a_1 + a_2 + \dots + ka_n^2} \geq 0.$$

3.17. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n \leq n$. If $k \geq 1 - \frac{1}{n}$, then

$$\frac{1 - a_1}{1 - a_1 + ka_1^2} + \frac{1 - a_2}{1 - a_2 + ka_2^2} + \dots + \frac{1 - a_n}{1 - a_n + ka_n^2} \geq 0.$$

3.18. Let $a_1, a_2, \dots, a_n$ be positive real numbers so that $a_1 + a_2 + \dots + a_n = n$. If $0 < k \leq \frac{n}{n-1}$, then

$$a_1^{k/a_1} + a_2^{k/a_2} + \dots + a_n^{k/a_n} \leq n.$$

3.19. If a, b, c, d, e are nonzero real numbers so that $a + b + c + d + e = 5$, then

$$\left(7 - \frac{5}{a}\right)^2 + \left(7 - \frac{5}{b}\right)^2 + \left(7 - \frac{5}{c}\right)^2 + \left(7 - \frac{5}{d}\right)^2 + \left(7 - \frac{5}{e}\right)^2 \geq 20.$$

3.20. If $a_1, a_2, \dots, a_7$ are real numbers so that $a_1 + a_2 + \dots + a_7 = 7$, then

$$(a_1^2 + 2)(a_2^2 + 2) \cdots (a_7^2 + 2) \geq 3^7.$$

3.21. Let $a_1, a_2, \dots, a_n$ be real numbers so that $a_1 + a_2 + \dots + a_n = n$. If $k \geq \frac{n^2}{4(n-1)}$, then

$$(a_1^2 + k)(a_2^2 + k) \cdots (a_n^2 + k) \geq (1 + k)^n.$$

3.22. Let $a_1, a_2, \dots, a_n$ be real numbers such that $a_1 + a_2 + \dots + a_n = n$. If $n \leq 10$, then

$$(a_1^2 - a_1 + 1)(a_2^2 - a_2 + 1) \cdots (a_n^2 - a_n + 1) \geq 1.$$

3.23. Let $a_1, a_2, \dots, a_n$ be real numbers such that $a_1 + a_2 + \dots + a_n = n$. If $n \leq 26$, then

$$(a_1^2 - a_1 + 2)(a_2^2 - a_2 + 2) \cdots (a_n^2 - a_n + 2) \geq 2^n.$$

3.24. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$(1 - a + a^4)(1 - b + b^4)(1 - c + c^4) \geq 1.$$

3.25. If a, b, c, d are nonnegative real numbers so that $a + b + c + d = 4$, then

$$(1 - a + a^3)(1 - b + b^3)(1 - c + c^3)(1 - d + d^3) \geq 1.$$

3.26. If a, b, c, d, e are nonzero real numbers so that $a + b + c + d + e = 5$, then

$$5 \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{1}{e^2} \right) + 45 \geq 14 \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} \right).$$

3.27. If a, b, c are positive real numbers so that $abc = 1$, then

$$\frac{7 - 6a}{2 + a^2} + \frac{7 - 6b}{2 + b^2} + \frac{7 - 6c}{2 + c^2} \geq 1.$$

3.28. If a, b, c are positive real numbers so that $abc = 1$, then

$$\frac{1}{a + 5bc} + \frac{1}{b + 5ca} + \frac{1}{c + 5ab} \leq \frac{1}{2}.$$

3.29. If a, b, c are positive real numbers so that $abc = 1$, then

$$\frac{1}{4 - 3a + 4a^2} + \frac{1}{4 - 3b + 4b^2} + \frac{1}{4 - 3c + 4c^2} \leq \frac{3}{5}.$$

3.30. If a, b, c are positive real numbers so that $abc = 1$, then

$$\frac{1}{(3a + 1)(3a^2 - 5a + 3)} + \frac{1}{(3b + 1)(3b^2 - 5b + 3)} + \frac{1}{(3c + 1)(3c^2 - 5c + 3)} \leq \frac{3}{4}.$$

3.31. Let $a_1, a_2, \dots, a_n$ ($n \geq 3$) be positive real numbers so that $a_1 a_2 \cdots a_n = 1$. If $p, q \geq 0$ so that $p + 4q \geq n - 1$, then

$$\frac{1 - a_1}{1 + pa_1 + qa_1^2} + \frac{1 - a_2}{1 + pa_2 + qa_2^2} + \cdots + \frac{1 - a_n}{1 + pa_n + qa_n^2} \geq 0.$$

3.32. If a, b, c are positive real numbers so that $abc = 1$, then

$$\frac{1-a}{17+4a+6a^2} + \frac{1-b}{17+4b+6b^2} + \frac{1-c}{17+4c+6c^2} \geq 0.$$

3.33. If $a_1, a_2, \dots, a_8$ are positive real numbers so that $a_1 a_2 \cdots a_8 = 1$, then

$$\frac{1-a_1}{(1+a_1)^2} + \frac{1-a_2}{(1+a_2)^2} + \cdots + \frac{1-a_8}{(1+a_8)^2} \geq 0.$$

3.34. Let a, b, c be positive real numbers so that $abc = 1$. If $k \in \left[\frac{-13}{3\sqrt{3}}, \frac{13}{3\sqrt{3}} \right]$, then

$$\frac{a+k}{a^2+1} + \frac{b+k}{b^2+1} + \frac{c+k}{c^2+1} \leq \frac{3(1+k)}{2}.$$

3.35. If a, b, c are positive real numbers and $0 < k \leq 2 + 2\sqrt{2}$, then

$$\frac{a^3}{ka^2+bc} + \frac{b^3}{kb^2+ca} + \frac{c^3}{kc^2+ab} \geq \frac{a+b+c}{k+1}.$$

3.36. If a, b, c, d, e are positive real numbers so that $abcde = 1$, then

$$2 \left(\frac{1}{a+1} + \frac{1}{b+1} + \cdots + \frac{1}{e+1} \right) \geq 3 \left(\frac{1}{a+2} + \frac{1}{b+2} + \cdots + \frac{1}{e+2} \right).$$

3.37. If $a_1, a_2, \dots, a_{14}$ are positive real numbers so that $a_1 a_2 \cdots a_{14} = 1$, then

$$3 \left(\frac{1}{2a_1+1} + \frac{1}{2a_2+1} + \cdots + \frac{1}{2a_{14}+1} \right) \geq 2 \left(\frac{1}{a_1+1} + \frac{1}{a_2+1} + \cdots + \frac{1}{a_{14}+1} \right).$$

3.38. Let $a_1, a_2, \dots, a_8$ be positive real numbers so that $a_1 a_2 \cdots a_8 = 1$. If $k > 1$, then

$$(k+1) \left(\frac{1}{ka_1+1} + \frac{1}{ka_2+1} + \cdots + \frac{1}{ka_8+1} \right) \geq 2 \left(\frac{1}{a_1+1} + \frac{1}{a_2+1} + \cdots + \frac{1}{a_8+1} \right).$$

3.39. If $a_1, a_2, \dots, a_9$ are positive real numbers so that $a_1 a_2 \cdots a_9 = 1$, then

$$\frac{1}{2a_1+1} + \frac{1}{2a_2+1} + \cdots + \frac{1}{2a_9+1} \geq \frac{1}{a_1+2} + \frac{1}{a_2+2} + \cdots + \frac{1}{a_9+2}.$$

3.40. If $a_1, a_2, \dots, a_n$ are real numbers so that

$$a_1, a_2, \dots, a_n \leq \pi, \quad a_1 + a_2 + \dots + a_n = \pi,$$

then

$$\cos a_1 + \cos a_2 + \dots + \cos a_n \leq n \cos \frac{\pi}{n}.$$

3.41. If $a_1, a_2, \dots, a_n$ ($n \geq 3$) are real numbers so that

$$a_1, a_2, \dots, a_n \geq \frac{-1}{n-2}, \quad a_1 + a_2 + \dots + a_n = n,$$

then

$$\frac{a_1^2}{a_1^2 - a_1 + 1} + \frac{a_2^2}{a_2^2 - a_2 + 1} + \dots + \frac{a_n^2}{a_n^2 - a_n + 1} \leq n.$$

3.42. If $a_1, a_2, \dots, a_n$ ($n \geq 3$) are nonzero real numbers so that

$$a_1, a_2, \dots, a_n \geq \frac{-n}{n-2}, \quad a_1 + a_2 + \dots + a_n = n,$$

then

$$\frac{1}{a_1^2} + \frac{1}{a_2^2} + \dots + \frac{1}{a_n^2} \geq \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}.$$

3.43. If $a_1, a_2, \dots, a_n \geq -1$ so that $a_1 + a_2 + \dots + a_n = n$, then

$$(n+1) \left(\frac{1}{a_1^2} + \frac{1}{a_2^2} + \dots + \frac{1}{a_n^2} \right) \geq 2n + (n-1) \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right).$$

3.44. If $a_1, a_2, \dots, a_n$ ($n \geq 3$) are real numbers so that

$$a_1, a_2, \dots, a_n \geq \frac{-(3n-2)}{n-2}, \quad a_1 + a_2 + \dots + a_n = n,$$

then

$$\frac{1-a_1}{(1+a_1)^2} + \frac{1-a_2}{(1+a_2)^2} + \dots + \frac{1-a_n}{(1+a_n)^2} \geq 0.$$

3.45. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$. If $n \geq 3$ and $k \geq 2 - \frac{2}{n}$, then

$$\frac{1-a_1}{(1-ka_1)^2} + \frac{1-a_2}{(1-ka_2)^2} + \dots + \frac{1-a_n}{(1-ka_n)^2} \geq 0.$$

3.46. If a, b, c are positive real numbers such that $abc = 1$, then

$$\frac{a}{(a+1)(a+25)} + \frac{b}{(b+1)(b+25)} + \frac{c}{(c+1)(c+25)} \leq \frac{3}{52}.$$

3.47. Let $a_1, a_2, \dots, a_n$ ($n \geq 3$) be nonnegative real numbers so that

$$a_1 + a_2 + \dots + a_n = n.$$

If $p \leq -4$ and $q = (p+1) \left(p + \frac{n}{n-1} \right)$, then

$$\frac{1}{a_1^2 + pa_1 + q} + \frac{1}{a_2^2 + pa_2 + q} + \dots + \frac{1}{a_n^2 + pa_n + q} \leq \frac{n}{1+p+q}.$$

3.48. Let $a_1, a_2, \dots, a_n$ ($n \geq 3$) be nonnegative real numbers so that

$$a_1 + a_2 + \dots + a_n = n.$$

If $p \in [-4, -2) \cup \left(-2, \frac{2}{n-2} \right]$ and $q = \frac{n^2 p^2 + 4(n^2 - n + 1)(p+1)}{4(n-1)}$, then

$$\frac{1}{a_1^2 + pa_1 + q} + \frac{1}{a_2^2 + pa_2 + q} + \dots + \frac{1}{a_n^2 + pa_n + q} \leq \frac{n}{1+p+q}.$$

3.3 Solutions

P 3.1. If a, b, c are real numbers so that $a + b + c = 3$, then

$$\frac{16a - 5}{32a^2 + 1} + \frac{16b - 5}{32b^2 + 1} + \frac{16c - 5}{32c^2 + 1} \leq 1.$$

(Vasile C., 2012)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) \geq 3f(s), \quad s = \frac{a + b + c}{3} = 1,$$

where

$$f(u) = \frac{5 - 16u}{32u^2 + 1}, \quad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{16(32u^2 - 20u - 1)}{(32u^2 + 1)^2},$$

it follows that f is increasing on

$$\left(-\infty, \frac{5 - \sqrt{33}}{16}\right] \cup [s_0, \infty)$$

and decreasing on

$$\left[\frac{5 - \sqrt{33}}{16}, s_0\right],$$

where

$$s_0 = \frac{5 + \sqrt{33}}{16} \approx 0.6715.$$

Also, from

$$\lim_{u \rightarrow -\infty} f(u) = 0$$

and

$$f(s_0) < 0,$$

it follows that $f(u) \geq f(s_0)$ for $u \in \mathbb{R}$. In addition, for $u \in [s_0, 1]$, we have

$$\begin{aligned} \frac{1}{64}f''(u) &= \frac{-512u^3 + 480u^2 + 48u - 5}{(32u^2 + 1)^3} \\ &= \frac{512u^2(1 - u) + 32u(1 - u) + (16u - 5)}{(32u^2 + 1)^3} > 0, \end{aligned}$$

hence f is convex on $[s_0, s]$. According to the LPCF-Theorem, we only need to show that $f(x) + 2f(y) \geq 3f(1)$ for all real x, y so that $x + 2y = 3$. Using Note 1, it suffices to prove that $h(x, y) \geq 0$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

Indeed, we have

$$g(u) = \frac{32(2u-1)}{3(32u^2+1)},$$

$$h(x, y) = \frac{64(1+16x+16y-32xy)}{3(32x^2+1)(32y^2+1)} = \frac{64(4x-5)^2}{3(32x^2+1)(32y^2+1)} \geq 0.$$

Thus, the proof is completed. From $x+2y=3$ and $h(x, y)=0$, we get

$$x = \frac{5}{4}, \quad y = \frac{7}{8}.$$

Therefore, in accordance with Note 3, the equality holds for $a=b=c=1$, and also for

$$a = \frac{5}{4}, \quad b = c = \frac{7}{8}$$

(or any cyclic permutation).

□

P 3.2. If a, b, c, d are real numbers so that $a+b+c+d=4$, then

$$\frac{18a-5}{12a^2+1} + \frac{18b-5}{12b^2+1} + \frac{18c-5}{12c^2+1} + \frac{18d-5}{12d^2+1} \leq 4.$$

(Vasile C., 2012)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) + f(d) \geq 4f(s), \quad s = \frac{a+b+c+d}{4} = 1,$$

where

$$f(u) = \frac{5-18u}{12u^2+1}, \quad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{6(36u^2-20u-3)}{(12u^2+1)^2},$$

it follows that f is increasing on

$$\left(-\infty, \frac{5-\sqrt{52}}{18}\right] \cup [s_0, \infty)$$

and decreasing on

$$\left[\frac{5-\sqrt{52}}{18}, s_0\right], \quad s_0 = \frac{5+\sqrt{52}}{18} \approx 0.678.$$

Also, from

$$\lim_{u \rightarrow -\infty} f(u) = 0$$

and

$$f(s_0) < 0,$$

it follows that $f(u) \geq f(s_0)$ for $u \in \mathbb{R}$. In addition, for $u \in [s_0, 1]$, we have

$$\begin{aligned} \frac{1}{24}f''(u) &= \frac{-216u^3 + 180u^2 + 54u - 5}{(12u^2 + 1)^3} \\ &= \frac{216u^2(1-u) + 36u(1-u) + (18u-5)}{(32u^2+1)^3} > 0, \end{aligned}$$

hence f is convex on $[s_0, s]$. According to the LPCF-Theorem and Note 1, we only need to show that $h(x, y) \geq 0$ for $x, y \in \mathbb{R}$ so that $x + 3y = 4$. We have

$$\begin{aligned} g(u) &= \frac{f(u) - f(1)}{u - 1} = \frac{6(2u - 1)}{12u^2 + 1}, \\ h(x, y) &= \frac{g(x) - g(y)}{x - y} = \frac{12(1 + 6x + 6y - 12xy)}{(12x^2 + 1)(12y^2 + 1)} = \frac{12(2x - 3)^2}{(12x^2 + 1)(12y^2 + 1)} \geq 0. \end{aligned}$$

Thus, the proof is completed. From $x + 3y = 4$ and $h(x, y) = 0$, we get $x = 3/2$ and $y = 5/6$. Therefore, in accordance with Note 3, the equality holds for $a = b = c = d = 1$, and also for

$$a = \frac{3}{2}, \quad b = c = d = \frac{5}{6}$$

(or any cyclic permutation).

□

P 3.3. If a, b, c, d, e, f are real numbers so that $a + b + c + d + e + f = 6$, then

$$\frac{5a-1}{5a^2+1} + \frac{5b-1}{5b^2+1} + \frac{5c-1}{5c^2+1} + \frac{5d-1}{5d^2+1} + \frac{5e-1}{5e^2+1} + \frac{5f-1}{5f^2+1} \leq 4.$$

(Vasile C., 2012)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) + f(d) + f(e) + f(f) \geq 4f(s), \quad s = \frac{a+b+c+d+e+f}{6} = 1,$$

where

$$f(u) = \frac{1-5u}{5u^2+1}, \quad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{5(5u^2 - 2u - 1)}{(5u^2 + 1)^2},$$

it follows that f is increasing on

$$\left(-\infty, \frac{1-\sqrt{6}}{5}\right] \cup [s_0, \infty)$$

and decreasing on

$$\left[\frac{1 - \sqrt{6}}{5}, s_0 \right], \quad s_0 = \frac{1 + \sqrt{6}}{5} \approx 0.69.$$

Also, from

$$\lim_{u \rightarrow -\infty} f(u) = 0$$

and

$$f(s_0) < 0,$$

it follows that $f(u) \geq f(s_0)$ for $u \in \mathbb{R}$. In addition, for $u \in [s_0, 1]$, we have

$$\begin{aligned} \frac{1}{24} f''(u) &= \frac{-216u^3 + 180u^2 + 54u - 5}{(12u^2 + 1)^3} \\ &= \frac{216u^2(1 - u) + 36u(1 - u) + (18u - 5)}{(32u^2 + 1)^3} > 0, \end{aligned}$$

hence f is convex on $[s_0, 1]$. According to the LPCF-Theorem and Note 1, we only need to show that $h(x, y) \geq 0$ for $x, y \in \mathbb{R}$ so that $x + 5y = 6$. We have

$$\begin{aligned} g(u) &= \frac{f(u) - f(1)}{u - 1} = \frac{5(2u - 1)}{3(5u^2 + 1)}, \\ h(x, y) &= \frac{g(x) - g(y)}{x - y} = \frac{5(2 + 5x + 5y - 10xy)}{3(5x^2 + 1)(5y^2 + 1)} = \frac{10(x - 2)^2}{3(5x^2 + 1)(5y^2 + 1)} \geq 0. \end{aligned}$$

In accordance with Note 3, the equality holds for $a = b = c = d = e = f = 1$, and also for

$$a = 2, \quad b = c = d = e = f = \frac{4}{5}$$

(or any cyclic permutation).

□

P 3.4. If $a_1, a_2, \dots, a_n$ ($n \geq 3$) are real numbers so that $a_1 + a_2 + \dots + a_n = n$, then

$$\frac{n(n+1) - 2a_1}{n^2 + (n-2)a_1^2} + \frac{n(n+1) - 2a_2}{n^2 + (n-2)a_2^2} + \dots + \frac{n(n+1) - 2a_n}{n^2 + (n-2)a_n^2} \leq n.$$

(Vasile C., 2008)

Solution. The desired inequality is true for $a_1 > \frac{n(n+1)}{2}$ since

$$\frac{n(n+1) - 2a_1}{n^2 + (n-2)a_1^2} < 0$$

and

$$\frac{n(n+1) - 2a_i}{n^2 + (n-2)a_i^2} < \frac{n}{n-1}, \quad i = 2, 3, \dots, n.$$

The last inequalities are equivalent to

$$n(n-2)a_i^2 + 2(n-1)a_i + n > 0,$$

which are true because

$$n(n-2)a_i^2 + 2(n-1)a_i + n \geq (n-1)a_i^2 + 2(n-1)a_i + n > (n-1)(a_i+1)^2 \geq 0.$$

Consider further that

$$a_1, a_2, \dots, a_n \leq \frac{n(n+1)}{2},$$

and rewrite the desired inequality as

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \dots + a_n}{n} = 1,$$

where

$$f(u) = \frac{2u - n(n+1)}{(n-2)u^2 + n^2}, \quad u \in \mathbb{I} = \left(-\infty, \frac{n(n+1)}{2}\right].$$

We have

$$\frac{f'(u)}{2(n-2)} = \frac{n^2 + n(n+1)u - u^2}{[(n-2)u^2 + n^2]^2}$$

and

$$\frac{f''(u)}{2(n-2)} = \frac{f_1(u)}{[(n-2)u^2 + n^2]^3},$$

where

$$f_1(u) = 2(n-2)u^3 - 3n(n+1)(n-2)u^2 - 2n^2(2n-3)u + n^3(n+1).$$

From the expression of f' , it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $\left[s_0, \frac{n(n+1)}{2}\right]$, where

$$s_0 = \frac{n}{2} \left(n+1 - \sqrt{n^2 + 2n + 5} \right) \in (-1, 0);$$

therefore,

$$\min_{u \in \mathbb{I}} f(u) = f(s_0).$$

On the other hand, for $-1 \leq u \leq 1$, we have

$$\begin{aligned} f_1(u) &> -2(n-2) - 3n(n+1)(n-2) - 2n^2(2n-3) + n^3(n+1) \\ &= n^2(n-3)^2 + 4(n+1) > 0, \end{aligned}$$

hence $f''(u) > 0$. Since $[s_0, s] \subset [-1, 1]$, f is convex on $[s_0, s]$. By the LPCF-Theorem and Note 1, we only need to show that $h(x, y) \geq 0$ for $x, y \in \mathbb{R}$ and $x + (n-1)y = n$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

Indeed, we have

$$g(u) = \frac{(n-2)u + n}{(n-2)u^2 + n^2}$$

and

$$\begin{aligned}\frac{h(x, y)}{n-2} &= \frac{n^2 - n(x+y) - (n-2)xy}{[(n-2)x^2 + n^2][(n-2)y^2 + n^2]} \\ &= \frac{(n-1)(n-2)y^2}{[(n-2)x^2 + n^2][(n-2)y^2 + n^2]} \geq 0.\end{aligned}$$

The proof is completed. By Note 3, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = n, \quad a_2 = \cdots = a_n = 0$$

(or any cyclic permutation).

□

P 3.5. If a, b, c, d are real numbers so that $a + b + c + d = 4$, then

$$\frac{a(a-1)}{3a^2+4} + \frac{b(b-1)}{3b^2+4} + \frac{c(c-1)}{3c^2+4} + \frac{d(d-1)}{3d^2+4} \geq 0.$$

(Vasile C., 2012)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) + f(d) \geq 4f(s), \quad s = \frac{a+b+c+d}{4} = 1,$$

where

$$f(u) = \frac{u^2 - u}{3u^2 + 4}, \quad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{3u^2 + 8u - 4}{(3u^2 + 4)^2},$$

it follows that f is increasing on $\left(-\infty, \frac{-4 - 2\sqrt{7}}{3}\right] \cup [s_0, \infty)$ and decreasing on $\left[\frac{-4 - 2\sqrt{7}}{3}, s_0\right]$,

where

$$s_0 = \frac{-4 + 2\sqrt{7}}{3} \approx 0.43.$$

Since

$$\lim_{u \rightarrow -\infty} f(u) = \frac{1}{3}$$

and $f(s_0) < 0$, it follows that

$$\min_{u \in \mathbb{R}} f(u) = f(s_0).$$

For $u \in [0, 1]$, we have

$$\begin{aligned}\frac{1}{2}f''(u) &= \frac{-9u^3 - 36u^2 + 36u + 14}{(3u^2 + 4)^3} \\ &= \frac{9u^2(1-u) + 45u(1-u) + (16-9u)}{(3u^2 + 4)^3} > 0.\end{aligned}$$

Therefore, f is convex on $[0, 1]$, hence on $[s_0, s]$. According to the LPCF-Theorem and Note 1, we only need to show that $h(x, y) \geq 0$ for $x, y \in \mathbb{R}$ so that $x + 3y = 4$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{u}{3u^2 + 4},$$

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{4 - 3xy}{(3x^2 + 4)(3y^2 + 4)}$$

$$= \frac{(x - 2)^2}{(3x^2 + 4)(3y^2 + 4)} \geq 0.$$

The proof is completed. From $x + 3y = 4$ and $h(x, y) = 0$, we get $x = 2$ and $y = 2/3$. By Note 3, the equality holds for $a = b = c = d = 1$, and also for

$$a = 2, \quad b = c = d = \frac{2}{3}$$

(or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization:

- If $a_1, a_2, \dots, a_n$ are real numbers so that $a_1 + a_2 + \dots + a_n = n$, then

$$\frac{a_1(a_1 - 1)}{4(n - 1)a_1^2 + n^2} + \frac{a_2(a_2 - 1)}{4(n - 1)a_2^2 + n^2} + \dots + \frac{a_n(a_n - 1)}{4(n - 1)a_n^2 + n^2} \geq 0,$$

with equality for $a_1 = a_2 = \dots = a_n = 1$, and also for

$$a_1 = \frac{n}{2}, \quad a_2 = a_3 = \dots = a_n = \frac{n}{2(n - 1)}$$

(or any cyclic permutation).

□

P 3.6. If a, b, c, d are real numbers so that $a + b + c + d = 4$, then

$$\frac{1}{3a^2 - 3a + 4} + \frac{1}{3b^2 - 3b + 4} + \frac{1}{3c^2 - 3c + 4} + \frac{1}{3d^2 - 3d + 4} \leq 1.$$

(Vasile C., 2015)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) + f(d) \geq 4f(s), \quad s = \frac{a + b + c + d}{4} = 1,$$

where

$$f(u) = \frac{-1}{3u^2 - 3u + 4}, \quad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{3(2u - 1)}{(3u^2 - 3u + 4)^2},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = \frac{1}{2} < 1 = s.$$

For $u \in [s_0, s] = [1/2, 1]$, we have

$$f''(u) = \frac{3(-6u^2 + 6u + 5)}{(3u^2 - 3u + 4)^3} > \frac{18u(1-u)}{(3u^2 - 3u + 4)^3} \geq 0.$$

Therefore, f is convex on $[s_0, s]$. According to the LPCF-Theorem and Note 1, we only need to show that $h(x, y) \geq 0$ for $x, y \in \mathbb{R}$ so that $x + 3y = 4$. We have

$$\begin{aligned} g(u) &= \frac{f(u) - f(1)}{u - 1} = \frac{3u}{4(3u^2 - 3u + 4)}, \\ h(x, y) &= \frac{g(x) - g(y)}{x - y} = \frac{3(4 - 3xy)}{4(3x^2 - 3x + 4)(3y^2 - 3y + 4)} \\ &= \frac{3(3y - 2)^2}{4(3x^2 - 3x + 4)(3y^2 - 3y + 4)} \geq 0. \end{aligned}$$

From $x + 3y = 4$ and $h(x, y) = 0$, we get

$$x = 2, \quad y = \frac{2}{3}.$$

In accord with Note 3, the equality holds for $a = b = c = d = 1$, and also for $a = 2$ and $b = c = d = \frac{2}{3}$ (or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- Let $a_1, a_2, \dots, a_n$ be real numbers so that $a_1 + a_2 + \dots + a_n = n$. If $k = \frac{n^2}{4(n-1)}$, then

$$\frac{1}{a_1^2 - a_1 + k} + \frac{1}{a_2^2 - a_2 + k} + \dots + \frac{1}{a_n^2 - a_n + k} \leq \frac{n}{k},$$

with equality for $a_1 = a_2 = \dots = a_n = 1$, and also for $a_1 = \frac{n}{2}$ and $a_2 = a_3 = \dots = a_n = \frac{n}{2(n-1)}$ (or any cyclic permutation).

□

P 3.7. If a, b, c, d are real numbers so that $a + b + c + d = 4$, then

$$\frac{1}{4a^2 - 5a + 4} + \frac{1}{4b^2 - 5b + 4} + \frac{1}{4c^2 - 5c + 4} + \frac{1}{4d^2 - 5d + 4} \leq \frac{4}{3}.$$

(Vasile C., 2015)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) + f(d) \geq 4f(s), \quad s = \frac{a+b+c+d}{4} = 1,$$

where

$$f(u) = \frac{-1}{4u^2 - 5u + 4}, \quad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{2(8u - 5)}{(4u^2 - 5u + 4)^2},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = \frac{5}{8} < 1 = s.$$

For $u \in [s_0, s] = [5/8, 1]$, we have

$$\begin{aligned} f''(u) &= \frac{4(-48u^2 + 60u - 9)}{(4u^2 - 5u + 4)^3} > \frac{4(-48u^2 + 60u - 12)}{(4u^2 - 5u + 4)^3} \\ &= \frac{48(-4u^2 + 5u - 1)}{(4u^2 - 5u + 4)^3} = \frac{48(1-u)(4u-1)}{(4u^2 - 5u + 4)^3} \geq 0. \end{aligned}$$

Therefore, f is convex on $[s_0, s]$. According to the LPCF-Theorem and Note 1, we only need to show that $h(x, y) \geq 0$ for $x, y \in \mathbb{R}$ so that $x + 3y = 4$. We have

$$\begin{aligned} g(u) &= \frac{f(u) - f(1)}{u - 1} = \frac{4u - 1}{3(4u^2 - 5u + 4)}, \\ h(x, y) &= \frac{g(x) - g(y)}{x - y} = \frac{4(x + y) - 16xy + 11}{3(4x^2 - 5x + 4)(4y^2 - 5y + 4)} \\ &= \frac{(4y - 3)^2}{(4x^2 - 5x + 4)(4y^2 - 5y + 4)} \geq 0. \end{aligned}$$

From $x + 3y = 4$ and $h(x, y) = 0$, we get

$$x = \frac{7}{4}, \quad y = \frac{3}{4}.$$

In accord with Note 3, the equality holds for $a = b = c = d = 1$, and also for $a = \frac{7}{4}$ and $b = c = d = \frac{3}{4}$ (or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- Let $a_1, a_2, \dots, a_n$ be real numbers so that $a_1 + a_2 + \dots + a_n = n$. If $k = 1 - \frac{2(n-1)}{n^2}$, then

$$\frac{1}{a_1^2 - 2ka_1 + 1} + \frac{1}{a_2^2 - 2ka_2 + 1} + \dots + \frac{1}{a_n^2 - 2ka_n + 1} \leq \frac{n}{2(1-k)},$$

with equality for $a_1 = a_2 = \dots = a_n = 1$, and also for $a_1 = \frac{3n^2 - 6n + 4}{n^2}$ and $a_2 = a_3 = \dots = a_n = \frac{n^2 - 2n + 4}{n^2}$ (or any cyclic permutation).

□

P 3.8. If a, b, c, d are real numbers so that $a + b + c + d = 0$, then

$$\frac{a+1}{a^2+3} + \frac{b+1}{b^2+3} + \frac{c+1}{c^2+3} + \frac{d+1}{d^2+3} \leq \frac{4}{3}.$$

(Marius Stanean, 2024)

Solution 1. Assume that $a \leq b \leq c \leq d$ and write the inequality as

$$f(a) + f(b) + f(c) + f(d) \geq 4f(s),$$

where $s = (a + b + c + d)/4$ and $f(u) = \frac{-u-1}{u^2+3}$.

Case 1: $a \geq -3$. Let $\mathbb{I} = [-3, \infty)$ and $s_0 = 1$. Since

$$f'(u) = \frac{(u-1)(u+3)}{(u^2+3)^2}, \quad f''(u) = \frac{2(3+9u-3u^2-u^3)}{(u^2+3)^3},$$

f is convex on $[s, s_0]$ and decreasing on $[-3, s_0]$. In addition, $f(u) \geq f(s_0)$ for $u \in \mathbb{I}$. So, by RPCF-Theorem, it suffices to prove the required inequality for $b = c = d$. It is easy to show that the inequality reduces to $a^2(a+3)^2 \geq 0$.

Case 2: $a \leq -3$ and $b \leq 0$. We have

$$f(a) + f(b) + f(c) + f(d) > 0 + f(0) + f(1) + f(1) = 0 - \frac{1}{3} - \frac{1}{2} - \frac{1}{2} = \frac{-4}{3} = 4f(s).$$

Case 3: $a \leq -3$ and $b \geq 0$. Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} f(a) + f(b) + f(c) + f(d) + \frac{3}{2} &= \frac{-a-1}{a^2+3} + \frac{(b-1)^2}{2(b^2+3)} + \frac{(c-1)^2}{2(c^2+3)} + \frac{(d-1)^2}{2(d^2+3)} \\ &\geq \frac{-a-1}{a^2+3} + \frac{(b+c+d-3)^2}{2(b^2+c^2+d^2)+18} \geq \frac{-a-1}{a^2+3} + \frac{(b+c+d-3)^2}{2(b+c+d)^2+18} \\ &= \frac{-a-1}{a^2+3} + \frac{(a+3)^2}{2(a^2+9)} = \frac{a^2}{3(a^2+3)(a^2+9)} + \frac{1}{6} > \frac{1}{6}, \end{aligned}$$

hence

$$f(a) + f(b) + f(c) + f(d) > \frac{1}{6} - \frac{3}{2} = \frac{-4}{3} = 4f(s).$$

The equality occurs for $a = b = c = d = 0$, and also for $a = -3$ and $b = c = d = 1$ (or any cyclic permutation).

Solution 2 (by Nguyen Ngoc Phuc). Let $a^2 = \max\{a^2, b^2, c^2, d^2\}$. Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \frac{b+1}{b^2+3} + \frac{c+1}{c^2+3} + \frac{d+1}{d^2+3} - \frac{3}{2} &= -\frac{(b-1)^2}{2(b^2+3)} - \frac{(c-1)^2}{2(c^2+3)} - \frac{(d-1)^2}{2(d^2+3)} \\ &= -\frac{(b-1)^2 + (c-1)^2 + (d-1)^2}{2(a^2+3)} \leq \frac{-[(b-1) + (c-1) + (d-1)]^2}{6(a^2+3)} = \frac{-(a+3)^2}{6(a^2+3)}, \end{aligned}$$

therefore

$$\frac{a+1}{a^2+3} + \frac{b+1}{b^2+3} + \frac{c+1}{c^2+3} + \frac{d+1}{d^2+3} \leq \frac{a+1}{a^2+3} - \frac{(a+3)^2}{6(a^2+3)} + \frac{3}{2} = \frac{-1}{6} + \frac{3}{2} = \frac{4}{3}.$$

□

P 3.9. Let $a_1, a_2, \dots, a_n \neq -k$ be real numbers so that $a_1 + a_2 + \dots + a_n = n$, where

$$k \geq \frac{n}{2\sqrt{n-1}}.$$

Then,

$$\frac{a_1(a_1 - 1)}{(a_1 + k)^2} + \frac{a_2(a_2 - 1)}{(a_2 + k)^2} + \dots + \frac{a_n(a_n - 1)}{(a_n + k)^2} \geq 0.$$

(Vasile C., 2008)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \dots + a_n}{n} = 1,$$

where

$$f(u) = \frac{u(u-1)}{(u+k)^2}, \quad u \in \mathbb{I} = \mathbb{R} \setminus \{-k\}.$$

From

$$f'(u) = \frac{(2k+1)u - k}{(u+k)^3},$$

it follows that f is increasing on $(-\infty, -k) \cup [s_0, \infty)$ and decreasing on $(-k, s_0]$, where

$$s_0 = \frac{k}{2k+1} < 1 = s.$$

Since

$$\lim_{u \rightarrow -\infty} f(u) = 1$$

and $f(s_0) < 0$, we have

$$\min_{u \in \mathbb{I}} f(u) = f(s_0).$$

From

$$\frac{1}{2}f''(u) = \frac{k(k+2) - (2k+1)u}{(u+k)^4},$$

it follows that f is convex on $\left[0, \frac{k(k+2)}{2k+1}\right]$, hence on $[s_0, 1]$. According to the LPCF-Theorem, Note 4 and Note 1, it suffices to show that $h(x, y) \geq 0$ for all $x, y \in \mathbb{I}$ which satisfy $x + (n-1)y = n$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

Indeed, we have

$$g(u) = \frac{u}{(u+k)^2}$$

and

$$\begin{aligned} h(x, y) &= \frac{k^2 - xy}{(x+k)^2(y+k)^2} \geq \frac{\frac{n^2}{4(n-1)} - xy}{(x+k)^2(y+k)^2} \\ &= \frac{[2(n-1)y - n]^2}{4(n-1)(x+k)^2(y+k)^2} \geq 0. \end{aligned}$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $k = \frac{n}{2\sqrt{n-1}}$, then the equality holds also for

$$a_1 = \frac{n}{2}, \quad a_2 = \cdots = a_n = \frac{n}{2(n-1)}$$

(or any cyclic permutation).

□

P 3.10. Let $a_1, a_2, \dots, a_n \neq -k$ be real numbers so that $a_1 + a_2 + \cdots + a_n = n$. If

$$k \geq 1 + \frac{n}{\sqrt{n-1}},$$

then

$$\frac{a_1^2 - 1}{(a_1 + k)^2} + \frac{a_2^2 - 1}{(a_2 + k)^2} + \cdots + \frac{a_n^2 - 1}{(a_n + k)^2} \geq 0.$$

(Vasile C., 2008)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{u^2 - 1}{(u + k)^2}, \quad u \in \mathbb{I} = \mathbb{R} \setminus \{-k\}.$$

From

$$f'(u) = \frac{2(ku + 1)}{(u + k)^3},$$

it follows that f is increasing on $(-\infty, -k) \cup [s_0, \infty)$ and decreasing on $(-k, s_0]$, where

$$s_0 = \frac{-1}{k} < 0 = s, \quad s_0 > -1.$$

Since

$$\lim_{u \rightarrow -\infty} f(u) = 1$$

and $f(s_0) < 0$, we have

$$\min_{u \in \mathbb{I}} f(u) = f(s_0).$$

For $u \in [-1, 1]$, we have

$$f''(u) = \frac{2(k^2 - 3 - 2ku)}{(u + k)^4} \geq \frac{2(k^2 - 3 - 2k)}{(u + k)^4} = \frac{2(k + 1)(k - 3)}{(u + k)^4} \geq 0,$$

hence f is convex on $[s_0, 1]$. According to the LPCF-Theorem, Note 4 and Note 1, it suffices to show that $h(x, y) \geq 0$ for $x, y \in \mathbb{I}$ which satisfy $x + (n - 1)y = n$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{u + 1}{(u + k)^2},$$

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{(k-1)^2 - 1 - x - y - xy}{(x+k)^2(y+k)^2} \geq 0,$$

since

$$(k-1)^2 - 1 - x - y - xy \geq \frac{n^2}{n-1} - 1 - x - y - xy = \frac{[(n-1)y-1]^2}{n-1} \geq 0.$$

The equality holds for $a_1 = a_2 = \dots = a_n = 1$. If $k = 1 + \frac{n}{\sqrt{n-1}}$, then the equality holds also for

$$a_1 = n-1, \quad a_2 = \dots = a_n = \frac{1}{n-1}$$

(or any cyclic permutation).

□

P 3.11. Let a_1, a_2, a_3, a_4, a_5 be real numbers so that $a_1 + a_2 + a_3 + a_4 + a_5 \geq 5$. If

$$k \in \left[\frac{1}{6}, \frac{25}{14} \right],$$

then

$$\sum \frac{1}{ka_1^2 + a_2 + a_3 + a_4 + a_5} \leq \frac{5}{k+4}.$$

(Vasile C., 2006)

Solution. We see that

$$ka_i^2 - a_i + (a_1 + a_2 + a_3 + a_4 + a_5) > \frac{1}{6}a_i^2 - a_i + \frac{3}{2} = \frac{(a_i-3)^2}{6} \geq 0$$

for all $i \in \{1, 2, \dots, n\}$. Since each term of the left hand side of the inequality decreases by increasing any number a_i , it suffices to consider the case

$$a_1 + a_2 + a_3 + a_4 + a_5 = 5,$$

when the desired inequality can be written as

$$f(a_1) + f(a_2) + f(a_3) + f(a_4) + f(a_5) \geq 5f(s), \quad s = \frac{a_1 + a_2 + a_3 + a_4 + a_5}{5} = 1,$$

where

$$f(u) = \frac{-1}{ku^2 - u + 5}, \quad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{2ku - 1}{(ku^2 - u + 5)^2},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = \frac{1}{2k}.$$

We have

$$f''(u) = \frac{2g(u)}{(ku^2 - u + 5)^3}, \quad g(u) = -3k^2u^2 + 3ku + 5k - 1.$$

For

$$\frac{1}{2} \leq k \leq \frac{25}{14},$$

we have

$$s_0 = \frac{1}{2k} \leq 1 = s,$$

and for $u \in [s_0, s]$, that is

$$\frac{1}{2k} \leq u \leq 1,$$

we have

$$\begin{aligned} (1-u)(2ku-1) &\geq 0, \\ -2ku^2 &\geq (2k+1)u+1, \\ -2k^2u^2 &\geq k(2k+1)u+k, \end{aligned}$$

therefore

$$\begin{aligned} g(u) &\geq \frac{3}{2}[k(2k+1)u+k] + 3ku + 5k - 1 = \frac{-3k(2k-1)u + 13k - 2}{2} \\ &\geq \frac{-3k(2k-1) + 13k - 2}{2} = -3k^2 + 8k - 1 = 3k(2-k) + (2k-1) > 0. \end{aligned}$$

Consequently, f is convex on $[s_0, s]$.

For

$$\frac{1}{6} \leq k \leq \frac{1}{2},$$

we have

$$s_0 = \frac{1}{2k} \geq 1 = s,$$

and for $u \in [s, s_0]$, that is

$$1 \leq u \leq \frac{1}{2k},$$

we have

$$\begin{aligned} g(u) &= -3k^2u^2 + 3ku + 5k - 1 \geq 3ku(1-k) + 5k - 1 \\ &\geq 3k(1-k) + 5k - 1 = -3k^2 + 8k - 1 \\ &> -6k^2 + 7k - 1 = (1-k)(6k-1) \geq 0. \end{aligned}$$

Consequently, f is convex on $[s, s_0]$.

In both cases, by the PCF-Theorem, it suffices to show that

$$\frac{1}{kx^2 - x + 5} + \frac{4}{ky^2 - y + 5} \leq \frac{5}{k+4}$$

for

$$x + 4y = 5, \quad x, y \in \mathbb{R}.$$

Write this inequality as follows:

$$\frac{1}{k+4} - \frac{1}{kx^2 - x + 5} + 4 \left[\frac{1}{k+4} - \frac{1}{ky^2 - y + 5} \right] \geq 0,$$

$$\frac{(x-1)(kx+k-1)}{kx^2 - x + 5} + \frac{4(y-1)(ky+k-1)}{ky^2 - y + 5} \geq 0.$$

Since

$$4(y-1) = 1-x,$$

the inequality is equivalent to

$$(x-1) \left(\frac{kx+k-1}{kx^2 - x + 5} - \frac{ky+k-1}{ky^2 - y + 5} \right) \geq 0,$$

$$\frac{5(x-1)^2 h(x, y)}{4(kx^2 - x + 5)(ky^2 - y + 5)} \geq 0,$$

where

$$\begin{aligned} h(x, y) &= -k^2 xy - k(k-1)(x+y) + 6k-1 \\ &= 4k^2 y^2 - k(2k+3)y - 5k^2 + 11k-1 \\ &= \left(2ky - \frac{2k+3}{4} \right)^2 + \frac{(25-14k)(6k-1)}{16} \geq 0. \end{aligned}$$

The equality holds for $a_1 = a_2 = a_3 = a_4 = a_5 = 1$. If $k = \frac{1}{6}$, then the equality holds also for

$$a_1 = -5, \quad a_2 = a_3 = a_4 = a_5 = \frac{5}{2}$$

(or any cyclic permutation). If $k = \frac{25}{14}$, then the equality holds also for

$$a_1 = \frac{79}{25}, \quad a_2 = a_3 = a_4 = a_5 = \frac{23}{50}$$

(or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization:

- Let $a_1, a_2, \dots, a_n$ be real numbers so that $a_1 + a_2 + \dots + a_n \leq n$. If $k \in [k_1, k_2]$, where

$$k_1 = \frac{(n-1)(\sqrt{53n^2 - 54n + 101} - 5n + 11)}{2(7n^2 + 14n - 5)},$$

$$k_2 = \frac{2n^2 - 2n + 1 + \sqrt{(n-1)(3n^3 - 4n^2 + 3n - 1)}}{2(n^2 - n + 1)},$$

then

$$\sum \frac{1}{ka_1^2 + a_2 + \dots + a_n} \leq \frac{n}{k+n-1},$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$. If $k = k_1$, then the equality holds also for

$$a_1 = -n, \quad a_2 = \cdots = a_n = \frac{2n}{n-1}$$

(or any cyclic permutation). If $k = k_2$, then the equality holds also for

$$a_1 = \frac{(2k-1)(n-1)+1}{2k}, \quad a_2 = \cdots = a_n = \frac{2k+n-2}{2k(n-1)}$$

(or any cyclic permutation).

□

P 3.12. Let $a_1, a_2, \dots, a_5$ be nonnegative numbers so that $a_1 + a_2 + a_3 + a_4 + a_5 \geq 5$. If $k \in [k_1, k_2]$, where

$$k_1 = \frac{29 - \sqrt{761}}{10} \approx 0.1414, \quad k_2 = \frac{25}{14} \approx 1.7857,$$

then

$$\sum \frac{1}{ka_1^2 + a_2 + a_3 + a_4 + a_5} \leq \frac{5}{k+4}.$$

(Vasile C., 2006)

Solution. Since all terms of the left hand side of the inequality decrease by increasing any number a_i , it suffices to consider the case

$$a_1 + a_2 + a_3 + a_4 + a_5 = 5.$$

The proof is similar to the one of the preceding P 3.11. Having in view P 3.11, it suffices to consider the case

$$k \in \left[k_1, \frac{1}{6} \right],$$

when

$$s_0 = \frac{1}{2k} > 1 = s.$$

For $u \in [s, s_0]$, that is

$$1 \leq u \leq \frac{1}{2k},$$

f is convex because

$$\begin{aligned} g(u) &= -3k^2u^2 + 3ku + 5k - 1 \geq 3ku(1-k) + 5k - 1 \\ &\geq 3k(1-k) + 5k - 1 = -3k^2 + 8k - 1 \\ &> -\frac{15}{4}k^2 + 87k - 1 = \frac{(2-k)(15k-2)}{4} > 0. \end{aligned}$$

Thus, by the RPCF-Theorem, it suffices to show that

$$\frac{1}{kx^2 - x + 5} + \frac{4}{ky^2 - y + 5} \leq \frac{5}{k+4}$$

for

$$x + 4y = 5, \quad 0 \leq x \leq 1 \leq y \leq \frac{5}{4}.$$

As shown at P 3.11, this inequality is true if $h(x, y) \geq 0$, where

$$h(x, y) = -k^2xy - k(k-1)(x+y) + 6k - 1.$$

We have

$$\begin{aligned} h(x, y) &= 4k^2y^2 - k(2k+3)y - 5k^2 + 11k - 1 \\ &= (5-4y)(A - k^2y) + B = x(A - k^2y) + B, \end{aligned}$$

where

$$A = \frac{3k(1-k)}{4}, \quad B = \frac{-5k^2 + 29k - 4}{4}.$$

Since $B \geq 0$, it suffices to show that $A - k^2y \geq 0$. Indeed, we have

$$A - k^2y \geq \frac{3k(1-k)}{4} - \frac{5k^2}{4} = \frac{k(3-8k)}{4} > 0.$$

The equality holds for $a_1 = a_2 = a_3 = a_4 = a_5 = 1$. If $k = k_1$, then the equality holds also for

$$a_1 = 0, \quad a_2 = a_3 = a_4 = a_5 = \frac{5}{4}$$

(or any cyclic permutation). If $k = k_2$, then the equality holds also for

$$a_1 = \frac{79}{25}, \quad a_2 = a_3 = a_4 = a_5 = \frac{23}{50}$$

(or any cyclic permutation)

Remark. Similarly, we can prove the following generalization:

• Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n \leq n$. If $k \in [k_1, k_2]$, where

$$\begin{aligned} k_1 &= \frac{n^2 + n - 1 - \sqrt{n^4 + 2n^3 - 5n^2 + 2n + 1}}{2n}, \\ k_2 &= \frac{2n^2 - 2n + 1 + \sqrt{(n-1)(3n^3 - 4n^2 + 3n - 1)}}{2(n^2 - n + 1)}, \end{aligned}$$

then

$$\sum \frac{1}{ka_1^2 + a_2 + \dots + a_n} \leq \frac{n}{k + n - 1},$$

with equality for $a_1 = a_2 = \dots = a_n = 1$. If $k = k_1$, then the equality holds also for

$$a_1 = 0, \quad a_2 = \dots = a_n = \frac{n}{n-1}$$

(or any cyclic permutation). If $k = k_2$, then the equality holds also for

$$a_1 = \frac{(2k-1)(n-1)+1}{2k}, \quad a_2 = \dots = a_n = \frac{2k+n-2}{2k(n-1)}$$

(or any cyclic permutation).

□

P 3.13. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n \geq n$. If $k > 1$, then

$$\frac{1}{a_1^k + a_2 + \dots + a_n} + \frac{1}{a_1 + a_2^k + \dots + a_n} + \dots + \frac{1}{a_1 + a_2 + \dots + a_n^k} \leq 1.$$

(Vasile C., 2006)

Solution. It suffices to consider the case $a_1 + a_2 + \dots + a_n = n$, when the desired inequality can be written as

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \dots + a_n}{n} = 1,$$

where

$$f(u) = \frac{-1}{u^k - u + n}, \quad u \in [0, n].$$

From

$$f'(u) = \frac{ku^{k-1} - 1}{(u^k - u + n)^2},$$

it follows that f is decreasing on $[0, s_0]$ and increasing on $[s_0, n]$, where

$$s_0 = k^{\frac{1}{1-k}} < 1 = s.$$

We will show that f is convex on $[s_0, 1]$. For $u \in [s_0, 1]$, we have

$$f''(u) = \frac{-k(k+1)u^{2k-2} + k(k+3)u^{k-1} + nk(k-1)u^{k-2} - 2}{(u^k - u + n)^3} > \frac{g(u)}{(u^k - u + n)^3},$$

where

$$g(u) = -k(k+1)u^{2k-2} + k(k+3)u^{k-1} - 2.$$

Denoting

$$t = ku^{k-1}, \quad 1 \leq t \leq k,$$

we get

$$\begin{aligned} kg(u) &= -(k+1)t^2 + k(k+3)t - 2k \\ &= (k+1)(t-1)(k-t) + (k-1)(t+k) > 0. \end{aligned}$$

By the LPCF-Theorem, it suffices to show that

$$\frac{1}{x^k - x + n} + \frac{n-1}{y^k - y + n} \leq 1$$

for $x \geq 1 \geq y \geq 0$ and $x + (n-1)y = n$. Since this inequality is trivial for $x = y = 1$, assume next that $x > 1 > y \geq 0$, and write the desired inequality as follows:

$$x^k - x + n \geq \frac{y^k - y + n}{y^k - y + 1},$$

$$x^k - x \geq \frac{(n-1)(y - y^k)}{y^k - y + 1},$$

$$\frac{x^k - x}{x - 1} \geq \frac{y - y^k}{(1 - y)(y^k - y + 1)}.$$

Let $h(x) = \frac{x^k - x}{x - 1}$, $x > 1$. By the weighted AM-GM inequality, we have

$$h'(x) = \frac{(k-1)x^k + 1 - kx^{k-1}}{(x-1)^2} > 0.$$

Therefore, h is increasing. Since

$$x - 1 = (n-1)(1-y) \geq 1-y, \quad x \geq 2-y > 1,$$

we get

$$h(x) \geq h(2-y) = \frac{(2-y)^k + y - 2}{1-y}.$$

Thus, it suffices to show that

$$(2-y)^k + y - 2 \geq \frac{y - y^k}{y^k - y + 1},$$

which is equivalent to

$$(2-y)^k + y - 1 \geq \frac{1}{y^k - y + 1}.$$

Using the substitution

$$t = 1 - y, \quad 0 < t \leq 1,$$

the inequality becomes

$$\begin{aligned} (1+t)^k - t &\geq \frac{1}{(1-t)^k + t}, \\ (1-t^2)^k + t(1+t)^k &\geq 1 + t^2 + t(1-t)^k. \end{aligned}$$

By Bernoulli's inequality,

$$(1-t^2)^k + t(1+t)^k \geq 1 - kt^2 + t(1+kt) = 1 + t.$$

So, we only need to show that

$$1 + t \geq 1 + t^2 + t(1-t)^k,$$

which is equivalent to the obvious inequality

$$t(1-t) [1 - (1-t)^{k-1}] \geq 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Remark. Using this result, we can formulate the following statement:

- Let $x_1, x_2, \dots, x_n$ be nonnegative real numbers so that $x_1 + x_2 + \cdots + x_n \geq n$. If $k > 1$, then

$$\frac{x_1^k - x_1}{x_1^k + x_2 + \cdots + x_n} + \frac{x_2^k - x_2}{x_1 + x_2^k + \cdots + x_n} + \cdots + \frac{x_n^k - x_n}{x_1 + x_2 + \cdots + x_n^k} \geq 0.$$

This inequality is equivalent to

$$\frac{1}{x_1^k + x_2 + \cdots + x_n} + \frac{1}{x_1 + x_2^k + \cdots + x_n} + \cdots + \frac{1}{x_1 + x_2 + \cdots + x_n^k} \leq \frac{n}{x_1 + x_2 + \cdots + x_n}.$$

Using the substitutions

$$s = \frac{x_1 + x_2 + \cdots + x_n}{n}, \quad s \geq 1,$$

and

$$a_i = \frac{x_i}{s}, \quad i = 1, 2, \dots, n,$$

which yields $a_1 + a_2 + \cdots + a_n = n$, the desired inequality becomes

$$\sum \frac{1}{s^{k-1}a_1^k + a_2 + \cdots + a_n} \leq 1.$$

Since $s^{k-1} \geq 1$, it suffices to show that

$$\sum \frac{1}{a_1^k + a_2 + \cdots + a_n} \leq 1,$$

which follows immediately from the inequality in P 3.13.

Since $x_1 x_2 \cdots x_n \geq 1$ involves $x_1 + x_2 + \cdots + x_n \geq n$, the inequality is also true under the more restrictive condition $x_1 x_2 \cdots x_n \geq 1$. For $n = 3$ and $k = 5/2$, we get the inequality from IMO-2005:

- If x, y, z are nonnegative real numbers so that $xyz \geq 1$, then

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0.$$

□

P 3.14. Let $a_1, a_2, \dots, a_5$ be nonnegative numbers so that $a_1 + a_2 + a_3 + a_4 + a_5 \geq 5$. If

$$k \in \left[\frac{4}{9}, \frac{61}{5} \right],$$

then

$$\sum \frac{a_1}{ka_1^2 + a_2 + a_3 + a_4 + a_5} \leq \frac{5}{k+4}.$$

(Vasile C., 2006)

Solution. Using the substitution

$$x_1 = \frac{a_1}{s}, \quad x_2 = \frac{a_2}{s}, \quad x_3 = \frac{a_3}{s}, \quad x_4 = \frac{a_4}{s}, \quad x_5 = \frac{a_5}{s},$$

where

$$s = \frac{a_1 + a_2 + a_3 + a_4 + a_5}{5} \geq 1,$$

we need to show that $x_1 + x_2 + x_3 + x_4 + x_5 = 5$ involves

$$\frac{x_1}{ksx_1^2 + x_2 + x_3 + x_4 + x_5} + \cdots + \frac{x_5}{x_1 + x_2 + x_3 + x_4 + ksx_5^2} \leq \frac{5}{k+4}.$$

Since $s \geq 1$, it suffices to prove the inequality for $s = 1$; that is, to show that

$$\frac{a_1}{ka_1^2 - a_1 + 5} + \frac{a_2}{ka_2^2 - a_1 + 5} + \cdots + \frac{a_5}{ka_5^2 - a_n + 5} \leq \frac{5}{k+4}$$

for

$$a_1 + a_2 + a_3 + a_4 + a_5 = 5.$$

Write the desired inequality as

$$f(a_1) + f(a_2) + f(a_3) + f(a_4) + f(a_5) \geq 5f(s),$$

where

$$s = \frac{a_1 + a_2 + a_3 + a_4 + a_5}{5} = 1$$

and

$$f(u) = \frac{-u}{ku^2 - u + 5}, \quad u \in [0, 5].$$

From

$$f'(u) = \frac{ku^2 - 5}{(ku^2 - u + 5)^2},$$

it follows that f is decreasing on $[0, s_0]$ and increasing on $[s_0, 5]$, where

$$s_0 = \sqrt{\frac{5}{k}}.$$

We have

$$f''(u) = \frac{2g(u)}{(u^2 - u + 5)^3}, \quad g(u) = -k^2u^3 + 15ku - 5, \quad g'(u) = 3k(5 - ku^2).$$

Case 1: $\frac{4}{9} \leq k \leq 5$. We have

$$s_0 = \sqrt{\frac{5}{k}} \geq 1 = s.$$

For $u \in [1, s_0]$, the derivative g' is nonnegative, g is increasing, hence

$$g(u) \geq g(1) = -k^2 + 15k - 5 = \left(k - \frac{4}{9}\right)(5 - k) + \frac{86k - 25}{9} > 0.$$

Consequently, $f''(u) > 0$ for $u \in [1, s_0]$, hence f is convex on $[s, s_0]$.

Case 2: $5 \leq k \leq \frac{61}{5}$. We have

$$s_0 = \sqrt{\frac{5}{k}} < 1 = s.$$

For $u \in [s_0, 1]$, we have $g'(u) \leq 0$, $g(u)$ is decreasing, hence

$$g(u) \geq g(1) = -k^2 + 15k - 5 = (k-1)(13-k) + k + 8 > 0.$$

Consequently, $f''(u) > 0$ for $u \in [s_0, 1]$, hence f is convex on $[s_0, s]$.

In both cases, by the PCF-Theorem, it suffices to show that

$$\frac{x}{kx^2 - x + 5} + \frac{4y}{ky^2 - y + 5} \leq \frac{5}{k+4}$$

for

$$x + 4y = 5, \quad x, y \geq 0.$$

Write this inequality as follows:

$$\begin{aligned} \frac{1}{k+4} - \frac{x}{kx^2 - x + 5} + 4 \left[\frac{1}{k+4} - \frac{y}{ky^2 - y + 5} \right] &\geq 0, \\ \frac{(x-1)(kx-5)}{kx^2 - x + 5} + \frac{4(y-1)(ky-5)}{ky^2 - y + 5} &\geq 0. \end{aligned}$$

Since

$$4(y-1) = 1-x,$$

the inequality is equivalent to

$$\begin{aligned} (x-1) \left(\frac{kx-5}{kx^2 - x + 5} - \frac{ky-5}{ky^2 - y + 5} \right) &\geq 0, \\ \frac{(x-1)^2 h(x, y)}{(kx^2 - x + 5)(ky^2 - y + 5)} &\geq 0, \end{aligned}$$

where

$$\begin{aligned} h(x, y) &= -k^2 xy + 5k(x+y) + 5k - 5 \\ &= 4k^2 y^2 - 5k(k+3)y + 5(6k-1). \end{aligned}$$

We need to show that $h(x, y) \geq 0$ for $k \in \left[\frac{4}{9}, \frac{61}{5} \right]$. For $k \in \left[\frac{4}{9}, 1 \right]$, we have

$$\begin{aligned} h(x, y) &= (5-4y) \left(-k^2 y + \frac{15k}{4} \right) + \frac{5(9k-4)}{4} \\ &= \frac{kx(15-4ky)}{4} + \frac{5(9k-4)}{4} \\ &= \frac{kx(kx+15-5k)}{4} + \frac{5(9k-4)}{4} \geq 0, \end{aligned}$$

while for $k \in \left[1, \frac{61}{5} \right]$, we have

$$h(x, y) = \left(2ky - \frac{5k+15}{4} \right)^2 + \frac{(61-5k)(k-1)}{16} \geq 0.$$

The equality holds for $a_1 = a_2 = a_3 = a_4 = a_5 = 1$. If $k = \frac{4}{9}$, then the equality holds also for

$$a_1 = 0, \quad a_2 = a_3 = a_4 = a_5 = \frac{5}{4}$$

(or any cyclic permutation). If $k = \frac{61}{5}$, then the equality holds also for

$$a_1 = \frac{115}{61}, \quad a_2 = a_3 = a_4 = a_5 = \frac{95}{122}$$

(or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization:

- Let $a_1, a_2, \dots, a_n$ be real numbers so that $a_1 + a_2 + \dots + a_n \leq n$. If $k \in [k_1, k_2]$, where

$$k_1 = \frac{n-1}{2n-1},$$

$$k_2 = \frac{n^2 + 2n - 2 + 2\sqrt{(n-1)(2n^2-1)}}{n},$$

then

$$\sum \frac{a_1}{ka_1^2 + a_2 + \dots + a_n} \leq \frac{n}{k+n-1},$$

with equality for $a_1 = a_2 = \dots = a_n = 1$. If $k = k_1$, then the equality holds also for

$$a_1 = 0, \quad a_2 = a_3 = a_4 = a_5 = \frac{n}{n-1}$$

(or any cyclic permutation). If $k = k_2$, then the equality holds also for

$$a_1 = \frac{n(k-n+2)}{2k}, \quad a_2 = \dots = a_n = \frac{n(k+n-2)}{2k(n-1)}$$

(or any cyclic permutation).

□

P 3.15. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n \geq n$. If $k > 1$, then

$$\frac{a_1}{a_1^k + a_2 + \dots + a_n} + \frac{a_2}{a_1 + a_2^k + \dots + a_n} + \dots + \frac{a_n}{a_1 + a_2 + \dots + a_n^k} \leq 1.$$

(Vasile C., 2006)

Solution. Using the substitution

$$x_1 = \frac{a_1}{s}, \quad x_2 = \frac{a_2}{s}, \quad \dots, \quad x_n = \frac{a_n}{s},$$

where

$$s = \frac{a_1 + a_2 + \dots + a_n}{n} \geq 1,$$

we need to show that $x_1 + x_2 + \cdots + x_n = n$ involves

$$\frac{x_1}{s^{k-1}x_1^k + x_2 + \cdots + x_n} + \cdots + \frac{x_n}{x_1 + x_2 + \cdots + s^{k-1}x_n^k} \leq 1.$$

Since $s^{k-1} \geq 1$, it suffices to prove the inequality for $s = 1$; that is, to show that

$$\frac{a_1}{a_1^k - a_1 + n} + \frac{a_2}{a_2^k - a_2 + n} + \cdots + \frac{a_n}{a_n^k - a_n + n} \leq 1$$

for

$$a_1 + a_2 + \cdots + a_n = n.$$

Case 1: $1 < k \leq n + 1$. By Bernoulli's inequality, we have

$$a_1^k \geq 1 + k(a_1 - 1), \quad a_1^k - a_1 + n \geq (k - 1)a_1 + n - k + 1.$$

Thus, it suffices to show that

$$\sum \frac{a_1}{(k - 1)a_1 + n - k + 1} \leq 1.$$

This is an equality for $k = n + 1$. If $1 < k < n + 1$, then the inequality is equivalent to

$$\sum \frac{1}{(k - 1)a_1 + n - k + 1} \geq 1,$$

which follows from the the AM-HM inequality

$$\sum \frac{1}{(k - 1)a_1 + n - k + 1} \geq \frac{n^2}{\sum [(k - 1)a_1 + n - k + 1]}.$$

Case 2: $k > n + 1$. Write the desired inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{-u}{u^k - u + n}, \quad u \in [0, n].$$

We have

$$f'(u) = \frac{(k - 1)u^k - n}{(u^k - u + n)^2}$$

and

$$f''(u) = \frac{f_1(u)}{(u^k - u + n)^3},$$

where

$$f_1(u) = k(k - 1)u^{k-1}(u^k - u + n) - 2(ku^{k-1} - 1)[(k - 1)u^k - n].$$

From the expression of f' , it follows that f is decreasing on $[0, s_0]$ and increasing on $[s_0, n]$, where

$$s_0 = \left(\frac{n}{k - 1} \right)^{1/k} < 1 = s.$$

For $u \in [s_0, 1]$, we have

$$(k-1)u^k - n \geq (k-1)s_0^k - n = 0,$$

hence

$$\begin{aligned} f_1(u) &\geq k(k-1)u^{k-1}(u^k - u + n) - 2ku^{k-1}[(k-1)u^k - n] \\ &= ku^{k-1}[-(k-1)(u^k + u) + n(k+1)] \\ &\geq ku^{k-1}[-2(k-1) + 2(k+1)] = 4ku^{k-1} > 0. \end{aligned}$$

Since $f''(u) > 0$, it follows that f is convex on $[s_0, s]$. By the LPCF-Theorem, we need to show that

$$f(x) + (n-1)f(y) \geq nf(1)$$

for

$$x \geq 1 \geq y \geq 0, \quad x + (n-1)y = n.$$

Consider the nontrivial case where $x > 1 > y \geq 0$ and write the required inequality as follows:

$$\begin{aligned} \frac{x}{x^k - x + n} + \frac{(n-1)y}{y^k - y + n} &\leq 1, \\ x^k - x + n &\geq \frac{x(y^k - y + n)}{y^k - ny + n}, \\ x^k - x &\geq \frac{(n-1)y(y - y^k)}{y^k - ny + n}. \end{aligned}$$

Since $y < 1$ and $y^k - ny + n > y^k - y + 1$, it suffices to show that

$$x^k - x \geq \frac{(n-1)(y - y^k)}{y^k - y + 1},$$

which has been proved at P 3.13.

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 3.16. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n \leq n$. If $k \geq 1 - \frac{1}{n}$, then

$$\frac{1 - a_1}{ka_1^2 + a_2 + \cdots + a_n} + \frac{1 - a_2}{a_1 + ka_2^2 + \cdots + a_n} + \cdots + \frac{1 - a_n}{a_1 + a_2 + \cdots + ka_n^2} \geq 0.$$

(Vasile C., 2006)

Solution. Let

$$s = \frac{a_1 + a_2 + \cdots + a_n}{n}, \quad s \leq 1.$$

We have three cases to consider.

Case 1: $s \leq \frac{1}{n}$. The inequality is trivial because

$$a_i \leq a_1 + a_2 + \cdots + a_n = ns \leq 1$$

for $i = 1, 2, \dots, n$.

Case 2: $\frac{1}{n} < s < 1$. Without loss of generality, assume that

$$a_1 \leq \dots \leq a_j < 1 \leq a_{j+1} \dots \leq a_n, \quad j \in \{1, 2, \dots, n\}.$$

Clearly, there are $b_1, b_2, \dots, b_n$ so that $b_1 + b_2 + \dots + b_n = n$ and

$$a_1 \leq b_1 \leq 1, \dots, a_j \leq b_j \leq 1, b_{j+1} = a_{j+1}, \dots, b_n = a_n.$$

Write the desired inequality as

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq 0,$$

where

$$f(u) = \frac{1-u}{ku^2 - u + ns}, \quad u \in [0, ns].$$

For $u \in [0, 1]$, we have

$$f'(u) = \frac{k[(1-u)^2 - 1] + (1 - ns)}{(ku^2 - u + ns)^2} < 0,$$

hence f is strictly decreasing on $[0, 1]$ and

$$f(b_1) \leq f(a_1), \dots, f(b_j) \leq f(a_j), f(b_{j+1}) = f(a_{j+1}), \dots, f(b_n) = f(a_n).$$

Since

$$f(b_1) + f(b_2) + \dots + f(b_n) \leq f(a_1) + f(a_2) + \dots + f(a_n),$$

it suffices to show that $f(b_1) + f(b_2) + \dots + f(b_n) \geq 0$ for $b_1 + b_2 + \dots + b_n = n$. This inequality is proved at Case 3.

Case 3: $s = 1$. Write the inequality as

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \dots + a_n}{n} = 1,$$

where

$$f(u) = \frac{1-u}{ku^2 - u + n}, \quad u \in [0, n].$$

From

$$f'(u) = \frac{k[(u-1)^2 - 1] - (n-1)}{(ku^2 - u + n)^2},$$

it follows that f is decreasing on $[0, s_0]$ and increasing on $[s_0, n]$, where

$$s_0 = 1 + \sqrt{1 + \frac{n-1}{k}} > 1 = s, \quad s_0 < n.$$

We will show that f is convex on $[1, s_0]$. We have

$$f''(u) = \frac{2g(u)}{(ku^2 - u + n)^3},$$

where

$$g(u) = -k^2u^3 + 3k^2u^2 + 3k(n-1)u - kn - n + 1, \quad g'(u) = 3k(-ku^2 + 2ku + n - 1).$$

For $u \in [1, s_0]$, we have $g'(u) \geq 0$, g is increasing, therefore

$$\begin{aligned} g(u) &\geq g(1) = 2k^2 + (2n-3)k - n + 1 \\ &\geq \frac{2(n-1)^2}{n^2} + \frac{(2n-3)(n-1)}{n} - n + 1 \\ &= \frac{(n^2-1)(n-2)}{n^2} \geq 0, \end{aligned}$$

$f''(u) \geq 0$, $f(u)$ is convex for $u \in [s, s_0]$. By the RPCF-Theorem, it suffices to show that

$$\frac{1-x}{kx^2-x+n} + \frac{(n-1)(1-y)}{ky^2-y+n} \geq 0$$

for $0 \leq x \leq 1 \leq y$ and $x + (n-1)y = n$. Since $(n-1)(1-y) = x-1$, we have

$$\begin{aligned} \frac{1-x}{kx^2-x+n} + \frac{(n-1)(1-y)}{ky^2-y+n} &= (x-1) \left(-\frac{1}{kx^2-x+n} + \frac{1}{ky^2-y+n} \right) \\ &= \frac{(x-1)(x-y)(kx+ky-1)}{(kx^2-x+n)(ky^2-y+n)} \\ &= \frac{n(x-1)^2(kx+ky-1)}{(n-1)(kx^2-x+n)(ky^2-y+n)} \geq 0, \end{aligned}$$

because

$$k(x+y) - 1 \geq \frac{n-1}{n}(x+y) - 1 = \frac{(n-2)x}{n} \geq 0.$$

The proof is completed. The equality holds for $a_1 = a_2 = \dots = a_n = 1$. If $k = 1 - \frac{1}{n}$, then the equality holds also for

$$a_1 = 0, \quad a_2 = a_3 = \dots = a_n = \frac{n}{n-1}$$

(or any cyclic permutation).

Remark. For $k = 1$, we get the following statement:

- If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \dots + a_n \leq n$, then

$$\frac{1-a_1}{a_1^2+a_2+\dots+a_n} + \frac{1-a_2}{a_1+a_2^2+\dots+a_n} + \dots + \frac{1-a_n}{a_1+a_2+\dots+a_n^2} \geq 0.$$

with equality for $a_1 = a_2 = \dots = a_n = 1$.

□

P 3.17. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n \leq n$. If $k \geq 1 - \frac{1}{n}$, then

$$\frac{1-a_1}{1-a_1+ka_1^2} + \frac{1-a_2}{1-a_2+ka_2^2} + \dots + \frac{1-a_n}{1-a_n+ka_n^2} \geq 0.$$

(Vasile C., 2006)

Solution. The proof is similar to the one of the preceding P 3.16. For the case 3, we need to show that

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{1-u}{1-u+ku^2}, \quad u \in [0, n].$$

From

$$f'(u) = \frac{ku(u-2)}{(1-u+ku^2)^2},$$

it follows that f is decreasing on $[0, s_0]$ and increasing on $[s_0, n]$, where

$$s_0 = 2 > s.$$

We will show that f is convex on $[1, s_0]$. For $u \in [1, s_0]$, we have

$$f''(u) = \frac{2kg(u)}{(1-u+ku^2)^3}, \quad g(u) = -ku^3 + 3ku^2 - 1.$$

Since

$$g'(u) = 3ku(2-u) \geq 0,$$

g is increasing, $g(u) \geq g(1) = 2k-1 \geq 0$, hence $f''(u) \geq 0$ for $u \in [1, s_0]$. By the RPCF-Theorem, it suffices to show that

$$\frac{1-x}{1-x+kx^2} + \frac{(n-1)(1-y)}{1-y+ky^2} \geq 0$$

for $0 \leq x \leq 1 \leq y$ and $x + (n-1)y = n$. Since $(n-1)(1-y) = x-1$, we have

$$\begin{aligned} \frac{1-x}{1-x+kx^2} + \frac{(n-1)(1-y)}{1-y+ky^2} &= (1-x) \left(\frac{1}{1-x+kx^2} - \frac{1}{1-y+ky^2} \right) \\ &= \frac{(1-x)(y-x)(kx+ky-1)}{(1-x+kx^2)(1-y+ky^2)} \\ &= \frac{n(x-1)^2(kx+ky-1)}{(n-1)(1-x+kx^2)(1-y+ky^2)}. \end{aligned}$$

Since

$$k(x+y)-1 \geq \frac{n-1}{n}(x+y)-1 = \frac{(n-2)x}{n} \geq 0,$$

the conclusion follows. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $k = 1 - \frac{1}{n}$, then the equality holds also for

$$a_1 = 0, \quad a_2 = a_3 = \cdots = a_n = \frac{n}{n-1}$$

(or any cyclic permutation).

□

P 3.18. Let $a_1, a_2, \dots, a_n$ be positive real numbers so that $a_1 + a_2 + \dots + a_n = n$. If $0 < k \leq \frac{n}{n-1}$, then

$$a_1^{k/a_1} + a_2^{k/a_2} + \dots + a_n^{k/a_n} \leq n.$$

(Vasile C., 2006)

Solution. According to the power mean inequality, we have

$$\left(\frac{a_1^{p/a_1} + a_2^{p/a_2} + \dots + a_n^{p/a_n}}{n} \right)^{1/p} \geq \left(\frac{a_1^{q/a_1} + a_2^{q/a_2} + \dots + a_n^{q/a_n}}{n} \right)^{1/q}$$

for all $p \geq q > 0$. Thus, it suffices to prove the desired inequality for

$$k = \frac{n}{n-1}, \quad 1 < k \leq 2.$$

Rewrite the desired inequality as

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \dots + a_n}{n} = 1,$$

where

$$f(u) = -u^{k/u}, \quad u \in \mathbb{I} = (0, n).$$

We have

$$\begin{aligned} f'(u) &= ku^{\frac{k}{u}-2}(\ln u - 1), \\ f''(u) &= ku^{\frac{k}{u}-4}[u + (1 - \ln u)(2u - k + k \ln u)]. \end{aligned}$$

For $n = 2$, when $k = 2$ and $\mathbb{I} = (0, 2)$, f is convex on $[1, 2)$ because

$$1 - \ln u > 0, \quad 2u - k + k \ln u = 2u - 2 + 2 \ln u \geq 2u - 2 \geq 0.$$

Therefore, we may apply the RHCF-Theorem. Consider now that $n \geq 3$. From the expression of f' , it follows that f is decreasing on $(0, s_0]$ and increasing on $[s_0, n)$, where

$$s_0 = e > 1 = s.$$

In addition, we claim that f is convex on $[1, s_0]$. Indeed, since

$$1 - \ln u \geq 0, \quad 2u - k + k \ln u \geq 2 - k > 0,$$

we have $f'' > 0$ for $u \in [1, s_0]$. Therefore, by the RHCF-Theorem (for $n = 2$) and the RPCF-Theorem (for $n \geq 3$), we only need to show that

$$x^{k/x} + (n-1)y^{k/y} \leq n$$

for

$$0 < x \leq 1 \leq y, \quad x + (n-1)y = n.$$

We have

$$\frac{k}{x} \geq k > 1.$$

Also, from

$$\frac{k}{y} = \frac{n}{(n-1)y} > \frac{n}{x + (n-1)y} = 1, \quad \frac{k}{y} = \frac{n}{(n-1)y} \leq \frac{2}{y} \leq 2,$$

we get

$$0 < \frac{k}{y} - 1 \leq 1.$$

Therefore, by Bernoulli's inequality, we have

$$\begin{aligned} x^{k/x} + (n-1)y^{k/y} - n &= \frac{1}{\left(\frac{1}{x}\right)^{k/x}} + (n-1)y \cdot y^{k/y-1} - n \\ &\leq \frac{1}{1 + \frac{k}{x}\left(\frac{1}{x} - 1\right)} + (n-1)y \left[1 + \left(\frac{k}{y} - 1\right)(y-1)\right] - n \\ &= \frac{x^2}{x^2 - kx + k} - (k-1)x^2 - (2-k)x \\ &= \frac{-x(x-1)^2[(k-1)x + k(2-k)]}{x^2 - kx + k} \leq 0. \end{aligned}$$

The proof is completed. The equality holds for $a_1 = a_2 = \dots = a_n = 1$.

□

P 3.19. If a, b, c, d, e are nonzero real numbers so that $a + b + c + d + e = 5$, then

$$\left(7 - \frac{5}{a}\right)^2 + \left(7 - \frac{5}{b}\right)^2 + \left(7 - \frac{5}{c}\right)^2 + \left(7 - \frac{5}{d}\right)^2 + \left(7 - \frac{5}{e}\right)^2 \geq 20.$$

(Vasile C., 2012)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) + f(d) + f(e) \geq 5f(s), \quad s = \frac{a+b+c+d+e}{5} = 1,$$

where

$$f(u) = \left(7 - \frac{5}{u}\right)^2, \quad u \in \mathbb{I} = \mathbb{R} \setminus \{0\}.$$

From

$$f'(u) = \frac{10(7u-5)}{u^3},$$

it follows that f is increasing on $(-\infty, 0) \cup [s_0, \infty)$ and decreasing on $(0, s_0]$, where

$$s_0 = \frac{5}{7} < 1 = s.$$

Since

$$\lim_{u \rightarrow -\infty} f(u) = 49$$

and $f(s_0) = 0$, we have

$$\min_{u \in \mathbb{I}} f(u) = f(s_0).$$

Also, f is convex on $[s_0, s] = [5/7, 1]$ because

$$f''(u) = \frac{10(15 - 14u)}{u^4} > 0.$$

According to the LPCF-Theorem and Note 4, we only need to show that

$$f(x) + 4f(y) \geq 5f(1)$$

for all nonzero real x, y so that $x + 4y = 5$. Using Note 1, it suffices to prove that $h(x, y) \geq 0$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

We have

$$g(u) = 5 \left(\frac{9}{u} - \frac{5}{u^2} \right),$$

$$h(x, y) = \frac{5(5x + 5y - 9xy)}{x^2y^2} = \frac{5(6y - 5)^2}{x^2y^2} \geq 0.$$

In accordance with Note 3, the equality holds for $a = b = c = d = e = 1$, and also for

$$a = \frac{5}{3}, \quad b = c = d = e = \frac{5}{6}$$

(or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

• Let $a_1, a_2, \dots, a_n$ be nonzero real numbers so that $a_1 + a_2 + \dots + a_n = n$. If $k = \frac{n}{n + \sqrt{n-1}}$, then

$$\left(1 - \frac{k}{a_1}\right)^2 + \left(1 - \frac{k}{a_2}\right)^2 + \dots + \left(1 - \frac{k}{a_n}\right)^2 \geq n(1 - k)^2,$$

with equality for $a_1 = a_2 = \dots = a_n = 1$, and also for

$$a_1 = \frac{n}{1 + \sqrt{n-1}}, \quad a_2 = a_3 = \dots = a_n = \frac{n}{n-1 + \sqrt{n-1}}$$

(or any cyclic permutation).

□

P 3.20. If $a_1, a_2, \dots, a_7$ are real numbers so that $a_1 + a_2 + \dots + a_7 = 7$, then

$$(a_1^2 + 2)(a_2^2 + 2) \cdots (a_7^2 + 2) \geq 3^7.$$

(Vasile C., 2007)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_7) \geq 7f(s), \quad s = \frac{a_1 + a_2 + \cdots + a_7}{7} = 1,$$

where

$$f(u) = \ln(u^2 + 2), \quad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{2u}{u^2 + 2},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty]$, where

$$s_0 = 0.$$

From

$$f''(u) = \frac{2(2 - u^2)}{(u^2 + 2)^2},$$

it follows that $f''(u) > 0$ for $u \in [0, 1]$, therefore f is convex on $[s_0, s]$. By the LPCF-Theorem, it suffices to prove that

$$f(x) + 6f(y) \geq 7f(1)$$

for $x, y \in \mathbb{R}$ so that $x + 6y = 7$. The inequality can be written as $g(y) \geq 0$, where

$$g(y) = \ln[(7 - 6y)^2 + 2] + 6 \ln(y^2 + 2) - 7 \ln 3, \quad y \in \mathbb{R}.$$

From

$$\begin{aligned} g'(y) &= \frac{4(6y - 7)}{12y^2 - 28y + 17} + \frac{12y}{y^2 + 2} \\ &= \frac{28(6y^3 - 13y^2 + 9y - 2)}{(12y^2 - 28y + 17)(y^2 + 2)} \\ &= \frac{28(2y - 1)(3y - 2)(y - 1)}{(12y^2 - 28y + 17)(y^2 + 2)}, \end{aligned}$$

it follows that g is decreasing on $\left(-\infty, \frac{1}{2}\right] \cup \left[\frac{2}{3}, 1\right]$ and increasing on $\left[\frac{1}{2}, \frac{2}{3}\right] \cup [1, \infty)$; therefore,

$$g \geq \min\{g(1/2), g(1)\}.$$

Since $g(1) = 0$, we only need to show that $g(1/2) \geq 0$; that is, to show that $x = 4$ and $y = 1/2$ involve

$$(x^2 + 2)(y^2 + 2)^6 \geq 3^7.$$

Indeed, we have

$$(x^2 + 2)(y^2 + 2)^6 - 3^7 = 3^7 \left(\frac{3^7}{2^{11}} - 1 \right) = \frac{139 \cdot 3^7}{2^{11}} > 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_7 = 1$.

□

P 3.21. Let $a_1, a_2, \dots, a_n$ be real numbers so that $a_1 + a_2 + \dots + a_n = n$. If $k \geq \frac{n^2}{4(n-1)}$, then

$$(a_1^2 + k)(a_2^2 + k) \cdots (a_n^2 + k) \geq (1 + k)^n.$$

(Vasile C., 2007)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \dots + a_n}{n} = 1,$$

where

$$f(u) = \ln(u^2 + k), \quad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{2u}{u^2 + k},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty]$, where

$$s_0 = 0.$$

From

$$f''(u) = \frac{2(k - u^2)}{(u^2 + k)^2},$$

it follows that $f''(u) \geq 0$ for $u \in [0, 1]$, therefore f is convex on $[s_0, s]$. By the LPCF-Theorem and Note 2, it suffices to prove that $H(x, y) \geq 0$ for $x, y \in \mathbb{R}$ so that $x + (n-1)y = n$, where

$$H(x, y) = \frac{f'(x) - f'(y)}{x - y}.$$

We have

$$\begin{aligned} \frac{1}{2}H(x, y) &= \frac{k - xy}{(x^2 + k)(y^2 + k)} \\ &\geq \frac{n^2 - 4(n-1)xy}{4(n-1)(x^2 + k)(y^2 + k)}, \\ &= \frac{[x + (n-1)y]^2 - 4(n-1)xy}{4(n-1)(x^2 + k)(y^2 + k)} \\ &= \frac{[x - (n-1)y]^2}{4(n-1)(x^2 + k)(y^2 + k)} \geq 0. \end{aligned}$$

The equality holds for $a_1 = a_2 = \dots = a_n = 1$.

□

P 3.22. Let $a_1, a_2, \dots, a_n$ be real numbers such that $a_1 + a_2 + \dots + a_n = n$. If $n \leq 10$, then

$$(a_1^2 - a_1 + 1)(a_2^2 - a_2 + 1) \cdots (a_n^2 - a_n + 1) \geq 1.$$

(Vasile C., 2007)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1$$

where

$$f(u) = \ln(u^2 - u + 1), \quad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{2u - 1}{u^2 - u + 1},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = \frac{1}{2} < 1 = s.$$

In addition, from

$$f''(u) = \frac{1 + 2u(1 - u)}{(u^2 - u + 1)^2},$$

it follows that $f''(u) > 0$ for $u \in [s_0, 1]$, hence f is convex on $[s_0, s]$. According to LPCF-Theorem, we only need to show that

$$f(x) + (n - 1)f(y) \geq nf(1)$$

for all real x, y so that $x + (n - 1)y = n$. Write this inequality as $g(x) \geq 0$, where

$$g(x) = \ln(x^2 - x + 1) + (n - 1)\ln(y^2 - y + 1), \quad y = \frac{n - x}{n - 1}.$$

Since $y'(x) = \frac{-1}{n - 1}$, we have

$$\begin{aligned} g'(x) &= \frac{2x - 1}{x^2 - x + 1} + (n - 1)y' \frac{2y - 1}{y^2 - y + 1} = \frac{2x - 1}{x^2 - x + 1} - \frac{2y - 1}{y^2 - y + 1} \\ &= \frac{(x - y)(1 + x + y - 2xy)}{(x^2 - x + 1)(y^2 - y + 1)} = \frac{(x - 1)[2x^2 - (n + 2)x + 2n - 1]}{(n - 1)^2(x^2 - x + 1)(y^2 - y + 1)}. \end{aligned}$$

Because $2x^2 - (n + 2)x + 2n - 1 > 0$ for $n \leq 10$, we have $g'(x) \leq 0$ for $x \in (-\infty, 1]$ and $g'(x) \geq 0$ for $x \in [1, \infty)$. Therefore, since $g(x)$ is decreasing on $(-\infty, 1]$ and increasing on $[1, \infty)$, we have

$$g(x) \geq g(1) = 0.$$

The equality occurs for $a_1 = a_2 = \cdots = a_n = 1$.

Remark 1. The inequality holds also for $n = 11$, $n = 12$ and $n = 13$, when the equation

$$2x^2 - (n + 2)x + 2n - 1 = 0$$

has two positive roots, namely

$$x_1 = \frac{n + 2 - \sqrt{n^2 - 12(n - 1)}}{4}, \quad x_2 = \frac{n + 2 + \sqrt{n^2 - 12(n - 1)}}{4},$$

satisfying $1 < x_1 < x_2$. Thus, $g(x)$ is decreasing on $(-\infty, 1] \cup [x_1, x_2]$ and increasing on $[1, x_1] \cup [x_2, \infty)$. Therefore, it suffices to show that

$$\min\{g(1), g(x_2)\} \geq 0.$$

We have $g(1) = 0$. For $n = 13$, we have

$$x_2 = 5, \quad y_2 = \frac{13 - x_2}{12} = \frac{2}{3},$$

hence

$$g(x_2) = \ln(x_2^2 - x_2 + 1) + (n - 1) \ln(y_2^2 - y_2 + 1) = \ln 21 + 12 \cdot \ln \frac{7}{9} = \ln \frac{7^{13}}{3^{23}} > 0.$$

For $n = 14$, the inequality does not hold.

Remark 2. By replacing $a_1, a_2, \dots, a_n$ respectively with $1 - a_1, 1 - a_2, \dots, 1 - a_n$, we get the following statement:

- Let $a_1, a_2, \dots, a_n$ be real numbers such that $a_1 + a_2 + \dots + a_n = 0$. If $n \leq 13$, then

$$(1 - a_1 + a_1^2)(1 - a_2 + a_2^2) \cdots (1 - a_n + a_n^2) \geq 1,$$

with equality for $a_1 = a_2 = \dots = a_n = 0$.

□

P 3.23. Let $a_1, a_2, \dots, a_n$ be real numbers such that $a_1 + a_2 + \dots + a_n = n$. If $n \leq 26$, then

$$(a_1^2 - a_1 + 2)(a_2^2 - a_2 + 2) \cdots (a_n^2 - a_n + 2) \geq 2^n.$$

(Vasile C., 2007)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \dots + a_n}{n} = 1,$$

where

$$f(u) = \ln(u^2 - u + 2), \quad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{2u - 1}{u^2 - u + 2},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = \frac{1}{2} < 1 = s.$$

In addition, from

$$f''(u) = \frac{3 + 2u(1 - u)}{(u^2 - u + 2)^2},$$

it follows that $f''(u) > 0$ for $u \in [s_0, 1]$, hence f is convex on $[s_0, s]$. According to LPCF-Theorem, we only need to show that

$$f(x) + (n-1)f(y) \geq nf(1)$$

for all real x, y so that $x + (n-1)y = n$. Write this inequality as $g(x) \geq 0$, where

$$g(x) = \ln(x^2 - x + 2) + (n-1)\ln(y^2 - y + 2), \quad y = \frac{n-x}{n-1}.$$

Since $y'(x) = \frac{-1}{n-1}$, we have

$$\begin{aligned} g'(x) &= \frac{2x-1}{x^2-x+2} + (n-1)y' \frac{2y-1}{y^2-y+2} = \frac{2x-1}{x^2-x+2} - \frac{2y-1}{y^2-y+2} \\ &= \frac{(x-y)(3+x+y-2xy)}{(x^2-x+2)(y^2-y+2)} = \frac{(x-1)[2x^2-(n+2)x+4n-3]}{(n-1)^2(x^2-x+1)(y^2-y+1)}. \end{aligned}$$

Because $2x^2 - (n+2)x + 4n - 3 > 0$ for $n \leq 26$, we have $g'(x) \leq 0$ for $x \in (-\infty, 1]$ and $g'(x) \geq 0$ for $x \in [1, \infty)$. Therefore, since $g(x)$ is decreasing on $(-\infty, 1]$ and increasing on $[1, \infty)$, we have

$$g(x) \geq g(1) = 0.$$

The equality occurs for $a_1 = a_2 = \dots = a_n = 1$.

Remark 1. The inequality holds also for $27 \leq n \leq 38$, when the equation

$$2x^2 - (n+2)x + 4n - 3 = 0$$

has two positive roots, namely

$$x_1 = \frac{n+2 - \sqrt{n^2 - 28(n-1)}}{4}, \quad x_2 = \frac{n+2 + \sqrt{n^2 - 28(n-1)}}{4},$$

satisfying $1 < x_1 < x_2$. Thus, $g(x)$ is decreasing on $(-\infty, 1] \cup [x_1, x_2]$ and increasing on $[1, x_1] \cup [x_2, \infty)$. Therefore, it suffices to show that

$$\min\{g(1), g(x_2)\} \geq 0.$$

We have $g(1) = 0$ and $g(x_2) > 0$ for $27 \leq n \leq 38$. For $n = 39$, the inequality does not hold.

Remark 2. By replacing $a_1, a_2, \dots, a_n$ respectively with $1 - a_1, 1 - a_2, \dots, 1 - a_n$, we get the following statement:

- Let $a_1, a_2, \dots, a_n$ be real numbers such that $a_1 + a_2 + \dots + a_n = 0$. If $n \leq 38$, then

$$(2 - a_1 + a_1^2)(2 - a_2 + a_2^2) \cdots (2 - a_n + a_n^2) \geq 2^n,$$

with equality for $a_1 = a_2 = \dots = a_n = 0$.

□

P 3.24. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$(1 - a + a^4)(1 - b + b^4)(1 - c + c^4) \geq 1.$$

Solution. Write the inequality as

$$f(a) + f(b) + f(c) \geq 3f(s), \quad s = \frac{a + b + c}{3} = 1,$$

where

$$f(u) = \ln(1 - u + u^4), \quad u \in [0, 3].$$

From

$$f'(u) = \frac{4u^3 - 1}{1 - u + u^4},$$

it follows that f is decreasing on $[0, s_0]$ and increasing on $[s_0, 3]$, where

$$s_0 = \frac{1}{\sqrt[3]{4}} < 1 = s.$$

Also, f is convex on $[s_0, 1]$ because

$$f''(u) = \frac{-4u^6 - 4u^3 + 12u^2 - 1}{(1 - u + u^4)^2} \geq \frac{-4u^2 - 4u^2 + 12u^2 - 1}{(1 - u + u^4)^2} = \frac{4u^2 - 1}{(1 - u + u^4)^2} > 0.$$

According to the LPCF-Theorem, we only need to show that

$$f(x) + 2f(y) \geq 3f(1)$$

for all $x, y \geq 0$ so that $x + 2y = 3$. Using Note 2, it suffices to prove that $H(x, y) \geq 0$, where

$$H(x, y) = \frac{f'(x) - f'(y)}{x - y}.$$

We have

$$\begin{aligned} H(x, y) &= \frac{(x + y)(x - y)^2 - 1 + 4(x^2 + y^2 + xy) - 2xy(x + y) - 4x^3y^3}{(1 - x + x^4)(1 - y + y^4)} \\ &\geq \frac{-1 + 4(x^2 + y^2 + xy) - 2xy(x + y) - 4x^3y^3}{(1 - x + x^4)(1 - y + y^4)} \\ &= \frac{h(x, y)}{(1 - x + x^4)(1 - y + y^4)}, \end{aligned}$$

where

$$h(x, y) = -1 + 2(x + y)[2(x + y) - xy] - 4xy - 4x^3y^3.$$

From $3 = x + 2y \geq 2\sqrt{2xy}$ and $(1 - x)(1 - y) \leq 0$, we get

$$xy \leq \frac{9}{8}, \quad x + y \geq 1 + xy.$$

Therefore,

$$\begin{aligned} h(x, y) &\geq -1 + 2(1 + xy)[2(1 + xy) - xy] - 4xy - 4x^3y^3 \\ &= 3 + 2xy + 2x^2y^2 - 4x^3y^3 \geq 3 + 2xy + 2x^2y^2 - 5x^2y^2 \\ &= 3 + 2xy - 3x^2y^2 \geq 3 + 2xy - 4xy = 3 - 2xy > 0. \end{aligned}$$

The proof is completed. The equality holds for $a = b = c = 1$.

□

P 3.25. If a, b, c, d are nonnegative real numbers so that $a + b + c + d = 4$, then

$$(1 - a + a^3)(1 - b + b^3)(1 - c + c^3)(1 - d + d^3) \geq 1.$$

(Vasile C., 2012)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) + f(d) \geq 4f(s), \quad s = \frac{a + b + c + d}{4} = 1,$$

where

$$f(u) = \ln(1 - u + u^3), \quad u \in [0, 4].$$

From

$$f'(u) = \frac{3u^2 - 1}{1 - u + u^3},$$

it follows that f is decreasing on $[0, s_0]$ and increasing on $[s_0, 4]$, where

$$s_0 = \frac{1}{\sqrt{3}} < 1 = s.$$

In addition, f is convex on $[s_0, 1]$ because

$$f''(u) = \frac{-3u^4 + 6u - 1}{(1 - u + u^3)^2} \geq \frac{-3u + 6u - 1}{(1 - u + u^3)^2} = \frac{3u - 1}{(1 - u + u^3)^2} > 0.$$

According to the LPCF-Theorem, we only need to show that

$$f(x) + 3f(y) \geq 4f(1)$$

for all $x, y \geq 0$ so that $x + 3y = 4$. Using Note 2, it suffices to prove that $H(x, y) \geq 0$, where

$$H(x, y) = \frac{f'(x) - f'(y)}{x - y}.$$

We have

$$H(x, y) = \frac{(x - y)^2 + 3(x + y) - 1 - 3x^2y^2}{(1 - x + x^3)(1 - y + y^3)} \geq \frac{3(x + y) - 1 - 3x^2y^2}{(1 - x + x^3)(1 - y + y^3)}.$$

From $4 = x + 3y \geq 2\sqrt{3xy}$ and $(1-x)(1-y) \leq 0$, we get

$$xy \leq \frac{4}{3}, \quad x + y \geq 1 + xy.$$

Therefore,

$$\begin{aligned} 3(x+y) - 1 - 3x^2y^2 &\geq 3(1+xy) - 1 - 3x^2y^2 \\ &\geq 3(1+xy) - 1 - 4xy = 2 - xy > 0, \end{aligned}$$

hence $H(x, y) > 0$. The equality holds for $a = b = c = d = 1$.

□

P 3.26. If a, b, c, d, e are nonzero real numbers so that $a + b + c + d + e = 5$, then

$$5 \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{1}{e^2} \right) + 45 \geq 14 \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} \right).$$

(Vasile C., 2013)

Solution. Write the desired inequality as

$$f(a) + f(b) + f(c) + f(d) + f(e) \geq 5f(s), \quad s = \frac{a+b+c+d+e}{5} = 1,$$

where

$$f(u) = \frac{5}{u^2} - \frac{14}{u} + 9, \quad u \in \mathbb{I} = \mathbb{R} \setminus \{0\}.$$

From

$$f'(u) = \frac{2(7u-5)}{u^3},$$

it follows that f is increasing on $(-\infty, 0) \cup [s_0, \infty)$ and decreasing on $(0, s_0]$, where

$$s_0 = \frac{5}{7} < 1 = s.$$

Since

$$\lim_{u \rightarrow -\infty} f(u) = 9$$

and $f(s_0) < f(1) = 0$, we have

$$\min_{u \in \mathbb{I}} f(u) = f(s_0).$$

From

$$f''(u) = \frac{2(15-14u)}{u^4},$$

it follows that f is convex on $[s_0, 1]$. By the LPCF-Theorem, Note 4 and Note 1, it suffices to show that $h(x, y) \geq 0$ for all $x, y \in \mathbb{I}$ which satisfy $x + 4y = 5$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

Indeed, we have

$$g(u) = \frac{9}{u} - \frac{5}{u^2},$$

$$h(x, y) = \frac{5x + 5y - 9xy}{x^2y^2} = \frac{(6y - 5)^2}{x^2y^2} \geq 0.$$

In accordance with Note 3, the equality holds for $a = b = c = d = e = 1$, and also for

$$a = \frac{5}{3}, \quad b = c = d = e = \frac{5}{6}$$

(or any cyclic permutation).

□

P 3.27. If a, b, c are positive real numbers so that $abc = 1$, then

$$\frac{7 - 6a}{2 + a^2} + \frac{7 - 6b}{2 + b^2} + \frac{7 - 6c}{2 + c^2} \geq 1.$$

(Vasile C., 2008)

Solution. Using the substitution

$$a = e^x, \quad b = e^y, \quad c = e^z,$$

we need to show that

$$f(x) + f(y) + f(z) \geq 3f(s),$$

where

$$s = \frac{x + y + z}{3} = 0$$

and

$$f(u) = \frac{7 - 6e^u}{2 + e^{2u}}, \quad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{2(3e^u + 2)(e^u - 3)}{(2 + e^{2u})^2},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = \ln 3 > s.$$

We have

$$f''(u) = \frac{2t \cdot h(t)}{(2 + t^2)^3}, \quad h(t) = -3t^4 + 14t^3 + 36t^2 - 28t - 12, \quad t = e^u.$$

We will show that $h(t) > 0$ for $t \in [1, 3]$, hence f is convex on $[0, s_0]$. We have

$$\begin{aligned} h(t) &= 3(t^2 - 1)(9 - t^2) + 14t^3 + 6t^2 - 28t + 15 \\ &\geq 14t^3 + 6t^2 - 28t + 15 \\ &= 14t^2(t - 1) + 14(t - 1)^2 + 6t^2 + 1 > 0. \end{aligned}$$

By the RPCF-Theorem, we only need to prove that

$$f(x) + 2f(y) \geq 3f(0)$$

for all real x, y so that $x + 2y = 0$. That is, to show that the original inequality holds for $b = c$ and $a = 1/c^2$. Write this inequality as

$$\frac{c^2(7c^2 - 6)}{2c^4 + 1} + \frac{2(7 - 6c)}{2 + c^2} \geq 1,$$

$$(c - 1)^2(c - 2)^2(5c^2 + 6c + 3) \geq 0.$$

By Note 3, the equality holds for $a = b = c = 1$, and also for

$$a = \frac{1}{4}, \quad b = c = 2$$

(or any cyclic permutation).

□

P 3.28. If a, b, c are positive real numbers so that $abc = 1$, then

$$\frac{1}{a + 5bc} + \frac{1}{b + 5ca} + \frac{1}{c + 5ab} \leq \frac{1}{2}.$$

(Vasile C., 2008)

Solution. Write the inequality as

$$\frac{a}{a^2 + 5} + \frac{b}{b^2 + 5} + \frac{c}{c^2 + 5} \leq \frac{1}{2}.$$

Using the substitution

$$a = e^x, \quad b = e^y, \quad c = e^z,$$

we need to show that

$$f(x) + f(y) + f(z) \geq 3f(s),$$

where

$$s = \frac{x + y + z}{3} = 0$$

and

$$f(u) = \frac{-e^u}{e^{2u} + 5}, \quad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{e^u(e^{2u} - 5)}{(e^{2u} + 5)^2},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = \frac{\ln 5}{2} > 0 = s.$$

Also, from

$$f''(u) = \frac{e^u(-e^{4u} + 30e^{2u} - 25)}{(e^{2u} + 5)^3},$$

it follows that f is convex on $[s, s_0]$, because $u \in [0, s_0]$ involves $e^u \in [1, \sqrt{5}]$ and $e^{2u} \in [1, 5]$, hence

$$-e^{4u} + 30e^{2u} - 25 = e^{2u}(5 - e^{2u}) + 25(e^{2u} - 1) > 0.$$

By the RPCF-Theorem, we only need to prove the original inequality for $b = c$ and $a = 1/c^2$. Write this inequality as

$$\begin{aligned} \frac{c^2}{5c^4 + 1} + \frac{2c}{c^2 + 5} &\leq \frac{1}{2}, \\ (c - 1)^2(5c^4 - 10c^3 - 2c^2 + 6c + 5) &\geq 0, \\ (c - 1)^2[5(c - 1)^4 + 2c(5c^2 - 16c + 13)] &\geq 0. \end{aligned}$$

The equality holds for $a = b = c = 1$.

□

P 3.29. If a, b, c are positive real numbers so that $abc = 1$, then

$$\frac{1}{4 - 3a + 4a^2} + \frac{1}{4 - 3b + 4b^2} + \frac{1}{4 - 3c + 4c^2} \leq \frac{3}{5}.$$

(Vasile Cîrtoaje, 2008)

Solution. Let

$$a = e^x, \quad b = e^y, \quad c = e^z.$$

We need to show that

$$f(x) + f(y) + f(z) \geq 3f(s),$$

where

$$s = \frac{x + y + z}{3} = 0$$

and

$$f(u) = \frac{-1}{4 - 3e^u + 4e^{2u}}, \quad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{e^u(8e^u - 3)}{(4 - 3e^u + 4e^{2u})^2},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = \ln \frac{3}{8} < 0 = s.$$

We claim that f is convex on $[s_0, 0]$. Since

$$f''(u) = \frac{e^u(-64e^{3u} + 36e^{2u} + 55e^u - 12)}{(4 - 3e^u + 4e^{2u})^3},$$

we need to show that

$$-64t^3 + 36t^2 + 55t - 12 \geq 0,$$

where

$$t = e^u \in \left[\frac{3}{8}, 1 \right].$$

Indeed, we have

$$\begin{aligned} -64t^3 + 36t^2 + 55t - 12 &> -72t^3 + 36t^2 + 48t - 12 \\ &= 12(1-t)(6t^2 + 3t - 1) \geq 0. \end{aligned}$$

By the LPCF-Theorem, we only need to prove the original inequality for $b = c$ and $a = 1/c^2$. Write this inequality as follows:

$$\begin{aligned} \frac{c^4}{4c^4 - 3c^2 + 4} + \frac{2}{4 - 3c + 4c^2} &\leq \frac{3}{5}, \\ 28c^6 - 21c^5 - 48c^4 + 27c^3 + 42c^2 - 36c + 8 &\geq 0, \\ (c-1)^2(28c^4 + 35c^3 - 6c^2 - 20c + 8) &\geq 0. \end{aligned}$$

It suffices to show that

$$7(4c^4 + 5c^3 - c^2 - 3c + 1) \geq 0.$$

Indeed,

$$4c^4 + 5c^3 - c^2 - 3c + 1 = c^2(2c-1)^2 + 9c^3 - 2c^2 - 3c + 1$$

and

$$9c^3 - 2c^2 - 3c + 1 = c(3c-1)^2 + (2c-1)^2 > 0.$$

The equality holds for $a = b = c = 1$.

Remark. Since

$$\frac{1}{4-3a+4a^2} \geq \frac{1}{4-3a+4a^2+(1-a)^2} = \frac{1}{5(1-a+a^2)},$$

we get the following known inequality

$$\frac{1}{1-a+a^2} + \frac{1}{1-b+b^2} + \frac{1}{1-c+c^2} \leq 3.$$

□

P 3.30. If a, b, c are positive real numbers so that $abc = 1$, then

$$\frac{1}{(3a+1)(3a^2-5a+3)} + \frac{1}{(3b+1)(3b^2-5b+3)} + \frac{1}{(3c+1)(3c^2-5c+3)} \leq \frac{3}{4}.$$

Solution. Let

$$a = e^x, \quad b = e^y, \quad c = e^z.$$

We need to show that

$$f(x) + f(y) + f(z) \geq 3f(s),$$

where

$$s = \frac{x + y + z}{3} = 0$$

and

$$f(u) = \frac{-1}{(3e^u + 1)(3e^{2u} - 5e^u + 3)}, \quad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{(3e^u - 2)(9e^u - 2)}{(3e^u + 1)^2(3e^{2u} - 5e^u + 3)^2},$$

it follows that f is increasing on $(-\infty, s_1] \cup [s_0, \infty)$ and decreasing on $[s_1, s_0]$, where

$$s_1 = \ln 2 - \ln 9, \quad s_0 = \ln 2 - \ln 3, \quad s_1 < s_0 < 0 = s.$$

Since

$$\lim_{u \rightarrow -\infty} f(u) = f(s_0) = \frac{-1}{3},$$

we get

$$\min_{u \in \mathbb{R}} f(u) = f(s_0).$$

We claim that f is convex on $[s_0, 0]$. We have

$$f''(u) = \frac{t \cdot h(t)}{(3t + 1)^3(3t^2 - 5t + 3)^3},$$

where

$$t = e^u \in \left[\frac{2}{3}, 1\right], \quad h(t) = -729t^5 + 1188t^4 - 648t^3 + 387t^2 - 160t + 12.$$

For $t \in [2/3, 1]$, we have $h(t) > 0$, hence f is convex on $[s_0, 0]$. Indeed,

$$h(t) > -729t^5 + 1188t^4 - 648t^3 + 387t^2 - 160t - 12 = (3t - 2)h_1(t),$$

$$\begin{aligned} h_1(t) &= -234t^4 + 234t^3 - 60t^2 + 89t + 6 > t(-243t^3 + 234t^2 - 60t + 89) \\ &> 3t(-81t^3 + 78t^2 - 20t + 23) = 3t(1 - t)(23 + 3t + 81t^2) \geq 0. \end{aligned}$$

By the LPCF-Theorem, we only need to prove the original inequality for $b = c \leq 1$ and $a = 1/c^2$. Write this inequality as follows:

$$\frac{c^6}{(c^2 + 3)(3c^4 - 5c^2 + 3)} + \frac{2}{(3c + 1)(3c^2 - 5c + 3)} \leq \frac{3}{4}.$$

Since

$$c^2 + 3 \geq 2(c + 1)$$

and

$$3c^4 - 5c^2 + 3 \geq c(3c^2 - 5c + 3),$$

it suffices to prove that

$$\frac{c^5}{2(c+1)(3c^2-5c+3)} + \frac{2}{(3c+1)(3c^2-5c+3)} \leq \frac{3}{4}.$$

This is equivalent to the obvious inequality

$$(1-c)^2(1+15c+5c^2-14c^3-6c^4) \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 3.31. Let $a_1, a_2, \dots, a_n$ ($n \geq 3$) be positive real numbers so that $a_1 a_2 \cdots a_n = 1$. If $p, q \geq 0$ so that $p + 4q \geq n - 1$, then

$$\frac{1-a_1}{1+pa_1+qa_1^2} + \frac{1-a_2}{1+pa_2+qa_2^2} + \cdots + \frac{1-a_n}{1+pa_n+qa_n^2} \geq 0.$$

(Vasile C., 2008)

Solution. For $q = 0$, we get a known inequality (see Remark 2 from the proof of P 1.63). Consider further that $q > 0$. Using the substitutions $a_i = e^{x_i}$ for $i = 1, 2, \dots, n$, we need to show that

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s),$$

where

$$s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0$$

and

$$f(u) = \frac{1 - e^u}{1 + pe^u + qe^{2u}}, \quad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{e^u(qe^{2u} - 2qe^u - p - 1)}{(1 + pe^u + qe^{2u})^2},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = \ln r_0 > 0 = s, \quad r_0 = 1 + \sqrt{1 + \frac{p+1}{q}}.$$

Also, we have

$$f''(u) = \frac{t \cdot h(t)}{(1 + pt + qt^2)^3},$$

where

$$h(t) = -q^2 t^4 + q(p+4q)t^3 + 3q(p+2)t^2 + (p-4q+p^2)t - p - 1, \quad t = e^u.$$

We will show that $h(t) \geq 0$ for $t \in [1, r_0]$, hence f is convex on $[0, s_0]$. We have

$$h'(t) = -4q^2 t^3 + 3q(p+4q)t^2 + 6q(p+2)t + p - 4q + p^2,$$

$$h''(t) = 6q[-2qt^2 + (p + 4q)t + p + 2].$$

Since

$$h''(t) = 6q[2(-qt^2 + 2qt + p + 1) + p(t - 1)] \geq 12q(-qt^2 + 2qt + p + 1) \geq 0,$$

$h'(t)$ is increasing,

$$h'(t) \geq h'(1) = p^2 + 9pq + 8q^2 + p + 8q > 0,$$

h is increasing, hence

$$\begin{aligned} h(t) &\geq h(1) = p^2 + 4pq + 3q^2 + 2q - 1 = (p + 2q)^2 - (q - 1)^2 \\ &= (p + q + 1)(p + 3q - 1). \end{aligned}$$

Since

$$p + 3q - 1 \geq p + 3q - \frac{p + 4q}{n - 1} = \frac{p + 2q}{2} > 0,$$

$f''(u) > 0$ for $u \in [0, s_0]$, therefore f is convex on $[s, s_0]$. By the RPCF-Theorem, we only need to prove the original inequality for

$$a_2 = \cdots = a_n := t, \quad a_1 = 1/t^{n-1}, \quad t \geq 1.$$

Write this inequality as

$$\frac{t^{n-1}(t^{n-1} - 1)}{t^{2n-2} + pt^{n-1} + q} + \frac{(n-1)(1-t)}{1 + pt + qt^2} \geq 0,$$

or

$$pA + qB \geq C,$$

where

$$\begin{aligned} A &= t^{n-1}(t^n - nt + n - 1), \\ B &= t^{2n} - t^{n+1} - (n-1)(t-1), \\ C &= t^{n-1}[(n-1)t^n + 1 - nt^{n-1}]. \end{aligned}$$

Since $p + 4q \geq n - 1$ and $C \geq 0$ (by the AM-GM inequality applied to n positive numbers), it suffices to show that

$$pA + qB \geq \frac{(p + 4q)C}{n - 1},$$

which is equivalent to

$$p[(n-1)A - C] + q[(n-1)B - 4C] \geq 0.$$

This is true if

$$(n-1)A - C \geq 0$$

and

$$(n-1)B - 4C \geq 0$$

for $t \geq 1$. By the AM-GM inequality, we have

$$(n-1)A - C = nt^{n-1}[t^{n-1} + n - 2 - (n-1)t] \geq 0.$$

For $n = 3$, we have

$$\begin{aligned} B &= (t-1)^2(t^4 + 2t^3 + 2t^2 + 2t + 2), \\ C &= t^2(t-1)^2(2t+1), \end{aligned}$$

$$\begin{aligned} B - 2C &= (t-1)^2(t^4 - 2t^3 + 2t + 2) \\ &= (t-1)^2[(t-1)^2(t^2 - 1) + 3] \geq 0. \end{aligned}$$

Consider further that

$$n \geq 4.$$

Since

$$t - 1 \leq t^{n-1}(t - 1),$$

we have

$$\begin{aligned} B &\geq t^{2n} - t^{n+1} - (n-1)t^{n-1}(t-1) \\ &= t^{n-1}[t^{n+1} - t^2 - (n-1)t + n-1]. \end{aligned}$$

Thus, the inequality $(n-1)B - 4C \geq 0$ is true if

$$(n-1)[t^{n+1} - t^2 - (n-1)t + n-1] - 4(n-1)t^n - 4 - 4nt^{n-1} \geq 0,$$

which is equivalent to $g(t) \geq 0$, where

$$g(t) = (n-1)t^{n+1} - 4(n-1)t^n + 4nt^{n-1} - (n-1)t^2 - (n-1)^2t + n^2 - 2n - 3.$$

We have

$$g'(t) = (n-1)g_1(t), \quad g_1(t) = (n+1)t^n - 4nt^{n-1} + 4nt^{n-2} - 2t - n + 1,$$

$$g'_1(t) = n(n+1)t^{n-1} - 4n(n-1)t^{n-2} + 4n(n-2)t^{n-3} - 2.$$

Since

$$n(n+1)t^{n-1} + 4n(n-2)t^{n-3} \geq 4n\sqrt{(n+1)(n-2)}t^{n-2},$$

we get

$$\begin{aligned} g'_1(t) &\geq 4n \left[\sqrt{(n+1)(n-2)} - n + 1 \right] t^{n-2} - 2 \\ &\geq 4n \left[\sqrt{(n+1)(n-2)} - n + 1 \right] - 2 \\ &= \frac{4n(n-3)}{\sqrt{(n+1)(n-2)} + n - 1} - 2 \\ &> \frac{4n(n-3)}{(n+1) + n - 1} - 2 = 2(n-4) \geq 0. \end{aligned}$$

Therefore, $g_1(t)$ is increasing for $t \geq 1$, $g_1(t) \geq g_1(1) = 0$, $g(t)$ is increasing for $t \geq 1$, hence

$$g(t) \geq g(1) = 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Remark. For $p = 0$ and $q = 1$, we get the inequality (Vasile C., 2006)

$$\frac{1-a}{1+a^2} + \frac{1-b}{1+b^2} + \frac{1-c}{1+c^2} + \frac{1-d}{1+d^2} + \frac{1-e}{1+e^2} \geq 0,$$

where a, b, c, d, e are positive real numbers so that $abcde = 1$. Replacing a, b, c, d, e by $1/a, 1/b, 1/c, 1/d, 1/e$, we get

$$\frac{1+a}{1+a^2} + \frac{1+b}{1+b^2} + \frac{1+c}{1+c^2} + \frac{1+d}{1+d^2} + \frac{1+e}{1+e^2} \leq 5,$$

where a, b, c, d, e are positive real numbers so that $abcde = 1$.

Notice that the inequality

$$\frac{1-a_1}{1+a_1^2} + \frac{1-a_2}{1+a_2^2} + \frac{1-a_3}{1+a_3^2} + \frac{1-a_4}{1+a_4^2} + \frac{1-a_5}{1+a_5^2} + \frac{1-a_6}{1+a_6^2} \geq 0$$

is not true for all positive numbers $a_1, a_2, a_3, a_4, a_5, a_6$ satisfying $a_1 a_2 a_3 a_4 a_5 a_6 = 1$. Indeed, for $a_2 = a_3 = a_4 = a_5 = a_6 = 2$, the inequality becomes

$$\frac{1-a_1}{1+a_1^2} - 1 \geq 0,$$

which is false for $a_1 > 0$.

□

P 3.32. If a, b, c are positive real numbers so that $abc = 1$, then

$$\frac{1-a}{17+4a+6a^2} + \frac{1-b}{17+4b+6b^2} + \frac{1-c}{17+4c+6c^2} \geq 0.$$

(Vasile C., 2008)

Solution. Using the substitution

$$a = e^x, \quad b = e^y, \quad c = e^z,$$

we need to show that

$$f(x) + g(y) + g(z) \geq 3f(s),$$

where

$$s = \frac{x+y+z}{3} = 0$$

and

$$f(u) = \frac{1-e^u}{1+pe^u+qe^{2u}}, \quad u \in \mathbb{R},$$

with

$$p = \frac{4}{17}, \quad q = \frac{6}{17}.$$

As we have shown in the proof of the preceding P 3.31, f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = \ln r_0 > 0 = s, \quad r_0 = 1 + \sqrt{1 + \frac{p+1}{q}} = 1 + \sqrt{\frac{9}{2}}.$$

In addition, since $p + 3q - 1 = \frac{5}{17} > 0$ (see the proof of P 3.31), f is convex on $[0, s_0]$. By the RPCF-Theorem, we only need to prove the original inequality for $b = c \geq 1$ and $a = 1/c^2$. Write this inequality as follows:

$$\frac{c^2(c^2 - 1)}{c^4 + pc^2 + q} + \frac{2(1 - c)}{1 + pc + qc^2} \geq 0,$$

$$pA + qB \geq C,$$

where

$$A = c^2(c - 1)^2(c + 2),$$

$$B = (c - 1)^2(c^4 + 2c^3 + 2c^2 + 2c + 2),$$

$$C = c^2(c - 1)^2(2c + 1).$$

Indeed, we have

$$pA + qB - C = \frac{3(c - 1)^2(c - 2)^2(2c^2 + 2c + 1)}{17} \geq 0.$$

In accordance with Note 3, the equality holds for $a = b = c = 1$, and also for

$$a = \frac{1}{4}, \quad b = c = 2$$

(or any cyclic permutation).

□

P 3.33. If $a_1, a_2, \dots, a_8$ are positive real numbers so that $a_1 a_2 \cdots a_8 = 1$, then

$$\frac{1 - a_1}{(1 + a_1)^2} + \frac{1 - a_2}{(1 + a_2)^2} + \cdots + \frac{1 - a_8}{(1 + a_8)^2} \geq 0.$$

(Vasile C., 2006)

Solution. Using the substitutions $a_i = e^{x_i}$ for $i = 1, 2, \dots, 8$, we need to show that

$$f(x_1) + f(x_2) + \cdots + f(x_8) \geq 8f(s),$$

where

$$s = \frac{x_1 + x_2 + \cdots + x_8}{8} = 0$$

and

$$f(u) = \frac{1 - e^u}{(1 + e^u)^2}, \quad u \in \mathbb{R}.$$

From

$$f'(t) = \frac{e^u(e^u - 3)}{(1 + e^u)^3},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = \ln 3 > 1 = s.$$

We have

$$f''(u) = \frac{e^u(8e^u - e^{2u} - 3)}{(1 + e^u)^4}.$$

For $u \in [0, \ln 3]$, that is $e^u \in [1, 3]$, we have

$$8e^u - e^{2u} - 3 > 8e^u - 3e^u - 7 = (e^u - 1)(7 - e^u) \geq 0;$$

therefore, f is convex on $[s, s_0]$. By the RPCF-Theorem, we only need to prove the original inequality for $a_2 = \dots = a_8 := t$ and $a_1 = 1/t^7$, where $t \geq 1$. For the nontrivial case $t > 1$, write this inequality as follows:

$$\begin{aligned} \frac{t^7(t^7 - 1)}{(t^7 + 1)^2} &\geq \frac{7(t - 1)}{(t + 1)^2}, \\ \frac{t^7(t^7 - 1)(t + 1)^2}{(t - 1)(t^7 + 1)^2} &\geq 7, \\ \frac{t^7(t^6 + t^5 + t^4 + t^3 + t^2 + t + 1)}{(t^6 - t^5 + t^4 - t^3 + t^2 - t + 1)^2} &\geq 7. \end{aligned}$$

Since

$$t^6 - t^5 + t^4 - t^3 + t^2 - t + 1 = t^4(t^2 - t + 1) - (t - 1)(t^2 + 1) < t^4(t^2 - t + 1),$$

it suffices to show that

$$\frac{t^6 + t^5 + t^4 + t^3 + t^2 + t + 1}{t(t^2 - t + 1)^2} \geq 7,$$

which is equivalent to the obvious inequality

$$(t - 1)^6 \geq 0.$$

Thus, the proof is completed. The equality holds for $a_1 = a_2 = \dots = a_8 = 1$.

Remark. The inequality

$$\frac{1 - a_1}{(1 + a_1)^2} + \frac{1 - a_2}{(1 + a_2)^2} + \dots + \frac{1 - a_9}{(1 + a_9)^2} \geq 0$$

is not true for all positive numbers $a_1, a_2, \dots, a_9$ satisfying $a_1 a_2 \dots a_9 = 1$. Indeed, for $a_2 = a_3 = \dots = a_9 = 3$, the inequality becomes

$$\frac{1 - a_1}{(1 + a_1)^2} - 1 \geq 0,$$

which is false for $a_1 > 0$.

□

P 3.34. Let a, b, c be positive real numbers so that $abc = 1$. If $k \in \left[\frac{-13}{3\sqrt{3}}, \frac{13}{3\sqrt{3}} \right]$, then

$$\frac{a + k}{a^2 + 1} + \frac{b + k}{b^2 + 1} + \frac{c + k}{c^2 + 1} \leq \frac{3(1 + k)}{2}.$$

(Vasile C., 2012)

Solution. The inequality is equivalent to

$$\begin{aligned} k \left(\sum \frac{1}{a^2 + 1} - \frac{3}{2} \right) &\leq \sum \left(\frac{1}{2} - \frac{a}{a^2 + 1} \right), \\ \sum \frac{(a-1)^2}{a^2 + 1} &\geq k \left(\sum \frac{2}{a^2 + 1} - 3 \right). \end{aligned} \quad (*)$$

Thus, it suffices to prove it for $|k| = \frac{13}{3\sqrt{3}}$. On the other hand, replacing a, b, c by $1/a, 1/b, 1/c$, the inequality becomes

$$\sum \frac{(a-1)^2}{a^2 + 1} \geq k \left(3 - \sum \frac{2}{a^2 + 1} \right). \quad (**)$$

Based on (*) and (**), we only need to prove the desired inequality for

$$k = \frac{13}{3\sqrt{3}}.$$

Using the substitution

$$a = e^x, \quad b = e^y, \quad c = e^z,$$

we need to show that

$$f(x) + g(y) + g(z) \geq 3f(s),$$

where

$$s = \frac{x + y + z}{3} = 0$$

and

$$f(u) = \frac{-e^u - k}{e^{2u} + 1}, \quad u \in \mathbb{R}.$$

From

$$f'(t) = \frac{e^{2u} + 2ke^u - 1}{(e^{2u} + 1)^2},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = \ln r_0 < 0 = s, \quad r_0 = -k + \sqrt{k^2 + 1} = \frac{1}{3\sqrt{3}}.$$

Also, we have

$$f''(u) = \frac{t \cdot h(t)}{(1 + t^2)^3},$$

where

$$h(t) = -t^4 - 4kt^3 + 6t^2 + 4kt - 1, \quad t = e^u.$$

We will show that $h(t) > 0$ for $t \in [r_0, 1]$, hence f is convex on $[s_0, s]$. Indeed, since

$$4kt = \frac{52t}{3\sqrt{3}} \geq \frac{52}{27} > 1,$$

we have

$$h(t) = -t^4 + 6t^2 - 1 + 4kt(1 - t^2) \geq -t^4 + 6t^2 - 1 + (1 - t^2) = t^2(5 - t^2) > 0.$$

By the LPCF-Theorem, we only need to prove the original inequality for $b = c := t$ and $a = 1/t^2$, where $t > 0$. Write this inequality as

$$\begin{aligned} \frac{t^2(kt^2 + 1)}{t^4 + 1} + \frac{2(t + k)}{t^2 + 1} &\leq \frac{3(1 + k)}{2}, \\ 3t^6 - 4t^5 + t^4 + t^2 - 4t + 3 - k(1 - t^2)^3 &\geq 0, \\ (t - 1)^2[(3 + k)t^4 + 2(1 + k)t^3 + 2t^2 + 2(1 - k)t + 3 - k] &\geq 0, \\ (t - 1)^2(t - 2 + \sqrt{3})^2 \left[(27 + 13\sqrt{3})t^2 + 24(2 + \sqrt{3})t + 33 + 17\sqrt{3} \right] &\geq 0. \end{aligned}$$

The equality holds for $a = b = c = 1$. If $k = \frac{13}{3\sqrt{3}}$, then the equality holds also for

$$a = 7 + 4\sqrt{3}, \quad b = c = 2 - \sqrt{3}$$

(or any cyclic permutation). If $k = \frac{-13}{3\sqrt{3}}$, then the equality holds also for

$$a = 7 - 4\sqrt{3}, \quad b = c = 2 + \sqrt{3}$$

(or any cyclic permutation).

□

P 3.35. If a, b, c are positive real numbers and $0 < k \leq 2 + 2\sqrt{2}$, then

$$\frac{a^3}{ka^2 + bc} + \frac{b^3}{kb^2 + ca} + \frac{c^3}{kc^2 + ab} \geq \frac{a + b + c}{k + 1}.$$

(Vasile C., 2011)

Solution. Due to homogeneity, we may assume that $abc = 1$. On this hypothesis, we write the inequality as follows:

$$\begin{aligned} \frac{a^4}{ka^3 + 1} + \frac{b^4}{kb^3 + 1} + \frac{b^4}{kb^3 + 1} &\geq \frac{a}{k + 1} + \frac{b}{k + 1} + \frac{c}{k + 1}, \\ \frac{a^4 - a}{ka^3 + 1} + \frac{b^4 - b}{kb^3 + 1} + \frac{c^4 - c}{kc^3 + 1} &\geq 0. \end{aligned}$$

Using the substitution

$$a = e^x, \quad b = e^y, \quad c = e^z,$$

we need to show that

$$f(x) + g(y) + g(z) \geq 3f(s),$$

where

$$s = \frac{x + y + z}{3} = 0$$

and

$$f(u) = \frac{e^{4u} - e^u}{ke^{3u} + 1}, \quad u \in \mathbb{R}.$$

From

$$f'(t) = \frac{ke^{6u} + 2(k+2)e^{3u} - 1}{(ke^{3u} + 1)^2},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = \ln r_0 < 0, \quad r_0 = \sqrt[3]{\frac{-k-2 + \sqrt{(k+1)(k+4)}}{k}} \in (0, 1).$$

Also, we have

$$f''(u) = \frac{t \cdot h(t)}{(kt^3 + 1)^3},$$

where

$$h(t) = k^2t^9 - k(4k+1)t^6 + (13k+16)t^3 - 1, \quad t = e^u.$$

If $h(t) > 0$ for $t \in [r_0, 1]$, then f is convex on $[s_0, 0]$. We will prove this only for $k = 2 + 2\sqrt{2}$, when $r_0 \approx 0.415$ and $h(t) \geq 0$ for $t \in [t_1, t_2]$, where $t_1 \approx 0.2345$ and $t_2 \approx 1.02$. Since $[r_0, 1] \subset [t_1, t_2]$, the conclusion follows. By the LPCF-Theorem, we only need to prove the original inequality for $b = c$. Due to homogeneity, we may consider that $b = c = 1$. Thus, we need to show that

$$\frac{a^3}{ka^2 + 1} + \frac{2}{a + k} \geq \frac{a + 2}{k + 1},$$

which is equivalent to the obvious inequality

$$(a - 1)^2[a^2 - (k - 2)a + 2] \geq 0.$$

For $k = 2 + 2\sqrt{2}$, this inequality has the form

$$(a - 1)^2(a - \sqrt{2})^2 \geq 0.$$

The equality holds for $a = b = c$. If $k = 2 + 2\sqrt{2}$, then the equality holds also for

$$\frac{a}{\sqrt{2}} = b = c$$

(or any cyclic permutation).

□

P 3.36. If a, b, c, d, e are positive real numbers so that $abcde = 1$, then

$$2 \left(\frac{1}{a+1} + \frac{1}{b+1} + \cdots + \frac{1}{e+1} \right) \geq 3 \left(\frac{1}{a+2} + \frac{1}{b+2} + \cdots + \frac{1}{e+2} \right).$$

(Vasile C., 2012)

Solution. Write the inequality as

$$\frac{1-a}{(a+1)(a+2)} + \frac{1-b}{(b+1)(b+2)} + \frac{1-c}{(c+1)(c+2)} + \frac{1-d}{(d+1)(d+2)} + \frac{1-e}{(e+1)(e+2)} \geq 0.$$

Using the substitution

$$a = e^x, \quad b = e^y, \quad c = e^z, \quad d = e^t, \quad e = e^w,$$

we need to show that

$$f(x) + f(y) + f(z) + f(t) + f(w) \geq 5f(s),$$

where

$$s = \frac{x+y+z+t+w}{5} = 0$$

and

$$f(u) = \frac{1-e^u}{(e^u+1)(e^u+2)}, \quad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{e^u(e^{2u} - 2e^u - 5)}{(e^u+1)^2(e^u+2)^2},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = \ln(1 + \sqrt{6}) < 2, \quad s < s_0.$$

Also, we have

$$f''(u) = \frac{t \cdot h(t)}{(t+1)^3(t+2)^3}, \quad t = e^u,$$

where

$$h(t) = -t^4 + 7t^3 + 21t^2 + 7t - 10.$$

We will show that $h(t) > 0$ for $t \in [1, 2]$, hence f is convex on $[0, s_0]$. We have

$$h(t) \geq -2t^3 + 7t^3 + 21t^2 + 7t - 10 = 5t^3 + 21t^2 + 7t - 10 > 0.$$

By the RPCF-Theorem, we only need to prove the original inequality for

$$b = c = d = e := t, \quad a = 1/t^4, \quad t \geq 1.$$

Write this inequality as

$$\frac{t^4(t^4-1)}{(t^4+1)(2t^4+1)} \geq \frac{4(t-1)}{(t+1)(t+2)},$$

which is true if

$$t^4(t+1)(t+2)(t^3+t^2+t+1) \geq 4(t^4+1)(2t^4+1).$$

Since

$$(t^4+1)(2t^4+1) = 2t^8 + 3t^4 + 1 \leq 2t^4(t^4+2),$$

it suffices to show that

$$(t+1)(t+2)(t^3+t^2+t+1) \geq 8(t^4+2).$$

This inequality is equivalent to

$$\begin{aligned} t^5 - 4t^4 + 6t^3 + 6t^2 + 5t - 14 &\geq 0, \\ t(t-1)^4 + 10(t^2-1) + 4(t-1) &\geq 0. \end{aligned}$$

The equality holds for $a = b = c = d = e = 1$.

□

P 3.37. If $a_1, a_2, \dots, a_{14}$ are positive real numbers so that $a_1 a_2 \cdots a_{14} = 1$, then

$$3 \left(\frac{1}{2a_1 + 1} + \frac{1}{2a_2 + 1} + \cdots + \frac{1}{2a_{14} + 1} \right) \geq 2 \left(\frac{1}{a_1 + 1} + \frac{1}{a_2 + 1} + \cdots + \frac{1}{a_{14} + 1} \right).$$

(Vasile C., 2012)

Solution. Write the inequality as

$$\frac{1 - a_1}{(a_1 + 1)(2a_1 + 1)} + \frac{1 - a_2}{(a_2 + 1)(2a_2 + 1)} + \cdots + \frac{1 - a_{14}}{(a_{14} + 1)(2a_{14} + 1)} \geq 0.$$

Using the substitutions $a_i = e^{x_i}$ for $i = 1, 2, \dots, 14$, we need to show that

$$f(x_1) + f(x_2) + \cdots + f(x_{14}) \geq 14f(s),$$

where

$$s = \frac{x_1 + x_2 + \cdots + x_{14}}{14} = 0$$

and

$$f(u) = \frac{1 - e^u}{(e^u + 1)(2e^u + 1)}, \quad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{2e^u(e^{2u} - 2e^u - 2)}{(e^u + 1)^2(2e^u + 1)^2},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = \ln(1 + \sqrt{3}) < 2, \quad s < s_0.$$

Also, we have

$$f''(u) = \frac{2t \cdot h(t)}{(t + 1)^3(2t + 1)^3}, \quad t = e^u,$$

where

$$h(t) = -2t^4 + 11t^3 + 15t^2 + 2t - 2.$$

We will show that $h(t) > 0$ for $t \in [1, 2]$, hence f is convex on $[0, s_0]$. We have

$$h(t) \geq -4t^3 + 11t^3 + 15t^2 + 2t - 2 = 7t^3 + 15t^2 + 2t - 2 > 0.$$

By the RPCF-Theorem, we only need to prove the original inequality for

$$a_2 = a_3 = \cdots = a_{14} := t, \quad a_1 = 1/t^{13}, \quad t \geq 1.$$

Write this inequality as

$$\frac{t^{13}(t^{13} - 1)}{(t^{13} + 1)(t^{13} + 2)} \geq \frac{13(t - 1)}{(t + 1)(2t + 1)}.$$

Since

$$(t^{13} + 1)(t^{13} + 2) = t^{26} + 3t^{13} + 2 \leq t^{13}(t^{13} + 5),$$

it suffices to show that

$$\frac{t^{13} - 1}{t^{13} + 5} \geq \frac{13(t - 1)}{(t + 1)(2t + 1)},$$

which is equivalent to

$$t^{13}(t^2 - 5t + 7) - t^2 - 34t + 32 \geq 0.$$

Substituting

$$t = 1 + x, \quad x \geq 0,$$

the inequality becomes

$$(1 + x)^{13}(x^2 - 3x + 3) - x^2 - 36x - 3 \geq 0.$$

Since

$$(1 + x)^{13} \geq 1 + 13x + 78x^2,$$

it suffices to show that

$$(78x^2 + 13x + 1)(x^2 - 3x + 3) - x^2 - 36x - 3 \geq 0.$$

This inequality, equivalent to

$$x^2(78x^2 - 221x + 196) \geq 0,$$

is true since

$$78x^2 - 221x + 196 \geq 64x^2 - 224x + 196 = 4(4x - 7)^2 \geq 0.$$

The equality holds for $a_1 = a_2 = \dots = a_{14} = 1$.

□

P 3.38. Let $a_1, a_2, \dots, a_8$ be positive real numbers so that $a_1 a_2 \dots a_8 = 1$. If $k > 1$, then

$$(k + 1) \left(\frac{1}{ka_1 + 1} + \frac{1}{ka_2 + 1} + \dots + \frac{1}{ka_8 + 1} \right) \geq 2 \left(\frac{1}{a_1 + 1} + \frac{1}{a_2 + 1} + \dots + \frac{1}{a_8 + 1} \right).$$

(Vasile C., 2012)

Solution. Write the inequality as

$$\frac{1 - a_1}{(a_1 + 1)(ka_1 + 1)} + \frac{1 - a_2}{(a_2 + 1)(ka_2 + 1)} + \dots + \frac{1 - a_8}{(a_8 + 1)(ka_8 + 1)} \geq 0.$$

Using the substitutions $a_i = e^{x_i}$ for $i = 1, 2, \dots, 8$, we need to show that

$$f(x_1) + f(x_2) + \dots + f(x_8) \geq 8f(s),$$

where

$$s = \frac{x_1 + x_2 + \cdots + x_8}{8} = 0$$

and

$$f(u) = \frac{1 - e^u}{(e^u + 1)(ke^u + 1)}, \quad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{e^u(ke^{2u} - 2ke^u - k - 2)}{(e^u + 1)^2(ke^u + 1)^2},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = \ln \left(1 + \sqrt{2 + \frac{2}{k}} \right) < 2, \quad s < s_0.$$

Also, we have

$$f''(u) = \frac{t \cdot h(t)}{(t+1)^3(kt+1)^3}, \quad t = e^u,$$

where

$$h(t) = -k^2t^4 + k(5k+1)t^3 + 3k(k+3)t^2 + (k^2 - k + 2)t - k - 2.$$

We will show that $h(t) > 0$ for $t \in [1, 2]$, hence f is convex on $[0, s_0]$. We have

$$\begin{aligned} h(t) &> -2k^2t^3 + k(5k+1)t^3 + 3k(k+3)t^2 + (k^2 - k + 2)t - k - 2 \\ &= k(3k+1)t^3 + 3k(k+3)t^2 + (k^2 - k + 2)t - k - 2 \\ &> 3k(k+3) + (k^2 - k + 2) - k - 2 > 0. \end{aligned}$$

By the RPCF-Theorem, we only need to prove the original inequality for

$$a_2 = a_3 = \cdots = a_8 := t, \quad a_1 = 1/t^7, \quad t \geq 1.$$

Write this inequality as

$$\frac{t^7(t^7 - 1)}{(t^7 + 1)(t^7 + k)} \geq \frac{7(t - 1)}{(t + 1)(kt + 1)}.$$

Since

$$(t^7 + 1)(t^7 + k) = t^{14} + (k + 1)t^7 + k \leq t^7(t^7 + 2k + 1),$$

it suffices to show that

$$\frac{t^7 - 1}{t^7 + 2k + 1} \geq \frac{7(t - 1)}{(t + 1)(kt + 1)},$$

which is equivalent to

$$k(t - 1)P(t) + Q(t) \geq 0,$$

where

$$P(t) = t(t + 1)(t^6 + t^5 + t^4 + t^3 + t^2 + t + 1) - 14,$$

$$Q(t) = (t + 1)(t^7 - 1) - 7(t - 1)(t^7 + 1).$$

Since $(t - 1)P(t) \geq 0$ for $t \geq 1$, it suffices to consider the case $k = 1$. So, we need to show that

$$\frac{t^7 - 1}{t^7 + 3} \geq \frac{7(t - 1)}{(t + 1)^2},$$

which is equivalent to

$$t^7(t^2 - 5t + 8) - t^2 - 23t + 20 \geq 0.$$

Substituting

$$t = 1 + x, \quad x \geq 0,$$

the inequality becomes

$$(1 + x)^7(x^2 - 3x + 4) - x^2 - 25x - 4 \geq 0.$$

Since

$$(1 + x)^7 \geq 1 + 7x + 21x^2,$$

it suffices to show that

$$(21x^2 + 7x + 1)(x^2 - 3x + 4) - x^2 - 25x - 4 \geq 0.$$

This inequality, equivalent to

$$x^2(21x^2 - 56x + 63) \geq 0.$$

is true since

$$21x^2 - 56x + 63 > 16x^2 - 56x + 49 = (4x - 7)^2 \geq 0.$$

The equality holds for $a_1 = a_2 = \dots = a_8 = 1$.

□

P 3.39. If $a_1, a_2, \dots, a_9$ are positive real numbers so that $a_1 a_2 \dots a_9 = 1$, then

$$\frac{1}{2a_1 + 1} + \frac{1}{2a_2 + 1} + \dots + \frac{1}{2a_9 + 1} \geq \frac{1}{a_1 + 2} + \frac{1}{a_2 + 2} + \dots + \frac{1}{a_9 + 2}.$$

(Vasile C., 2012)

Solution. Write the inequality as

$$\frac{1 - a_1}{(2a_1 + 1)(a_1 + 2)} + \frac{1 - a_2}{(2a_2 + 1)(a_2 + 2)} + \dots + \frac{1 - a_9}{(2a_9 + 1)(a_9 + 2)} \geq 0.$$

Using the substitutions $a_i = e^{x_i}$ for $i = 1, 2, \dots, 9$, we need to show that

$$f(x_1) + f(x_2) + \dots + f(x_9) \geq 9f(s),$$

where

$$s = \frac{x_1 + x_2 + \dots + x_9}{9} = 0$$

and

$$f(u) = \frac{1 - e^u}{(2e^u + 1)(e^u + 2)}, \quad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{e^u(2e^{2u} - 4e^u - 7)}{(2e^u + 1)^2(e^u + 2)^2},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = \ln \left(1 + \frac{3\sqrt{2}}{2} \right) < 2, \quad s < s_0.$$

Also, we have

$$f''(u) = \frac{t \cdot h(t)}{(2t+1)^3(t+2)^3}, \quad t = e^u,$$

where

$$h(t) = -4t^4 + 26t^3 + 54t^2 + 19t - 14.$$

We will show that $h(t) > 0$ for $t \in [1, 2]$, hence f is convex on $[0, s_0]$. We have

$$h(t) \geq -8t^3 + 26t^3 + 54t^2 + 19t - 14 = 18t^3 + 54t^2 + 19t - 14 > 0.$$

By the RPCF-Theorem, we only need to prove the original inequality for

$$a_2 = a_3 = \cdots = a_9 := t, \quad a_1 = 1/t^8, \quad t \geq 1.$$

Write this inequality as

$$\frac{t^8(t^8 - 1)}{(t^8 + 2)(2t^8 + 1)} \geq \frac{8(t - 1)}{(2t + 1)(t + 2)}.$$

Since

$$(t^8 + 2)(2t^8 + 1) = 2t^{16} + 5t^8 + 2 \leq t^8(2t^8 + 7),$$

it suffices to show that

$$\frac{t^8 - 1}{2t^8 + 7} \geq \frac{8(t - 1)}{(2t + 1)(t + 2)},$$

which is equivalent to

$$t^8(2t^2 - 11t + 18) - 2t^2 - 61t + 54 \geq 0.$$

Substituting

$$t = 1 + x, \quad x \geq 0,$$

the inequality becomes

$$(1 + x)^8(2x^2 - 7x + 9) - 2x^2 - 65x - 9 \geq 0.$$

Since

$$(1 + x)^8 \geq 1 + 8x + 28x^2,$$

it suffices to show that

$$(28x^2 + 8x + 1)(2x^2 - 7x + 9) - 2x^2 - 65x - 9 \geq 0.$$

This inequality, equivalent to

$$x^2(56x^2 - 180x + 196) \geq 0.$$

is true since

$$56x^2 - 180x + 196 \geq 49x^2 - 196x + 196 = 49(x - 2)^2 \geq 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_9 = 1$.

□

P 3.40. If $a_1, a_2, \dots, a_n$ are real numbers so that

$$a_1, a_2, \dots, a_n \leq \pi, \quad a_1 + a_2 + \dots + a_n = \pi,$$

then

$$\cos a_1 + \cos a_2 + \dots + \cos a_n \leq n \cos \frac{\pi}{n}.$$

(Vasile C., 2000)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \dots + a_n}{n} = \frac{\pi}{n},$$

where

$$f(u) = -\cos u, \quad u \in \mathbb{I} = [-(n-2)\pi, \pi].$$

Let

$$s_0 = 0 < s.$$

We see that f is increasing on $[s_0, \pi] = \mathbb{I}_{\geq s_0}$ and $f(u) \geq f(s_0) = -1$ for $u \in \mathbb{I}$. In addition, f is convex on $[s_0, s]$. Thus, by the LPCF-Theorem, we only need to prove that $g(x) \leq 0$, where

$$g(x) = \cos x + (n-1)\cos y - n\cos s, \quad x + (n-1)y = \pi, \quad \pi \geq x \geq s \geq y \geq 0.$$

Since $y' = \frac{-1}{n-1}$, we get

$$g'(x) = -\sin x + \sin y = -2 \sin \frac{x-y}{2} \cos \frac{x+y}{2}.$$

We have $g'(x) \leq 0$ because

$$0 < \frac{x+y}{2} \leq \frac{x+(n-1)y}{2} = \frac{\pi}{2}$$

and

$$0 \leq \frac{x-y}{2} < \frac{\pi}{2}.$$

From $g' \leq 0$, it follows that g is decreasing, hence $g(x) \leq g(s) = 0$.

The equality holds for $a_1 = a_2 = \dots = a_n = \frac{\pi}{n}$. If $n = 2$, then the inequality is an identity.

Remark. In the same manner, we can prove the following generalization:

- If $a_1, a_2, \dots, a_n$ are real numbers so that

$$a_1, a_2, \dots, a_n \leq \pi, \quad \frac{a_1 + a_2 + \dots + a_n}{n} = s, \quad 0 < s \leq \frac{\pi}{4},$$

then

$$\cos a_1 + \cos a_2 + \dots + \cos a_n \leq n \cos s,$$

with equality for $a_1 = a_2 = \dots = a_n = s$.

□

P 3.41. If $a_1, a_2, \dots, a_n$ ($n \geq 3$) are real numbers so that

$$a_1, a_2, \dots, a_n \geq \frac{-1}{n-2}, \quad a_1 + a_2 + \dots + a_n = n,$$

then

$$\frac{a_1^2}{a_1^2 - a_1 + 1} + \frac{a_2^2}{a_2^2 - a_2 + 1} + \dots + \frac{a_n^2}{a_n^2 - a_n + 1} \leq n.$$

(Vasile Cîrtoaje, 2012)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \dots + a_n}{n} = 1,$$

where

$$f(u) = \frac{1-u}{u^2-u+1}, \quad u \in \mathbb{I} = \left[\frac{-1}{n-2}, \frac{n^2-n-1}{n-2} \right].$$

Let $s_0 = 2$. We have $s < s_0$ and

$$\min_{u \in \mathbb{I}} f(u) = f(s_0)$$

because

$$f(u) - f(2) = \frac{1-u}{u^2-u+1} + \frac{1}{3} = \frac{(u-2)^2}{3(u^2-u+1)} \geq 0.$$

From

$$f'(u) = \frac{u(u-2)}{(u^2-u+1)^2},$$

$$f''(u) = \frac{2(3u^2-u^3-1)}{(u^2-u+1)^3} = \frac{2u^2(2-u) + 2(u^2-1)}{(u^2-u+1)^3},$$

it follows that f is convex on $[1, s_0]$. However, we can't apply the RPCF-Theorem in its original form because f is not decreasing on $\mathbb{I}_{\leq s_0}$. According to Theorem 1, we may replace this condition with $ns - (n-1)s_0 \leq \inf \mathbb{I}$. Indeed, we have

$$ns - (n-1)s_0 = n - 2(n-1) = -n + 2 \leq \frac{-1}{n-2} = \inf \mathbb{I}.$$

So, it suffices to show that $f(x) + (n-1)f(y) \geq nf(1)$ for all $x, y \in \mathbb{I}$ so that

$$x + (n-1)y = n.$$

According to Note 1, we only need to show that $h(x, y) \geq 0$, where

$$g(u) = \frac{f(u) - f(1)}{u - 1}, \quad h(x, y) = \frac{g(x) - g(y)}{x - y}.$$

We have

$$g(u) = \frac{-1}{u^2 - u + 1},$$

$$h(x, y) = \frac{x + y - 1}{(x^2 - x + 1)(y^2 - y + 1)} = \frac{(n-2)x + 1}{(n-1)(x^2 - x + 1)(y^2 - y + 1)} \geq 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = \frac{-1}{n-2}, \quad a_2 = a_3 = \cdots = a_n = \frac{n-1}{n-2}$$

(or any cyclic permutation).

□

P 3.42. If $a_1, a_2, \dots, a_n$ ($n \geq 3$) are nonzero real numbers so that

$$a_1, a_2, \dots, a_n \geq \frac{-n}{n-2}, \quad a_1 + a_2 + \cdots + a_n = n,$$

then

$$\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2} \geq \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}.$$

(Vasile Cîrtoaje, 2012)

Solution. According to P 2.25-(a) in Volume 1, the inequality is true for $n = 3$. Assume further that $n \geq 4$ and write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{1}{u^2} - \frac{1}{u}, \quad u \in \mathbb{I} = \left[\frac{-n}{n-2}, \frac{n(2n-3)}{n-2} \right] \setminus \{0\}.$$

Let

$$s_0 = 2, \quad s < s_0.$$

From

$$f(u) - f(2) = \frac{1}{u^2} - \frac{1}{u} + \frac{1}{4} = \frac{(u-2)^2}{4u^2} \geq 0,$$

it follows that

$$\min_{u \in \mathbb{I}} f(u) = f(s_0),$$

while from

$$f'(u) = \frac{u-2}{u^3}, \quad f''(u) = \frac{2(3-u)}{u^4},$$

it follows that f is convex on $[s, s_0]$. However, we can't apply the RPCF-Theorem because f is not decreasing on $\mathbb{I}_{\leq s_0}$. According to Theorem 1 and Note 6, we may replace this condition with $ns - (n-1)s_0 \leq \inf \mathbb{I}$. For $n \geq 4$, we have

$$ns - (n-1)s_0 = n - 2(n-1) = -n + 2 \leq \frac{-n}{n-2} = \inf \mathbb{I}.$$

So, according to Note 1, it suffices to show that $h(x, y) \geq 0$ for all $x, y \in \mathbb{I}$ so that $x + (n-1)y = n$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{-1}{u^2},$$

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{x + y}{x^2 y^2} = \frac{(n-2)x + n}{(n-1)x^2 y^2} \geq 0.$$

The proof is completed. By Note 3, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = \frac{-n}{n-2}, \quad a_2 = a_3 = \cdots = a_n = \frac{n}{n-2}$$

(or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- Let $a_1, a_2, \dots, a_n \geq \frac{-n}{n-2}$ so that $a_1 + a_2 + \cdots + a_n = n$. If $n \geq 3$ and $k \geq 0$, then

$$\frac{1 - a_1}{k + a_1^2} + \frac{1 - a_2}{k + a_2^2} + \cdots + \frac{1 - a_n}{k + a_n^2} \geq 0,$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = \frac{-n}{n-2}, \quad a_2 = a_3 = \cdots = a_n = \frac{n}{n-2}$$

(or any cyclic permutation).

□

P 3.43. If $a_1, a_2, \dots, a_n \geq -1$ so that $a_1 + a_2 + \cdots + a_n = n$, then

$$(n+1) \left(\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2} \right) \geq 2n + (n-1) \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right).$$

(Vasile C., 2013)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{n+1}{u^2} - \frac{n-1}{u}, \quad u \in \mathbb{I} = [-1, 2n-1] \setminus \{0\}.$$

Let

$$s_0 = \frac{2(n+1)}{n-1} \in \mathbb{I}, \quad s < s_0.$$

Since

$$f(u) - f(s_0) = \frac{[(n-1)u - 2(n+1)]^2}{4(n+1)u^2} \geq 0,$$

we have

$$\min_{u \in \mathbb{I}} f(u) = f(s_0).$$

From

$$f'(u) = \frac{(n-1)u - 2(n+1)}{u^3}, \quad f''(u) = \frac{6(n+1) - 2(n-1)u}{u^4},$$

it follows that f is convex on $[1, s_0]$. Since f is not decreasing on $\mathbb{I}_{\leq s_0}$, according to Theorem 1 and Note 6, we may replace this condition in RPCF-Theorem with $ns - (n-1)s_0 \leq \inf \mathbb{I}$. We have

$$ns - (n-1)s_0 = n - 2(n+1) = -n - 2 < -1 = \inf \mathbb{I}.$$

According to Note 1, we only need to show that $h(x, y) \geq 0$ for $-1 \leq x \leq 1 \leq y$ and $x + (n-1)y = n$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = -\frac{2}{u} - \frac{n+1}{u^2}$$

and

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{2xy + (n+1)(x+y)}{x^2y^2} = \frac{(x+1)(n^2 + n - 2x)}{(n-1)x^2y^2} \geq 0.$$

According to Note 4, the equality holds for $a_1 = a_2 = \dots = a_n = 1$, and also for

$$a_1 = -1, \quad a_2 = \dots = a_n = \frac{n+1}{n-1}$$

(or any cyclic permutation).

□

P 3.44. If $a_1, a_2, \dots, a_n$ ($n \geq 3$) are real numbers so that

$$a_1, a_2, \dots, a_n \geq \frac{-(3n-2)}{n-2}, \quad a_1 + a_2 + \dots + a_n = n,$$

then

$$\frac{1-a_1}{(1+a_1)^2} + \frac{1-a_2}{(1+a_2)^2} + \dots + \frac{1-a_n}{(1+a_n)^2} \geq 0.$$

(Vasile C., 2014)

Solution. According to P 2.25-(b) in Volume 1, the inequality is true for $n = 3$. Assume further that $n \geq 4$ and write the inequality as

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \dots + a_n}{n} = 1,$$

where

$$f(u) = \frac{1-u}{(1+u)^2}, \quad u \in \mathbb{I} = \left[\frac{-(3n-2)}{n-2}, \frac{4n^2-7n+2}{n-2} \right] \setminus \{-1\}.$$

Let

$$s_0 = 3, \quad s < s_0.$$

From

$$f(u) - f(3) = \frac{1-u}{(1+u)^2} + \frac{1}{8} = \frac{(u-3)^2}{8(u+1)^2} \geq 0,$$

it follows that

$$\min_{u \in \mathbb{I}} f(u) = f(s_0).$$

From

$$f'(u) = \frac{u-3}{(u+1)^3}, \quad f''(u) = \frac{2(5-u)}{(u+1)^4},$$

it follows that f is convex on $[1, s_0]$. We can't apply the RPCF-Theorem in its original form because f is not decreasing on $\mathbb{I}_{\leq s_0}$. However, according to Theorem 1 and Note 6, we may replace this condition with $ns - (n-1)s_0 \leq \inf \mathbb{I}$. Indeed, for $n \geq 4$, we have

$$ns - (n-1)s_0 = n - 3(n-1) = -2n + 3 \leq \frac{-(3n-2)}{n-2} = \inf \mathbb{I}.$$

According to Note 1, it suffices to show that $h(x, y) \geq 0$ for all $x, y \in \mathbb{I}$ so that $x \leq 1 \leq y$ and $x + (n-1)y = n$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{-1}{(u+1)^2},$$

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{x + y + 2}{(x+1)^2(y+1)^2} = \frac{(n-2)x + 3n - 2}{(n-1)(x+1)^2(y+1)^2} \geq 0.$$

In accordance with Note 3, the equality holds for $a_1 = a_2 = \dots = a_n = 1$, and also for

$$a_1 = \frac{-(3n-2)}{n-2}, \quad a_2 = a_3 = \dots = a_n = \frac{n+2}{n-2}$$

(or any cyclic permutation).

□

P 3.45. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$. If $n \geq 3$ and $k \geq 2 - \frac{2}{n}$, then

$$\frac{1-a_1}{(1-ka_1)^2} + \frac{1-a_2}{(1-ka_2)^2} + \dots + \frac{1-a_n}{(1-ka_n)^2} \geq 0.$$

(Vasile C., 2012)

Solution. According to P 3.99 in Volume 1, the inequality is true for $n = 3$. Assume further that $n \geq 4$ and write the inequality as

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \dots + a_n}{n} = 1,$$

where

$$f(u) = \frac{1-u}{(1-ku)^2}, \quad u \in \mathbb{I} = [0, n] \setminus \{1/k\}.$$

Let

$$s_0 = 2 - 1/k, \quad 1 = s < s_0.$$

Since

$$f(u) - f(s_0) = \frac{1-u}{(1-ku)^2} + \frac{1}{4k(k-1)} = \frac{(ku-2k+1)^2}{4k(k-1)(1-ku)^2} \geq 0,$$

we have

$$\min_{u \in \mathbb{I}} f(u) = f(s_0).$$

From

$$f'(u) = \frac{ku - 2k + 1}{(ku - 1)^3}, \quad f''(u) = \frac{2k(-ku + 3k - 2)}{(1 - ku)^4},$$

it follows that f is convex on $[1, s_0]$. We can't apply the RPCF-Theorem because f is not decreasing on $\mathbb{I}_{\leq s_0}$. According to Theorem 1 and Note 6, we may replace this condition with $ns - (n - 1)s_0 \leq \inf \mathbb{I}$. Indeed, we have

$$ns - (n - 1)s_0 \leq n - (n - 1) \cdot \frac{3n - 4}{2(n - 1)} = \frac{4 - n}{2} \leq 0 = \inf \mathbb{I}.$$

So, it suffices to show that $f(x) + (n - 1)f(y) \geq nf(1)$ for all $x, y \in \mathbb{I}$ so that $x \leq 1 \leq y$ and $x + (n - 1)y = n$. According to Note 1, we only need to show that $h(x, y) \geq 0$, where

$$g(u) = \frac{f(u) - f(1)}{u - 1}, \quad h(x, y) = \frac{g(x) - g(y)}{x - y}.$$

Since

$$g(u) = \frac{-1}{(1 - ku)^2}, \quad h(x, y) = \frac{k[k(x + y) - 2]}{(1 - kx)^2(1 - ky)^2},$$

we need to show that $k(x + y) - 2 \geq 0$. Indeed, we have

$$\frac{k(x + y) - 2}{2} \geq \frac{(n - 1)(x + y)}{n} - 1 = \frac{(n - 1)(x + y)}{n} - \frac{x + (n - 1)y}{n} = \frac{(n - 2)x}{n} \geq 0.$$

The equality holds for $a_1 = a_2 = \dots = a_n = 1$. If $k = 2 - \frac{2}{n}$, then the equality also holds for

$$a_1 = 0, \quad a_2 = a_3 = \dots = a_n = \frac{n}{n - 1}$$

(or any cyclic permutation).

□

P 3.46. If a, b, c are positive real numbers such that $abc = 1$, then

$$\frac{a}{(a + 1)(a + 25)} + \frac{b}{(b + 1)(b + 25)} + \frac{c}{(c + 1)(c + 25)} \leq \frac{3}{52}.$$

(Vasile C., 2022)

Solution. Using the substitution

$$a = e^x, \quad b = e^y, \quad c = e^z,$$

we need to show that

$$f(x) + f(y) + f(z) \geq 3f(s), \quad s = \frac{x + y + z}{3} = 0,$$

where

$$f(u) = \frac{-e^u}{e^{2u} + 26e^u + 25}, \quad u \in \mathbb{I} = \mathbb{R}.$$

From

$$f'(u) = \frac{e^u(e^{2u} - 25)}{(e^{2u} + 26e^u + 25)^2},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = \ln 5.$$

Also, we have

$$f''(u) = \frac{v \cdot h(v)}{(e^{2u} + 26e^u + 25)^3},$$

where

$$h(v) = -v^4 + 26v^3 + 150v^2 + 650v - 625, \quad v = e^u.$$

For $v \in [1, 5]$, we have

$$h(v) = v^3(26 - v) + 150v^2 + 25v + 625(v - 1) > 0,$$

hence $f(u)$ is convex on $[s, s_0]$. By the RPCF-Theorem, we only need to prove the original inequality for

$$b = c := t, \quad a = 1/t^2, \quad t > 0,$$

that is,

$$\frac{t^2}{25t^4 + 26t^2 + 1} + \frac{2t}{t^2 + 26t + 25} \leq \frac{3}{52}.$$

Since

$$25t^4 + 26t^2 + 1 = 24t^4 + (t^4 + 1) + 26t^2 \geq 24t^4 + 28t^2 = 4t^2(6t^2 + 7),$$

it suffices to show that

$$\frac{1}{4(6t^2 + 7)} + \frac{2t}{t^2 + 26t + 25} \leq \frac{3}{52},$$

which is equivalent to

$$\begin{aligned} \frac{2t}{t^2 + 26t + 25} &\leq \frac{9t^2 + 4}{26(6t^2 + 7)}, \\ 9t^4 - 78t^3 + 229t^2 - 260t + 100 &\geq 0, \\ (t - 1)^2(3t - 10)^2 &\geq 0. \end{aligned}$$

The equality holds for $a = b = c = 1$.

□

P 3.47. Let $a_1, a_2, \dots, a_n$ ($n \geq 3$) be nonnegative real numbers so that

$$a_1 + a_2 + \dots + a_n = n.$$

If $p \leq \frac{-2(n-1)}{n-2}$ and $q \geq q_0 = (p+1) \left(p + \frac{n}{n-1} \right)$, then

$$\frac{1}{a_1^2 + pa_1 + q} + \frac{1}{a_2^2 + pa_2 + q} + \dots + \frac{1}{a_n^2 + pa_n + q} \leq \frac{n}{1 + p + q}.$$

(Vasile C., 2023)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{-1}{u^2 + pu + q}, \quad u \in [0, n].$$

Let $s_0 = -p/2$, $s_0 > s = 1$. We have

$$f'(u) = \frac{2u + p}{(u^2 + pu + q)^2} = \frac{2(u - s_0)}{(u^2 + pu + q)^2},$$

$$f''(u) = \frac{2A(u)}{(u^2 + pu + q)^3}, \quad A(u) = q - p^2 - 3pu - 3u^2.$$

From the expression of $f'(u)$, it follows that f is decreasing for $u \leq s_0$ and increasing for $u \geq s_0$, hence $f(u) \geq f(s_0)$ for all real u . On the other hand, since $A(u)$ is concave, $A(u)$ has the maximum value for $u = s_0$ and

$$A(1) = q - p^2 - 3p - 3 \geq q_0 - p^2 - 3p - 3 = \frac{-(n-2)p - 2n + 3}{n-1} \geq \frac{2(n-1) - 2n + 3}{n-1} = \frac{1}{n-1} > 0,$$

we have $A(u) > 0$ for all $u \in [1, s_0]$, hence f is convex on $[s, s_0]$. If $s_0 \geq n$, we may apply RHCF-Theorem from chapter 1 (because f is convex on $[s, n]$), and if $s_0 < n$, we may apply RPCF-Theorem (because $f(u)$ is decreasing on $[0, s_0]$ and $f(u) \geq f(s_0)$ for all $u \in [0, n]$). In both cases, it suffices to show that $h(x, y) \geq 0$ for $0 \leq x \leq 1 \leq y$ such that $x + (n-1)y = n$. We have

$$\begin{aligned} g(u) &= \frac{f(u) - f(1)}{u - 1} = \frac{u + 1 + p}{(1 + p + q)(u^2 + pu + q)}, \\ h(x, y) &= \frac{g(x) - g(y)}{x - y} = \frac{q - p(p+1) - (p+1)(x+y) - xy}{(1 + p + q)(x^2 + px + q)(y^2 + py + q)} \\ &\geq \frac{q_0 - p(p+1) - (p+1)(x+y) - xy}{(1 + p + q)(x^2 + px + q)(y^2 + py + q)} = \frac{x[x - p(n-2) - 2(n-1)]}{(n-1)(1 + p + q)(x^2 + px + q)(y^2 + py + q)} \geq 0. \end{aligned}$$

The equality occurs for $a_1 = a_2 = \cdots = a_n = 1$. In addition, if $q = q_0$, then the equality also occurs for $a_1 = 0$ and $a_2 = a_3 = \cdots = a_n = \frac{n}{n-1}$ (or any cyclic permutation). □

P 3.48. Let $a_1, a_2, \dots, a_n$ ($n \geq 3$) be nonnegative real numbers so that

$$a_1 + a_2 + \cdots + a_n = n.$$

If $p \in \left[\frac{-2(n-1)}{n-2}, -2 \right) \cup \left(-2, \frac{2}{n-2} \right]$ and $q \geq q_0 = \frac{n^2 p^2 + 4(n^2 - n + 1)(p+1)}{4(n-1)}$, then

$$\frac{1}{a_1^2 + pa_1 + q} + \frac{1}{a_2^2 + pa_2 + q} + \cdots + \frac{1}{a_n^2 + pa_n + q} \leq \frac{n}{1 + p + q}.$$

(Vasile C., 2023)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{-1}{u^2 + pu + q}, \quad u \in [0, n].$$

We have

$$f'(u) = \frac{2u + p}{(u^2 + pu + q)^2} = \frac{2(u - s_0)}{(u^2 + pu + q)^2},$$

$$f''(u) = \frac{2A(u)}{(u^2 + pu + q)^3}, \quad A(u) = q - p^2 - 3pu - 3u^2.$$

Let $s_0 = -p/2$. From $f'(u)$, it follows that f is decreasing for $u \leq s_0$ and increasing for $u \geq s_0$, hence $f(u) \geq f(s_0)$ for all real u . On the other hand, since $A(u)$ is concave, $A(u)$ has the maximum value for $u = s_0$ and

$$A(1) \geq q_0 - p^2 - 3pu - 3u^2 = \frac{(n-2)^2(p+2)^2}{4(n-1)} > 0,$$

we have $A(u) > 0$ for all u between 1 and s_0 , hence f is convex on the interval between $s = 1$ and s_0 .

Case 1: $p \in \left(-2, \frac{2}{n-2}\right]$. Since $s_0 < s$, f is convex on $[s_0, s]$. If $s_0 \leq 0$, we may apply LHCF-Theorem from chapter 1 (because f is convex on $[0, s]$), and if $s_0 > 0$, we may apply LPCF-Theorem (because f is convex on $[s_0, s]$, $f(u)$ is increasing on $[s_0, n]$ and $f(u) \geq f(s_0)$ for all $u \in [0, n]$). In both cases, it suffices to show that $h(x, y) \geq 0$ for $x \geq 1 \geq y$ such that $x + (n-1)y = n$. As shown in the preceding P 3.47, we need to show that $q_0 - p(p+1) - (p+1)(x+y) - xy \geq 0$, which is equivalent to

$$(n-1)y^2 + [(n-2)p-2]y + \frac{[(n-2)p-2]^2}{4(n-1)} \geq 0,$$

$$[2(n-1)y + (n-2)p-2]^2 \geq 0.$$

The equality occurs for $a_1 = a_2 = \cdots = a_n = 1$. In addition, if $q = q_0$, then the equality also occurs for $a_1 = n-1 + \frac{(n-2)p}{2}$ and $a_2 = a_3 = \cdots = a_n = \frac{2-(n-2)p}{2(n-1)}$ (or any cyclic permutation).

Case 2: $p \in \left[\frac{-2(n-1)}{n-2}, -2\right)$. Since $s < s_0$, f is convex on $[s, s_0]$. In addition, $f(u)$ is decreasing on $[0, s_0]$ and $f(u) \geq f(s_0)$ for all $u \in [0, n]$. So, we may apply RPCF-Theorem. By Note 1, it suffices to show that $h(x, y) \geq 0$ for $x \leq 1 \leq y$ such that $x + (n-1)y = n$. As shown in the preceding P 3.47, we need to show that $q_0 - p(p+1) - (p+1)(x+y) - xy \geq 0$, which is equivalent to

$$[2(n-1)y + (n-2)p-2]^2 \geq 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. In addition, if $q = q_0$, then the equality also occurs for $a_1 = n - 1 + \frac{(n-2)p}{2}$ and $a_2 = a_3 = \cdots = a_n = \frac{2 - (n-2)p}{2(n-1)}$ (or any cyclic permutation).

□

Chapter 4

Partially Convex Function Method for Ordered Variables

4.1 Theoretical Basis

The following statement is known as Right Partially Convex Function Theorem for Ordered Variables (RPCF-OV Theorem).

RPCF-OV Theorem (Vasile Cîrtoaje, 2014). *Let f be a real function defined on an interval $\mathbb{I}$ and convex on $[s, s_0]$, where $s, s_0 \in \mathbb{I}$, $s < s_0$. In addition, f is decreasing on $\mathbb{I}_{\leq s_0}$ and $f(u) \geq f(s_0)$ for $u \in \mathbb{I}$. The inequality*

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \dots, a_n \in \mathbb{I}$ satisfying

$$a_1 + a_2 + \cdots + a_n = ns$$

and

$$a_1 \leq a_2 \leq \cdots \leq a_m \leq s, \quad m \in \{1, 2, \dots, n-1\},$$

if and only if

$$f(x) + (n-m)f(y) \geq (1+n-m)f(s)$$

for all $x, y \in \mathbb{I}$ so that $x \leq s \leq y$ and $x + (n-m)y = (1+n-m)s$.

Proof. For

$$a_1 = x, \quad a_2 = \cdots = a_m = s, \quad a_{m+1} = \cdots = a_n = y,$$

the inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s)$$

becomes

$$f(x) + (n-m)f(y) \geq (1+n-m)f(s);$$

therefore, the necessity is obvious. By Lemma from Chapter 3, to prove the sufficiency, it suffices to consider that $a_1, a_2, \dots, a_n \in \mathbb{J}$, where

$$\mathbb{J} = \mathbb{I}_{\leq s_0}.$$

Because f is convex on $\mathbb{J}_{\geq s}$, the desired inequality follows from HCF-OV Theorem applied to the interval $\mathbb{J}$.

Similarly, we can prove Left Partially Convex Function Theorem for Ordered Variables (LPCF-OV Theorem).

LPCF-OV Theorem. *Let f be a real function defined on an interval $\mathbb{I}$ and convex on $[s_0, s]$, where $s_0, s \in \mathbb{I}$, $s_0 < s$. In addition, f is increasing on $\mathbb{I}_{\geq s_0}$ and $f(u) \geq f(s_0)$ for $u \in \mathbb{I}$. The inequality*

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf \left(\frac{a_1 + a_2 + \cdots + a_n}{n} \right)$$

holds for all $a_1, a_2, \dots, a_n \in \mathbb{I}$ satisfying

$$a_1 + a_2 + \cdots + a_n = ns$$

and

$$a_1 \geq a_2 \geq \cdots \geq a_m \geq s, \quad m \in \{1, 2, \dots, n-1\},$$

if and only if

$$f(x) + (n-m)f(y) \geq (1+n-m)f(s)$$

for all $x, y \in \mathbb{I}$ so that $x \geq s \geq y$ and $x + (n-m)y = (1+n-m)s$.

The RPCF-OV Theorem and the LPCF-OV Theorems are respectively generalizations of the RPCF Theorem and LPCF Theorem, because the last theorems can be obtained from the first theorems for $m = 1$.

Note 1. Let us denote

$$g(u) = \frac{f(u) - f(s)}{u - s}, \quad h(x, y) = \frac{g(x) - g(y)}{x - y}.$$

We may replace the hypothesis condition in the RPCF-OV Theorem and the LPCF-OV Theorem, namely

$$f(x) + mf(y) \geq (1+m)f(s),$$

by the condition

$$h(x, y) \geq 0 \quad \text{for all } x, y \in \mathbb{I} \quad \text{so that } x + my = (1+m)s.$$

Note 2. Assume that f is differentiable on $\mathbb{I}$, and let

$$H(x, y) = \frac{f'(x) - f'(y)}{x - y}.$$

The desired inequality of Jensen's type in the RPCF-OV Theorem and the LPCF-OV Theorem holds true by replacing the hypothesis

$$f(x) + mf(y) \geq (1+m)f(s)$$

with the more restrictive condition

$$H(x, y) \geq 0 \quad \text{for all } x, y \in \mathbb{I} \quad \text{so that } x + my = (1+m)s.$$

Note 3. The desired inequalities in the RPCF-OV Theorem and the LPCF-OV Theorem become equalities for

$$a_1 = a_2 = \cdots = a_n = s.$$

In addition, if there exist $x, y \in \mathbb{I}$ so that

$$x + (n - m)y = (1 + n - m)s, \quad f(x) + (n - m)f(y) = (1 + n - m)f(s), \quad x \neq y,$$

then the equality holds also for

$$a_1 = x, \quad a_2 = \cdots = a_m = s, \quad a_{m+1} = \cdots = a_n = y$$

(or any cyclic permutation). Notice that these equality conditions are equivalent to

$$x + (n - m)y = (1 + n - m)s, \quad h(x, y) = 0$$

($x < y$ for RHCF-OV Theorem, and $x > y$ for LHCF-OV Theorem).

Note 4. The RPCF-OV Theorem is also valid in the case where f is defined on $\mathbb{I} \setminus \{u_0\}$, where u_0 is an interior point of $\mathbb{I}$ so that $u_0 > s_0$. Similarly, LPCF Theorem is also valid in the case in which f is defined on $\mathbb{I} \setminus \{u_0\}$, where u_0 is an interior point of $\mathbb{I}$ so that $u_0 < s_0$.

Note 5. The RPCF-Theorem holds true by replacing the condition

$$f \text{ is decreasing on } \mathbb{I}_{\leq s_0}$$

with

$$ns - (n - 1)s_0 \leq \inf \mathbb{I}.$$

More precisely, the following theorem holds:

Theorem 1. Let f be a function defined on a real interval $\mathbb{I}$, convex on $[s, s_0]$ and satisfying

$$\min_{u \in \mathbb{I}_{\geq s}} f(u) = f(s_0),$$

where

$$s, s_0 \in \mathbb{I}, \quad s < s_0, \quad (1 + n - m)s - (n - m)s_0 \leq \inf \mathbb{I}.$$

The inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf \left(\frac{a_1 + a_2 + \cdots + a_n}{n} \right)$$

holds for all $a_1, a_2, \dots, a_n \in \mathbb{I}$ satisfying

$$a_1 + a_2 + \cdots + a_n = ns$$

and

$$a_1 \leq a_2 \leq \cdots \leq a_m \leq s, \quad m \in \{1, 2, \dots, n - 1\},$$

if and only if

$$f(x) + (n - m)f(y) \geq (1 + n - m)f(s)$$

for all $x, y \in \mathbb{I}$ so that $x \leq s \leq y$ and $x + (n - m)y = (1 + n - m)s$.

The proof of this theorem is similar to the one of Theorem 1 from chapter 3.

Similarly, the LPCF-Theorem holds true by replacing the condition

$$f \text{ is increasing on } \mathbb{I}_{\geq s_0}$$

with

$$ns - (n-1)s_0 \geq \sup \mathbb{I}.$$

More precisely, the following theorem holds:

Theorem 2. *Let f be a function defined on a real interval $\mathbb{I}$, convex on $[s_0, s]$ and satisfying*

$$\min_{u \in \mathbb{I}_{\leq s}} f(u) = f(s_0),$$

where

$$s, s_0 \in \mathbb{I}, \quad s > s_0, \quad (1+n-m)s - (n-m)s_0 \geq \sup \mathbb{I}.$$

The inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \dots, a_n \in \mathbb{I}$ satisfying

$$a_1 + a_2 + \cdots + a_n = ns$$

and

$$a_1 \geq a_2 \geq \cdots \geq a_m \geq s, \quad m \in \{1, 2, \dots, n-1\},$$

if and only if

$$f(x) + (n-m)f(y) \geq (1+n-m)f(s)$$

for all $x, y \in \mathbb{I}$ so that $x \geq s \geq y$ and $x + (n-m)y = (1+n-m)s$.

Note 6. Theorem 1 is also valid in the case in which f is defined on $\mathbb{I} \setminus \{u_0\}$, where u_0 is an interior point of $\mathbb{I}$ so that $u_0 \notin [s, s_0]$. Similarly, Theorem 2 is also valid in the case in which f is defined on $\mathbb{I} \setminus \{u_0\}$, where u_0 is an interior point of $\mathbb{I}$ so that $u_0 \notin [s_0, s]$.

Note 7. We can extend *weighted* Jensen's inequality to right and left partially convex functions with ordered variables establishing the WRPCF-OV Theorem and the WLPCF-OV Theorem (Vasile Cîrtoaje, 2014).

WRPCF-OV Theorem. *Let $p_1, p_2, \dots, p_n$ be positive real numbers so that*

$$p_1 + p_2 + \cdots + p_n = 1,$$

and let f be a real function defined on an interval $\mathbb{I}$ and convex on $[s, s_0]$, where $s, s_0 \in \text{int}(\mathbb{I})$, $s < s_0$. In addition, f is decreasing on $\mathbb{I}_{\leq s_0}$ and $f(u) \geq f(s_0)$ for $u \in \mathbb{I}$. The inequality

$$p_1 f(x_1) + p_2 f(x_2) + \cdots + p_n f(x_n) \geq f(p_1 x_1 + p_2 x_2 + \cdots + p_n x_n)$$

holds for all $x_1, x_2, \dots, x_n \in \mathbb{I}$ so that $p_1 x_1 + p_2 x_2 + \cdots + p_n x_n = s$ and

$$x_1 \leq x_2 \leq \cdots \leq x_n, \quad x_m \leq s, \quad m \in \{1, 2, \dots, n-1\},$$

if and only if

$$f(x) + kf(y) \geq (1+k)f(s)$$

for all $x, y \in \mathbb{I}$ satisfying

$$x \leq s \leq y, \quad x + ky = (1 + k)s,$$

where

$$k = \frac{p_{m+1} + p_{m+2} + \cdots + p_n}{p_1}.$$

WLPCF-OV Theorem. Let $p_1, p_2, \dots, p_n$ be positive real numbers so that

$$p_1 + p_2 + \cdots + p_n = 1,$$

and let f be a real function defined on an interval $\mathbb{I}$ and convex on $[s_0, s]$, where $s_0, s \in \mathbb{I}$, $s_0 < s$. In addition, f is increasing on $\mathbb{I}_{\geq s_0}$ and $f(u) \geq f(s_0)$ for $u \in \mathbb{I}$. The inequality

$$p_1 f(x_1) + p_2 f(x_2) + \cdots + p_n f(x_n) \geq f(p_1 x_1 + p_2 x_2 + \cdots + p_n x_n)$$

holds for all $x_1, x_2, \dots, x_n \in \mathbb{I}$ so that $p_1 x_1 + p_2 x_2 + \cdots + p_n x_n = s$ and

$$x_1 \geq x_2 \geq \cdots \geq x_n, \quad x_m \geq s, \quad m \in \{1, 2, \dots, n-1\},$$

if and only if

$$f(x) + kf(y) \geq (1 + k)f(s)$$

for all $x, y \in \mathbb{I}$ satisfying

$$x \geq s \geq y, \quad x + ky = (1 + k)s,$$

where

$$k = \frac{p_{m+1} + p_{m+2} + \cdots + p_n}{p_1}.$$

For the most commonly used case

$$p_1 = p_2 = \cdots = p_n = \frac{1}{n},$$

the WRPCF-OV Theorem and the WLPCF-OV Theorem yield the RPCF-OV Theorem and the LPCF-OV Theorem, respectively.

4.2 Applications

4.1. If a, b, c, d are real numbers so that

$$a \leq 1 \leq b \leq c \leq d, \quad a + b + c + d = 4,$$

then

$$\frac{a}{3a^2 + 1} + \frac{b}{3b^2 + 1} + \frac{c}{3c^2 + 1} + \frac{d}{3d^2 + 1} \leq 1.$$

4.2. If a, b, c, d are real numbers so that

$$a \geq b \geq 1 \geq c \geq d, \quad a + b + c + d = 4,$$

then

$$\frac{16a - 5}{32a^2 + 1} + \frac{16b - 5}{32b^2 + 1} + \frac{16c - 5}{32c^2 + 1} + \frac{16d - 5}{32d^2 + 1} \leq \frac{4}{3}.$$

4.3. If a, b, c, d, e are real numbers so that

$$a \geq b \geq 1 \geq c \geq d \geq e, \quad a + b + c + d + e = 5,$$

then

$$\frac{18a - 5}{12a^2 + 1} + \frac{18b - 5}{12b^2 + 1} + \frac{18c - 5}{12c^2 + 1} + \frac{18d - 5}{12d^2 + 1} + \frac{18e - 5}{12e^2 + 1} \leq 5.$$

4.4. If a, b, c, d, e are real numbers so that

$$a \geq b \geq 1 \geq c \geq d \geq e, \quad a + b + c + d + e = 5,$$

then

$$\frac{a(a-1)}{3a^2 + 4} + \frac{b(b-1)}{3b^2 + 4} + \frac{c(c-1)}{3c^2 + 4} + \frac{d(d-1)}{3d^2 + 4} + \frac{e(e-1)}{3e^2 + 4} \geq 0.$$

4.5. Let $a_1, a_2, \dots, a_{2n} \neq -k$ be real numbers so that

$$a_1 \geq \dots \geq a_n \geq 1 \geq a_{n+1} \geq \dots \geq a_{2n}, \quad a_1 + a_2 + \dots + a_{2n} = 2n.$$

If $k \geq \frac{n+1}{2\sqrt{n}}$, then

$$\frac{a_1(a_1-1)}{(a_1+k)^2} + \frac{a_2(a_2-1)}{(a_2+k)^2} + \dots + \frac{a_{2n}(a_{2n}-1)}{(a_{2n}+k)^2} \geq 0.$$

4.6. Let $a_1, a_2, \dots, a_{2n} \neq -k$ be real numbers so that

$$a_1 \geq \dots \geq a_n \geq 1 \geq a_{n+1} \geq \dots \geq a_{2n}, \quad a_1 + a_2 + \dots + a_{2n} = 2n.$$

If $k \geq 1 + \frac{n+1}{\sqrt{n}}$, then

$$\frac{a_1^2 - 1}{(a_1 + k)^2} + \frac{a_2^2 - 1}{(a_2 + k)^2} + \dots + \frac{a_{2n}^2 - 1}{(a_{2n} + k)^2} \geq 0.$$

4.7. If $a_1, a_2, \dots, a_n$ are positive real numbers so that

$$a_1 \geq 1 \geq a_2 \geq \dots \geq a_n, \quad a_1 + a_2 + \dots + a_n = n,$$

then

$$a_1^{3/a_1} + a_2^{3/a_2} + \dots + a_n^{3/a_n} \leq n.$$

4.8. If $a_1, a_2, \dots, a_{11}$ are real numbers so that

$$a_1 \geq a_2 \geq 1 \geq a_3 \geq \dots \geq a_{11}, \quad a_1 + a_2 + \dots + a_{11} = 11,$$

then

$$(1 - a_1 + a_1^2)(1 - a_2 + a_2^2) \dots (1 - a_{11} + a_{11}^2) \geq 1.$$

4.9. If $a_1, a_2, \dots, a_8$ are nonzero real numbers so that

$$a_1 \geq a_2 \geq a_3 \geq a_4 \geq 1 \geq a_5 \geq a_6 \geq a_7 \geq a_8, \quad a_1 + a_2 + \dots + a_8 = 8,$$

then

$$5 \left(\frac{1}{a_1^2} + \frac{1}{a_2^2} + \dots + \frac{1}{a_8^2} \right) + 72 \geq 14 \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_8} \right).$$

4.10. If a, b, c, d are positive real numbers so that

$$a \leq b \leq 1 \leq c \leq d, \quad abcd = 1,$$

then

$$\frac{7-6a}{2+a^2} + \frac{7-6b}{2+b^2} + \frac{7-6c}{2+c^2} + \frac{7-6d}{2+d^2} \geq \frac{4}{3}.$$

4.11. If a, b, c are positive real numbers so that

$$a \leq b \leq 1 \leq c, \quad abc = 1,$$

then

$$\frac{7-4a}{2+a^2} + \frac{7-4b}{2+b^2} + \frac{7-4c}{2+c^2} \geq 3.$$

4.12. If a, b, c are positive real numbers so that

$$a \geq 1 \geq b \geq c, \quad abc = 1,$$

then

$$\frac{23-8a}{3+2a^2} + \frac{23-8b}{3+2b^2} + \frac{23-8c}{3+2c^2} \geq 9.$$

4.13. Let $a_1, a_2, \dots, a_n$ be positive real numbers so that

$$a_1 \leq \dots \leq a_{n-1} \leq 1 \leq a_n, \quad a_1 a_2 \dots a_n = 1.$$

If $p, q \geq 0$ so that $p + 3q \geq 1$, then

$$\frac{1-a_1}{1+pa_1+qa_1^2} + \frac{1-a_2}{1+pa_2+qa_2^2} + \dots + \frac{1-a_n}{1+pa_n+qa_n^2} \geq 0.$$

4.14. If a, b, c, d, e are real numbers so that

$$-2 \leq a \leq b \leq 1 \leq c \leq d \leq e, \quad a + b + c + d + e = 5,$$

then

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{1}{e^2} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e}.$$

4.3 Solutions

P 4.1. If a, b, c, d are real numbers so that

$$a \leq 1 \leq b \leq c \leq d, \quad a + b + c + d = 4,$$

then

$$\frac{a}{3a^2 + 1} + \frac{b}{3b^2 + 1} + \frac{c}{3c^2 + 1} + \frac{d}{3d^2 + 1} \leq 1.$$

Solution. Write the inequality as

$$f(a) + f(b) + f(c) + f(d) \geq 4f(s), \quad s = \frac{a + b + c + d}{4} = 1,$$

where

$$f(u) = \frac{-u}{3u^2 + 1}, \quad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{3u^2 - 1}{(3u^2 + 1)^2},$$

it follows that f is increasing on $(-\infty, -s_0] \cup [s_0, \infty)$ and decreasing on $[-s_0, s_0]$, where $s_0 = 1/\sqrt{3}$. Since

$$\lim_{u \rightarrow -\infty} f(u) = 0$$

and $f(s_0) < 0$, it follows that

$$\min_{u \in \mathbb{R}} f(u) = f(s_0).$$

From

$$f''(u) = \frac{18u(1 - u^2)}{(3u^2 + 1)^3},$$

it follows that f is convex on $[0, 1]$, hence on $[s_0, 1]$. Therefore, we may apply the LPCF-OV Theorem for $n = 4$ and $m = 1$. We only need to show that $f(x) + f(y) \geq 2f(1)$ for all real x, y so that $x + y = 2$. Using Note 1, it suffices to prove that $h(x, y) \geq 0$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

Indeed, we have

$$g(u) = \frac{3u - 1}{4(3u^2 + 1)},$$

$$h(x, y) = \frac{3(1 + x + y - 3xy)}{4(3x^2 + 1)(3y^2 + 1)} = \frac{9(1 - xy)}{4(3x^2 + 1)(3y^2 + 1)} \geq 0,$$

since

$$4(1 - xy) = (x + y)^2 - 4xy = (x - y)^2 \geq 0.$$

Thus, the proof is completed. The equality holds for $a = b = c = d = 1$.

Remark. Similarly, we can prove the following generalization:

- If $a_1, a_2, \dots, a_n$ are real numbers so that

$$a_1 \leq 1 \leq a_2 \leq \dots \leq a_n, \quad a_1 + a_2 + \dots + a_n = n,$$

then

$$\frac{a_1}{3a_1^2 + 1} + \frac{a_2}{3a_2^2 + 1} + \dots + \frac{a_n}{3a_n^2 + 1} \leq \frac{n}{4},$$

with equality for $a_1 = a_2 = \dots = a_n = 1$.

□

P 4.2. If a, b, c, d are real numbers so that

$$a \geq b \geq 1 \geq c \geq d, \quad a + b + c + d = 4,$$

then

$$\frac{16a - 5}{32a^2 + 1} + \frac{16b - 5}{32b^2 + 1} + \frac{16c - 5}{32c^2 + 1} + \frac{16d - 5}{32d^2 + 1} \leq \frac{4}{3}.$$

(Vasile C., 2012)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) + f(d) \geq 4f(s), \quad s = \frac{a + b + c + d}{4} = 1,$$

where

$$f(u) = \frac{5 - 16u}{32u^2 + 1}, \quad u \in \mathbb{R}.$$

As shown in the proof of P 3.1, f is convex on $[s_0, 1]$, increasing for $u \geq s_0$ and

$$\min_{u \in \mathbb{R}} f(u) = f(s_0),$$

where

$$s_0 = \frac{5 + \sqrt{33}}{16} \approx 0.6715.$$

Therefore, we may apply the LPCF-OV Theorem for $n = 4$ and $m = 2$. We only need to show that $f(x) + 2f(y) \geq 3f(1)$ for all real x, y so that $x + 2y = 3$. Using Note 1, it suffices to prove that $h(x, y) \geq 0$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

Indeed, we have

$$g(u) = \frac{32(2u - 1)}{3(32u^2 + 1)},$$

$$h(x, y) = \frac{64(1 + 16x + 16y - 32xy)}{3(32x^2 + 1)(32y^2 + 1)} = \frac{64(4x - 5)^2}{3(32x^2 + 1)(32y^2 + 1)} \geq 0.$$

From $x + 2y = 3$ and $h(x, y) = 0$, we get $x = 5/4$ and $y = 7/8$. Therefore, in accordance with Note 3, the equality holds for $a = b = c = d = 1$, and also for

$$a = \frac{5}{4}, \quad b = 1, \quad c = d = \frac{7}{8}.$$

Remark. Similarly, we can prove the following generalization:

- If $a_1, a_2, \dots, a_n$ ($n \geq 3$) are real numbers so that

$$a_1 \geq \dots \geq a_{n-2} \geq 1 \geq a_{n-1} \geq a_n, \quad a_1 + a_2 + \dots + a_n = n,$$

then

$$\frac{16a_1 - 5}{32a_1^2 + 1} + \frac{16a_2 - 5}{32a_2^2 + 1} + \dots + \frac{16a_n - 5}{32a_n^2 + 1} \leq \frac{n}{3},$$

with equality for $a_1 = a_2 = \dots = a_n = 1$, and also for

$$a_1 = \frac{5}{4}, \quad a_2 = \dots = a_{n-2} = 1, \quad a_{n-1} = a_n = \frac{7}{8}.$$

□

P 4.3. If a, b, c, d, e are real numbers so that

$$a \geq b \geq 1 \geq c \geq d \geq e, \quad a + b + c + d + e = 5,$$

then

$$\frac{18a - 5}{12a^2 + 1} + \frac{18b - 5}{12b^2 + 1} + \frac{18c - 5}{12c^2 + 1} + \frac{18d - 5}{12d^2 + 1} + \frac{18e - 5}{12e^2 + 1} \leq 5.$$

(Vasile C., 2012)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) + f(d) + f(e) \geq 5f(s), \quad s = \frac{a + b + c + d + e}{5} = 1,$$

where

$$f(u) = \frac{5 - 18u}{12u^2 + 1}, \quad u \in \mathbb{R}.$$

As shown in the proof of P 3.2, f is convex on $[s_0, 1]$, increasing for $u \geq s_0$ and

$$\min_{u \in \mathbb{R}} f(u) = f(s_0),$$

where

$$s_0 = \frac{5 + \sqrt{52}}{18} \approx 0.678.$$

Therefore, applying the LPCF-OV Theorem for $n = 5$ and $m = 3$, we only need to show that $f(x) + 3f(y) \geq 4f(1)$ for all real x, y so that $x + 3y = 4$. Using Note 1, it suffices to prove that $h(x, y) \geq 0$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

Indeed, we have

$$g(u) = \frac{6(2u-1)}{12u^2+1},$$

$$h(x, y) = \frac{12(1+6x+6y-12xy)}{(12x^2+1)(12y^2+1)} = \frac{12(2x-3)^2}{(12x^2+1)(12y^2+1)} \geq 0.$$

From $x+3y=4$ and $h(x, y)=0$, we get $x=3/2$ and $y=5/6$. Therefore, in accordance with Note 3, the equality holds for $a=b=c=d=e=1$, and also for

$$a = \frac{3}{2}, \quad b = 1, \quad c = d = e = \frac{5}{6}.$$

Remark. Similarly, we can prove the following generalization:

- If $a_1, a_2, \dots, a_n$ ($n \geq 4$) are real numbers so that

$$a_1 \geq \dots \geq a_{n-3} \geq 1 \geq a_{n-2} \geq a_{n-1} \geq a_n, \quad a_1 + a_2 + \dots + a_n = n,$$

then

$$\frac{18a_1-5}{12a_1^2+1} + \frac{18a_2-5}{12a_2^2+1} + \dots + \frac{18a_n-5}{12a_n^2+1} \leq n,$$

with equality for $a_1 = a_2 = \dots = a_n = 1$, and also for

$$a_1 = \frac{3}{2}, \quad a_2 = \dots = a_{n-3} = 1, \quad a_{n-2} = a_{n-1} = a_n = \frac{5}{6}.$$

□

P 4.4. If a, b, c, d, e are real numbers so that

$$a \geq b \geq 1 \geq c \geq d \geq e, \quad a + b + c + d + e = 5,$$

then

$$\frac{a(a-1)}{3a^2+4} + \frac{b(b-1)}{3b^2+4} + \frac{c(c-1)}{3c^2+4} + \frac{d(d-1)}{3d^2+4} + \frac{e(e-1)}{3e^2+4} \geq 0.$$

(Vasile C., 2012)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) + f(d) + f(e) \geq 5f(s), \quad s = \frac{a+b+c+d+e}{5} = 1,$$

where

$$f(u) = \frac{u^2-u}{3u^2+4}, \quad u \in \mathbb{R}.$$

As shown in the proof of P 3.5, f is convex on $[s_0, 1]$, increasing for $u \geq s_0$ and

$$\min_{u \in \mathbb{R}} f(u) = f(s_0),$$

where

$$s_0 = \frac{-4 + 2\sqrt{7}}{3} \approx 0.43.$$

Therefore, we may apply the LPCF-OV Theorem for $n = 5$ and $m = 2$. We only need to show that $f(x) + 3f(y) \geq 4f(1)$ for all real x, y so that $x + 3y = 4$. Using Note 1, it suffices to prove that $h(x, y) \geq 0$. Indeed, we have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{u}{3u^2 + 4},$$

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{4 - 3xy}{(3x^2 + 4)(3y^2 + 4)} = \frac{(x - 2)^2}{(12x^2 + 1)(12y^2 + 1)} \geq 0.$$

From $x + 3y = 4$ and $h(x, y) = 0$, we get $x = 2$ and $y = 2/3$. Therefore, in accordance with Note 3, the equality holds for

$$a = b = c = d = e = 1,$$

and also for

$$a = 2, \quad b = 1, \quad c = d = e = \frac{2}{3}.$$

Remark. Similarly, we can prove the following generalizations:

- If $a_1, a_2, \dots, a_n$ ($n \geq 4$) are real numbers so that

$$a_1 \geq \dots \geq a_{n-3} \geq 1 \geq a_{n-2} \geq a_{n-1} \geq a_n, \quad a_1 + a_2 + \dots + a_n = n,$$

then

$$\frac{a_1(a_1 - 1)}{3a_1^2 + 4} + \frac{a_2(a_2 - 1)}{3a_2^2 + 4} + \dots + \frac{a_n(a_n - 1)}{3a_n^2 + 4} \geq 0,$$

with equality for $a_1 = a_2 = \dots = a_n = 1$, and also for

$$a_1 = 2, \quad a_2 = \dots = a_{n-3} = 1, \quad a_{n-2} = a_{n-1} = a_n = \frac{2}{3}.$$

- If $a_1, a_2, \dots, a_n$ ($n \geq 3$) are real numbers so that

$$a_1 \geq a_2 \geq 1 \geq a_3 \geq \dots \geq a_n, \quad a_1 + a_2 + \dots + a_n = n,$$

then

$$\frac{a_1(a_1 - 1)}{4(n - 2)a_1^2 + (n - 1)^2} + \frac{a_2(a_2 - 1)}{4(n - 2)a_2^2 + (n - 1)^2} + \dots + \frac{a_n(a_n - 1)}{4(n - 2)a_n^2 + (n - 1)^2} \geq 0,$$

with equality for $a_1 = a_2 = \dots = a_n = 1$, and also for

$$a_1 = \frac{n - 1}{2}, \quad a_2 = 1, \quad a_3 = \dots = a_n = \frac{n - 1}{2(n - 2)}.$$

□

P 4.5. Let $a_1, a_2, \dots, a_{2n} \neq -k$ be real numbers so that

$$a_1 \geq \dots \geq a_n \geq 1 \geq a_{n+1} \geq \dots \geq a_{2n}, \quad a_1 + a_2 + \dots + a_{2n} = 2n.$$

If $k \geq \frac{n+1}{2\sqrt{n}}$, then

$$\frac{a_1(a_1 - 1)}{(a_1 + k)^2} + \frac{a_2(a_2 - 1)}{(a_2 + k)^2} + \dots + \frac{a_{2n}(a_{2n} - 1)}{(a_{2n} + k)^2} \geq 0.$$

(Vasile C., 2012)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \dots + f(a_{2n}) \geq 2nf(s), \quad s = \frac{a_1 + a_2 + \dots + a_{2n}}{2n} = 1,$$

where

$$f(u) = \frac{u(u-1)}{(u+k)^2}, \quad u \in \mathbb{I} = \mathbb{R} \setminus \{-k\}.$$

As shown in the proof of P 3.8, f is convex on $[s_0, 1]$, increasing for $u \geq s_0$ and

$$\min_{u \in \mathbb{I}} f(u) = f(s_0),$$

where

$$s_0 = \frac{k}{2k+1} < 1.$$

Having in view Note 4, we may apply the LPCF-OV Theorem for $2n$ real numbers and $m = n$. We only need to show that $f(x) + nf(y) \geq (n+1)f(1)$ for $x, y \in \mathbb{I}$ so that $x + ny = n+1$. Using Note 1, it suffices to prove that $h(x, y) \geq 0$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{u}{(u+k)^2},$$

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{k^2 - xy}{(x+k)^2(y+k)^2} \geq 0,$$

because

$$k^2 - xy \geq \frac{(n+1)^2}{4n} - xy = \frac{(x+ny)^2}{4n} - xy = \frac{(x-ny)^2}{4n} \geq 0.$$

The equality holds for $a_1 = a_2 = \dots = a_n = 1$. If $k = \frac{n+1}{2\sqrt{n}}$, then the equality holds also for

$$a_1 = \frac{n+1}{2}, \quad a_2 = \dots = a_n = 1, \quad a_{n+1} = \dots = a_{2n} = \frac{n+1}{2n}.$$

□

P 4.6. Let $a_1, a_2, \dots, a_{2n} \neq -k$ be real numbers so that

$$a_1 \geq \dots \geq a_n \geq 1 \geq a_{n+1} \geq \dots \geq a_{2n}, \quad a_1 + a_2 + \dots + a_{2n} = 2n.$$

If $k \geq 1 + \frac{n+1}{\sqrt{n}}$, then

$$\frac{a_1^2 - 1}{(a_1 + k)^2} + \frac{a_2^2 - 1}{(a_2 + k)^2} + \dots + \frac{a_{2n}^2 - 1}{(a_{2n} + k)^2} \geq 0.$$

(Vasile C., 2012)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \dots + f(a_{2n}) \geq 2nf(s), \quad s = \frac{a_1 + a_2 + \dots + a_{2n}}{2n} = 1,$$

where

$$f(u) = \frac{u^2 - 1}{(u + k)^2}, \quad u \in \mathbb{I} = \mathbb{R} \setminus \{-k\}.$$

As shown in the proof of P 3.9, f is convex on $[s_0, 1]$, increasing for $u \geq s_0$ and

$$\min_{u \in \mathbb{I}} f(u) = f(s_0),$$

where

$$s_0 = \frac{-1}{k} \in (-1, 0).$$

According to Note 4, we may apply the LPCF-OV Theorem for $2n$ real numbers and $m = n$. Thus, we only need to show that $f(x) + nf(y) \geq (n+1)f(1)$ for $x, y \in \mathbb{I}$ so that $x + ny = n+1$. Using Note 1, it suffices to prove that $h(x, y) \geq 0$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{u + 1}{(u + k)^2},$$

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{(k-1)^2 - 1 - x - y - xy}{(x+k)^2(y+k)^2} \geq 0,$$

because

$$(k-1)^2 - 1 - x - y - xy \geq \frac{(n+1)^2}{n} - 1 - x - y - xy = \frac{(ny-1)^2}{n} \geq 0.$$

The equality holds for $a_1 = a_2 = \dots = a_n = 1$. If $k = 1 + \frac{n+1}{\sqrt{n}}$, then the equality holds also for

$$a_1 = n, \quad a_2 = \dots = a_n = 1, \quad a_{n+1} = \dots = a_{2n} = \frac{1}{n}.$$

□

P 4.7. If $a_1, a_2, \dots, a_n$ are positive real numbers so that

$$a_1 \geq 1 \geq a_2 \geq \dots \geq a_n, \quad a_1 + a_2 + \dots + a_n = n,$$

then

$$a_1^{3/a_1} + a_2^{3/a_2} + \dots + a_n^{3/a_n} \leq n.$$

(Vasile C., 2012)

Solution. Rewrite the desired inequality as

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \dots + a_n}{n} = 1,$$

where

$$f(u) = -u^{3/u}, \quad u \in \mathbb{I} = (0, n).$$

We have

$$\begin{aligned} f'(u) &= 3u^{\frac{3}{u}-2}(\ln u - 1), \\ f''(u) &= 3u^{\frac{3}{u}-4}g(u), \quad g(u) = u + (1 - \ln u)(2u - 3 + 3 \ln u). \end{aligned}$$

From the expression of f' , it follows that f is decreasing on $(0, s_0]$ and increasing on $[s_0, n)$, where

$$s_0 = e.$$

In addition, we claim that $f''(u) \geq 0$ for $u \in [1, e]$. If $u \in [3/2, e]$, then

$$g(u) > (1 - \ln u)(2u - 3) \geq 0.$$

Also, for $u \in [1, 3/2]$, we have

$$g(u) = 3(u - 1) + (6 - 2u - 3 \ln u) \ln u \geq (6 - 2u - 3 \ln u) \ln u \geq 3 \left(1 - \ln \frac{3}{2}\right) \ln u > 0.$$

Since f is convex on $[1, s_0]$, we may apply the RPCF-OV Theorem for $m = n - 1$. We only need to show that $f(x) + f(y) \geq 2f(1)$ for all $x, y > 0$ so that $x + y = 2$. The inequality $f(x) + f(y) \geq 2f(1)$ is equivalent to

$$x^{3/x} + y^{3/y} \leq 2,$$

which is just the inequality in P 3.32 from Volume 2. The equality holds for

$$a_1 = a_2 = \dots = a_n = 1.$$

□

P 4.8. If $a_1, a_2, \dots, a_{11}$ are real numbers so that

$$a_1 \geq a_2 \geq 1 \geq a_3 \geq \dots \geq a_{11}, \quad a_1 + a_2 + \dots + a_{11} = 11,$$

then

$$(1 - a_1 + a_1^2)(1 - a_2 + a_2^2) \dots (1 - a_{11} + a_{11}^2) \geq 1.$$

(Vasile C., 2012)

Solution. Rewrite the desired inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_{11}) \geq 11f(s), \quad s = \frac{a_1 + a_2 + \cdots + a_{11}}{11} = 1,$$

where

$$f(u) = \ln(1 - u + u^2), \quad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{2u - 1}{1 - u + u^2},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = 1/2.$$

Also, from

$$f''(u) = \frac{1 + 2u(1 - u)}{(1 - u + u^2)^2},$$

it follows that $f''(u) > 0$ for $u \in [s_0, 1]$, hence f is convex on $[s_0, 1]$. Therefore, applying the LPCF-OV Theorem for $n = 11$ and $m = 2$, we only need to show that $f(x) + 9f(y) \geq 9f(1)$ for all real x, y so that $x + 9y = 10$. Using Note 2, it suffices to prove that $H(x, y) \geq 0$, where

$$H(x, y) = \frac{f'(x) - f'(y)}{x - y} = \frac{1 + x + y - 2xy}{(1 - x + x^2)(1 - y + y^2)}.$$

Since

$$1 + x + y - 2xy = 18y^2 - 8y + 1 = 2y^2 + (4y - 1)^2 > 0,$$

the conclusion follows. The equality holds for $a_1 = a_2 = \cdots = a_{11} = 1$.

Remark. By replacing $a_1, a_2, \dots, a_{11}$ respectively with $1 - a_1, 1 - a_2, \dots, 1 - a_{11}$, we get the following statement.

- If $a_1, a_2, \dots, a_{11}$ are real numbers so that

$$a_1 \leq a_2 \leq 0 \leq a_3 \leq \cdots \leq a_{11}, \quad a_1 + a_2 + \cdots + a_{11} = 0,$$

then

$$(1 - a_1 + a_1^2)(1 - a_2 + a_2^2) \cdots (1 - a_{11} + a_{11}^2) \geq 1,$$

with equality for $a_1 = a_2 = \cdots = a_n = 0$.

□

P 4.9. If $a_1, a_2, \dots, a_8$ are nonzero real numbers so that

$$a_1 \geq a_2 \geq a_3 \geq a_4 \geq 1 \geq a_5 \geq a_6 \geq a_7 \geq a_8, \quad a_1 + a_2 + \cdots + a_8 = 8,$$

then

$$5 \left(\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_8^2} \right) + 72 \geq 14 \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_8} \right).$$

(Vasile C., 2012)

Solution. Write the desired inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_8) \geq 8f(s), \quad s = \frac{a_1 + a_2 + \cdots + a_8}{8} = 1,$$

where

$$f(u) = \frac{5}{u^2} - \frac{14}{u} + 9, \quad u \in \mathbb{I} = \mathbb{R} \setminus \{0\}.$$

As shown in the proof of P 3.25, f is convex on $[s_0, 1]$, increasing for $u \geq s_0$ and

$$\min_{u \in \mathbb{I}} f(u) = f(s_0),$$

where

$$s_0 = \frac{5}{7}.$$

Taking into account Note 4, we may apply the LPCF-OV Theorem for $n = 8$ and $m = 4$. We only need to show that $f(x) + 4f(y) \geq 5f(1)$ for $x, y \in \mathbb{I}$ so that $x + 4y = 5$. Using Note 1, it suffices to prove that $h(x, y) \geq 0$. Indeed, we have

$$\begin{aligned} g(u) &= \frac{f(u) - f(1)}{u - 1} = \frac{9}{u} - \frac{5}{u^2}, \\ h(x, y) &= \frac{g(x) - g(y)}{x - y} = \frac{5(x + y) - 9xy}{x^2y^2} \\ &= \frac{(x + 4y)(x + y) - 9xy}{x^2y^2} = \frac{(x - 2y)^2}{x^2y^2} \geq 0. \end{aligned}$$

In accordance with Note 3, the equality holds for $a_1 = a_2 = \cdots = a_8 = 1$, and also for

$$a_1 = \frac{5}{3}, \quad a_2 = a_3 = a_4 = 1, \quad a_5 = a_6 = a_7 = a_8 = \frac{5}{6}.$$

□

P 4.10. If a, b, c, d are positive real numbers so that

$$a \leq b \leq 1 \leq c \leq d, \quad abcd = 1,$$

then

$$\frac{7 - 6a}{2 + a^2} + \frac{7 - 6b}{2 + b^2} + \frac{7 - 6c}{2 + c^2} + \frac{7 - 6d}{2 + d^2} \geq \frac{4}{3}.$$

(Vasile C., 2012)

Solution. Using the substitution

$$a = e^x, \quad b = e^y, \quad c = e^z, \quad d = e^w,$$

we need to show that

$$f(x) + f(y) + f(z) + f(w) \geq 4f(s),$$

where

$$x \leq y \leq 0 \leq z \leq w, \quad s = \frac{x + y + z + w}{4} = 0,$$

$$f(u) = \frac{7 - 6e^u}{2 + e^{2u}}, \quad u \in \mathbb{R}.$$

As shown in the proof of P 3.26, f is convex on $[0, s_0]$, is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = \ln 3.$$

Therefore, we may apply the RPCF-OV Theorem for $n = 4$ and $m = 2$. We only need to show that $f(x) + 2f(y) \geq 3f(0)$ for all real x, y so that $x + 2y = 0$; that is, to prove that

$$\frac{7 - 6a}{2 + a^2} + \frac{2(7 - 6d)}{2 + d^2} \geq 1$$

for $a, d > 0$ so that $ad^2 = 1$. This is equivalent to

$$(d - 1)^2(d - 2)^2(5d^2 + 6d + 3) \geq 0,$$

which is clearly true. In accordance with Note 3, the equality holds for $a = b = c = d = 1$, and also for

$$a = \frac{1}{4}, \quad b = 1, \quad c = d = 2.$$

□

P 4.11. *If a, b, c are positive real numbers so that*

$$a \leq b \leq 1 \leq c, \quad abc = 1,$$

then

$$\frac{7 - 4a}{2 + a^2} + \frac{7 - 4b}{2 + b^2} + \frac{7 - 4c}{2 + c^2} \geq 3.$$

(Vasile C., 2012)

Solution. Using the substitution

$$a = e^x, \quad b = e^y, \quad c = e^z,$$

we need to show that

$$f(x) + f(y) + f(z) \geq 3f(s),$$

where

$$x \leq y \leq 0 \leq z, \quad s = \frac{x + y + z}{3} = 0,$$

$$f(u) = \frac{7 - 4e^u}{2 + e^{2u}}, \quad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{2e^u(2e^u + 1)(e^u - 4)}{(2 + e^{2u})^2},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = \ln 4.$$

Also, we have

$$f''(u) = \frac{4t \cdot h(t)}{(2 + t^2)^3}, \quad t = e^u,$$

where

$$h(t) = -t^4 + 7t^3 + 12t^2 - 14t - 4.$$

We will show that $h(t) \geq 0$ for $t \in [1, 4]$, hence f is convex on $[0, s_0]$. Indeed,

$$h(t) = (t - 1)[t^2(-t + 6) + 18t + 4] \geq 0.$$

Therefore, we may apply the RPCF-OV Theorem for $n = 3$ and $m = 2$. We only need to show that $f(x) + f(y) \geq 2f(0)$ for all real x, y so that $x + y = 0$. That is, to prove that

$$\frac{7 - 4a}{2 + a^2} + \frac{7 - 4b}{2 + b^2} \geq 2$$

for all $a, b > 0$ so that $ab = 1$. This is equivalent to

$$(a - 1)^4 \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 4.12. *If a, b, c are positive real numbers so that*

$$a \geq 1 \geq b \geq c, \quad abc = 1,$$

then

$$\frac{23 - 8a}{3 + 2a^2} + \frac{23 - 8b}{3 + 2b^2} + \frac{23 - 8c}{3 + 2c^2} \geq 9.$$

(Vasile C., 2012)

Solution. Using the substitution

$$a = e^x, \quad b = e^y, \quad c = e^z,$$

we need to show that

$$f(x) + f(y) + f(z) \geq 3f(s),$$

where

$$x \geq 1 \geq y \geq z, \quad s = \frac{x + y + z}{3} = 0,$$

$$f(u) = \frac{23 - 8e^u}{3 + 2e^{2u}}, \quad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{4e^u(4e^u + 1)(e^u - 6)}{(3 + 2e^{2u})^2},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where $s_0 = \ln 6$. Also, we have

$$f''(u) = \frac{8t \cdot h(t)}{(3 + 2t^2)^3}, \quad t = e^u,$$

where

$$h(t) = -4t^4 + 46t^3 + 36t^2 - 69t - 9.$$

We will show that $h(t) \geq 0$ for $t \in [1, 6]$, hence f is convex on $[0, s_0]$. Indeed,

$$h(t) = (t - 1)(2t + 3)[2t(-t + 12) + 3] \geq 0.$$

Therefore, we may apply the RPCF-OV Theorem for $n = 3$ and $m = 2$. We only need to show that $f(x) + f(y) \geq 2f(0)$ for all real x, y so that $x + y = 0$. That is, to prove that

$$\frac{23 - 8a}{3 + 2a^2} + \frac{23 - 8b}{3 + 2b^2} \geq 6.$$

for all $a, b > 0$ so that $ab = 1$. This is equivalent to

$$(a - 1)^4 \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 4.13. Let $a_1, a_2, \dots, a_n$ be positive real numbers so that

$$a_1 \leq \dots \leq a_{n-1} \leq 1 \leq a_n, \quad a_1 a_2 \cdots a_n = 1.$$

If $p, q \geq 0$ so that $p + 3q \geq 1$, then

$$\frac{1 - a_1}{1 + pa_1 + qa_1^2} + \frac{1 - a_2}{1 + pa_2 + qa_2^2} + \dots + \frac{1 - a_n}{1 + pa_n + qa_n^2} \geq 0.$$

(Vasile C., 2012)

Solution. For $q = 0$, we need to show that $p \geq 1$ involves

$$\frac{1 - a_1}{1 + pa_1} + \frac{1 - a_2}{1 + pa_2} + \dots + \frac{1 - a_n}{1 + pa_n} \geq 0.$$

This is just the inequality from P 2.25. Consider next that $q > 0$. Using the substitutions $a_i = e^{x_i}$ for $i = 1, 2, \dots, n$, we need to show that

$$f(x_1) + f(x_2) + \dots + f(x_n) \geq nf(s),$$

where

$$x_1 \leq \dots \leq x_{n-1} \leq 0 \leq x_n, \quad s = \frac{x_1 + x_2 + \dots + x_n}{n} = 0,$$

$$f(u) = \frac{1 - e^u}{1 + pe^u + qe^{2u}}, \quad u \in \mathbb{R}.$$

As shown in the proof of P 3.30, if $p + 3q - 1 \geq 0$, then f is convex on $[0, s_0]$, where

$$s_0 = \ln r_0 > 0, \quad r_0 = 1 + \sqrt{1 + \frac{p+1}{q}}.$$

In addition, f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$. Therefore, we may apply the RPCF-OV Theorem for $m = n - 1$. We only need to show that $f(x) + f(y) \geq 2f(0)$ for all real x, y so that $x + y = 0$; that is, to prove that

$$\frac{1-a}{1+pa+qa^2} + \frac{1-b}{1+pb+qb^2} \geq 0$$

for $a, b > 0$ so that $ab = 1$. This is equivalent to

$$(a-1)^2[(p-1)a + q(a^2 + a + 1)] \geq 0,$$

which is true because

$$(p-1)a + q(a^2 + a + 1) \geq (p-1)a + q(3a) = (p+3q-1)a \geq 0.$$

The equality holds for $a_1 = a_2 = \dots = a_n = 1$.

□

P 4.14. If a, b, c, d, e are real numbers so that

$$-2 \leq a \leq b \leq 1 \leq c \leq d \leq e, \quad a + b + c + d + e = 5,$$

then

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{1}{e^2} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e}.$$

Solution. Write the inequality as

$$f(a) + f(b) + f(c) + f(d) + f(e) \geq 5f(s), \quad s = \frac{a+b+c+d+e}{5} = 1,$$

where

$$f(u) = \frac{1}{u^2} - \frac{1}{u}, \quad u \in \mathbb{I} = [-2, 7] \setminus \{0\}.$$

Let

$$s_0 = 2, \quad s < s_0.$$

From

$$f(u) - f(2) = \frac{1}{u^2} - \frac{1}{u} + \frac{1}{4} = \frac{(u-2)^2}{4u^2} \geq 0,$$

it follows that

$$\min_{u \in \mathbb{I}} f(u) = f(s_0),$$

while from

$$f'(u) = \frac{u-2}{u^3}, \quad f''(u) = \frac{2(3-u)}{u^4},$$

it follows that f is convex on $[s, s_0]$. We can't apply the the RPCF-OV Theorem because f is not decreasing on $\mathbb{I}_{\leq s_0}$. According to Theorem 1 (applied for $n = 5$ and $m = 2$) and Note 6, we may replace this condition with $(1 + n - m)s - (n - m)s_0 \leq \inf \mathbb{I}$. Indeed, we have

$$(1 + n - m)s - (n - m)s_0 = 4 - 6 = -2 = \inf \mathbb{I}.$$

So, according to Note 1, it suffices to show that $h(x, y) \geq 0$ for all $x, y \in \mathbb{I}$ so that $x + 3y = 4$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{-1}{u^2},$$

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{x + y}{x^2 y^2} = \frac{2(x + 2)}{3x^2 y^2} \geq 0.$$

The proof is completed. By Note 3, the equality holds for $a = b = c = d = e = 1$, and also for

$$a = -2, \quad b = 1, \quad c = d = e = 2.$$

□

Chapter 5

EV Method for Nonnegative Variables

5.1 Theoretical Basis

The Equal Variables Method is an effective tool for solving some difficult symmetric inequalities.

EV-Theorem (Vasile Cîrtoaje, 2005). Let $x_1, x_2, \dots, x_n$ ($n \geq 3$) be nonnegative real numbers such that $x_1 \leq x_2 \leq \dots \leq x_n$, and let f be a real-valued function, continuous on $[0, \infty)$ and differentiable on $(0, \infty)$, so that the joined function

$$g(x) = f' \left(x^{\frac{1}{k-1}} \right)$$

is strictly convex on $(0, \infty)$. For fixed $x_1 + x_2 + \dots + x_n$ and $x_1^k + x_2^k + \dots + x_n^k$, where $k \neq 1$ ($k = 0$ means that the product $x_1 x_2 \dots x_n$ is fixed), the sum

$$S_n = f(x_1) + f(x_2) + \dots + f(x_n)$$

has the maximum value only for

$$x_1 = x_2 = \dots = x_{n-1} \leq x_n$$

and the minimum value only for $x_1 = 0$ or $0 < x_1 \leq x_2 = x_3 = \dots = x_n$.

To prove the EV-Theorem, we need the EV-Lemma and the EV-Proposition below.

EV-Lemma. Let x, y, z be nonnegative real numbers such that $x \leq y \leq z$. For fixed $x + y + z$ and $x^k + y^k + z^k$, where $k \neq 1$ ($k = 0$ means that the product xyz is fixed), the range of y is an interval $[m, M]$; in addition,

- (1) $y = m$ if and only if $x = y \leq z$;
- (2) $y = M$ if and only if $x = 0$ or $0 < x \leq y = z$.

Proof. We may consider that x and z are functions of y . From

$$x' + z' = -1, \quad x^{k-1}x' + z^{k-1}z' = -y^{k-1},$$

we get

$$x' = \frac{y^{k-1} - z^{k-1}}{z^{k-1} - x^{k-1}} \leq 0, \quad z' = \frac{y^{k-1} - x^{k-1}}{x^{k-1} - z^{k-1}} \leq 0. \quad (*)$$

Let us define the nonnegative functions

$$f_1(y) = y - x(y), \quad f_2(y) = z(y) - y, \quad f_3(y) = x(y).$$

Since

$$f'_1(y) = 1 - x'(y) > 0, \quad f'_2(y) = z'(y) - 1 < 0, \quad f'_3(y) = x'(y) \leq 0,$$

these functions are strictly increasing, decreasing and decreasing, respectively. Thus, the inequality $f_1(y) \geq 0$ (with f_1 increasing) involves $y \geq m$, where m is a root of the equation $x(y) = y$, and the inequality $f_2(y) \geq 0$ (with f_2 decreasing) involves $y \leq y_2$, where y_2 is a root of the equation $z(y) = y$. If $x(y_2) \geq 0$, then y_2 is the maximum value M of y . Otherwise, the maximum value of y is given by the inequality $f_3(y) \geq 0$ (with f_3 decreasing), which involves $y \leq M$, where M is a root of the equation $x(y) = 0$. Therefore, $y \in [m, M]$, with $y = m$ for $x = y$, and $y = M$ for either $y = z$ or $x = 0$.

EV-Proposition. *Let x, y, z be nonnegative real numbers such that $x \leq y \leq z$, and let f be a real-valued function, continuous on $[0, \infty)$ and differentiable on $(0, \infty)$, so that the joined function*

$$g(x) = f'\left(x^{\frac{1}{k-1}}\right)$$

is strictly convex on $(0, \infty)$. For fixed $x + y + z$ and $x^k + y^k + z^k$, where $k \neq 1$ ($k = 0$ means that the product xyz is fixed), the sum

$$S_3 = f(x) + f(y) + f(z)$$

has the maximum value only when $x = y \leq z$, and the minimum value only when $x = 0$ or $0 < x \leq y = z$.

Proof. We may consider x and z as functions of y . Thus, we have

$$S_3 = f(x(y)) + f(y) + f(z(y)) := F(y).$$

According to the EV-Lemma, it suffices to show that F is maximum for $y = m$ and is minimum for $y = M$. Using (*), we have

$$\begin{aligned} F'(y) &= x'f'(x) + f'(y) + z'f'(z) \\ &= \frac{y^{k-1} - z^{k-1}}{z^{k-1} - x^{k-1}}g(x^{k-1}) + g(y^{k-1}) + \frac{y^{k-1} - x^{k-1}}{x^{k-1} - z^{k-1}}g(z^{k-1}), \end{aligned}$$

which, for $x < y < z$, is equivalent to

$$\begin{aligned} \frac{F'(y)}{(y^{k-1} - x^{k-1})(y^{k-1} - z^{k-1})} &= \frac{g(x^{k-1})}{(x^{k-1} - y^{k-1})(x^{k-1} - z^{k-1})} \\ &+ \frac{g(y^{k-1})}{(y^{k-1} - z^{k-1})(y^{k-1} - x^{k-1})} + \frac{g(z^{k-1})}{(z^{k-1} - x^{k-1})(z^{k-1} - y^{k-1})}. \end{aligned}$$

Since g is strictly convex, the right hand side is positive. Moreover, since

$$(y^{k-1} - x^{k-1})(y^{k-1} - z^{k-1}) < 0,$$

we have $F'(y) < 0$ for $y \in (m, M)$ (see the EV-Lemma), hence F is strictly decreasing on $[m, M]$. Therefore, F is maximum for $y = m$ (when $x = y \leq z$) and is minimum for $y = M$ (when $x = 0$ or $0 < x \leq y = z$).

Proof of the EV-Theorem. Since $X = \{x_1, x_2, \dots, x_n\}$ is defined as a compact set in $\mathbf{R}_n^+$, S_n attains its minimum and maximum. For $n = 3$, the EV-Theorem follows immediately from the EV-Proposition. To prove the theorem for $n \geq 4$, we use the contradiction method.

(a) For the sake of contradiction, assume that the sum S_n has the maximum value when $x_1 < x_{n-1}$. According to the EV-Proposition, the sum $f(x_1) + f(x_{n-1}) + f(x_n)$ increases by replacing the set (x_1, x_{n-1}, x_n) with the set (y_1, y_{n-1}, y_n) such that $y_1 = y_{n-1} < y_n$ and

$$y_1 + y_{n-1} + y_n = x_1 + x_{n-1} + x_n, \quad y_1^2 + y_{n-1}^2 + y_n^2 = x_1^2 + x_{n-1}^2 + x_n^2,$$

which is a contradiction.

(b) Similarly, we can prove that S_n is minimum for $n \geq 4$ when either $x_1 = 0$ or

$$0 < x_1 \leq x_2 = \dots = x_n.$$

Corollary 1. Let $x_1, x_2, \dots, x_n$ ($n \geq 3$) be nonnegative real numbers such that $x_1 \leq x_2 \leq \dots \leq x_n$, and let f be a real-valued function, continuous on $[0, \infty)$ and differentiable on $(0, \infty)$, so that the joined function

$$g(x) = f'(x)$$

is strictly convex on $(0, \infty)$. For fixed $x_1 + x_2 + \dots + x_n$ and $x_1^2 + x_2^2 + \dots + x_n^2$, the sum

$$S_n = f(x_1) + f(x_2) + \dots + f(x_n)$$

has the maximum value only when

$$x_1 = x_2 = \dots = x_{n-1} \leq x_n,$$

and the minimum value only when $x_1 = 0$ or $0 < x_1 \leq x_2 = x_3 = \dots = x_n$.

Corollary 2. Let $x_1, x_2, \dots, x_n$ ($n \geq 3$) be positive real numbers such that $x_1 \leq x_2 \leq \dots \leq x_n$, and let f be a real-valued function, continuous and differentiable on $(0, \infty)$, so that the joined function

$$g(x) = f' \left(\frac{1}{\sqrt{x}} \right)$$

is strictly convex on $(0, \infty)$. For fixed $x_1 + x_2 + \dots + x_n$ and $\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n}$, the sum

$$S_n = f(x_1) + f(x_2) + \dots + f(x_n)$$

has the maximum value only when

$$x_1 = x_2 = \dots = x_{n-1} \leq x_n,$$

and the minimum value only when

$$x_1 \leq x_2 = x_3 = \dots = x_n.$$

Corollary 3. Let $x_1, x_2, \dots, x_n$ ($n \geq 3$) be nonnegative real numbers such that $x_1 \leq x_2 \leq \dots \leq x_n$, and let f be a real-valued function, continuous on $[0, \infty)$ and differentiable on $(0, \infty)$, so that the joined function

$$g(x) = f'(1/x)$$

is strictly convex on $(0, \infty)$. For fixed $x_1 + x_2 + \dots + x_n$ and $x_1 x_2 \dots x_n$, the sum

$$S_n = f(x_1) + f(x_2) + \dots + f(x_n)$$

has the maximum value only when

$$x_1 = x_2 = \dots = x_{n-1} \leq x_n,$$

and the minimum value only when $x_1 = 0$ or $0 < x_1 \leq x_2 = x_3 = \dots = x_n$.

Corollary 4. Suppose that $x_1, x_2, \dots, x_n$ ($n \geq 3$) are nonnegative real numbers such that $x_1 \leq x_2 \leq \dots \leq x_n$, and the sums $x_1 + x_2 + \dots + x_n$ and $x_1^k + x_2^k + \dots + x_n^k$ are fixed, where k is a real number ($k \neq 1$).

(1) For $k < 0$, the product $P_n = x_1 x_2 \dots x_n$ has the maximum value only when

$$0 < x_1 \leq x_2 = x_3 = \dots = x_n,$$

and the minimum value only when

$$0 < x_1 = x_2 = \dots = x_{n-1} \leq x_n.$$

(2) For $k > 0$, the product $P_n = x_1 x_2 \dots x_n$ has the maximum value only when

$$x_1 = x_2 = \dots = x_{n-1} \leq x_n,$$

and the minimum value only when $x_1 = 0$ or $0 < x_1 \leq x_2 = x_3 = \dots = x_n$.

Note 1. The EV-Theorem, Corollary 1 and Corollary 3 are also valid when $x_1, x_2, \dots, x_n > 0$, f is continuous and differentiable on $(0, \infty)$, $f(0+) = \pm\infty$ and the sum S_n has a global maximum (minimum).

From the EV-Theorem and Note 1, we can obtain some interesting particular results, which are useful in many applications.

Corollary 5. Suppose that $x_1, x_2, \dots, x_n$ are nonnegative real numbers such that $x_1 \leq x_2 \leq \dots \leq x_n$, and the sums $x_1 + x_2 + \dots + x_n$ and $x_1^k + x_2^k + \dots + x_n^k$ are fixed, where k is a real number. Let us denote

$$S_n = x_1^m + x_2^m + \dots + x_n^m.$$

Case 1: $k > 1$.

(a) If $m \in (0, 1) \cup (k, \infty)$, then S_n has the maximum value only for

$$0 \leq x_1 = x_2 = \dots = x_{n-1} \leq x_n,$$

and the minimum value only for either $x_1 = 0$ or $0 < x_1 \leq x_2 = x_3 = \cdots = x_n$.

(b) If $m \in (-\infty, 0)$, then S_n has the minimum value only for

$$0 < x_1 = x_2 = \cdots = x_{n-1} \leq x_n,$$

and the maximum value (if there is a global maximum) only for

$$0 < x_1 \leq x_2 = x_3 = \cdots = x_n.$$

(c) If $m \in (1, k)$, then S_n has the minimum value only for

$$0 \leq x_1 = x_2 = \cdots = x_{n-1} \leq x_n,$$

and the maximum value only for either $x_1 = 0$ or $0 < x_1 \leq x_2 = x_3 = \cdots = x_n$.

Case 2 : $0 \leq k < 1$ ($k = 0$ means $x_1 x_2 \cdots x_n = a_1 a_2 \cdots a_n$).

(a) If $m \in (0, k) \cup (1, \infty)$, then S_n has the maximum value only for

$$0 \leq x_1 = x_2 = \cdots = x_{n-1} \leq x_n,$$

and the minimum value only for either $x_1 = 0$ or $0 < x_1 \leq x_2 = x_3 = \cdots = x_n$.

(b) If $m \in (-\infty, 0)$, then S_n has the minimum value only for

$$0 < x_1 = x_2 = \cdots = x_{n-1} \leq x_n,$$

and the maximum value (if there is a global maximum) only for

$$0 < x_1 \leq x_2 = x_3 = \cdots = x_n.$$

(c) If $m \in (k, 1)$, then S_n has the minimum value only for

$$0 \leq x_1 = x_2 = \cdots = x_{n-1} \leq x_n,$$

and the maximum value only for either $x_1 = 0$ or $0 < x_1 \leq x_2 = x_3 = \cdots = x_n$.

Case 3 : $k < 0$.

(a) If $m \in (k, 0) \cup (1, \infty)$, then S_n has the maximum value only for

$$0 < x_1 = x_2 = \cdots = x_{n-1} \leq x_n,$$

and the minimum value only for

$$0 < x_1 \leq x_2 = x_3 = \cdots = x_n.$$

(b) If $m \in (-\infty, k) \cup (0, 1)$, then S_n has the minimum value only for

$$0 < x_1 = x_2 = \cdots = x_{n-1} \leq x_n,$$

and the maximum value only for

$$0 < x_1 \leq x_2 = x_3 = \cdots = x_n.$$

Proof. We apply the EV-Theorem and Note 1 to the function

$$f(u) = m(m-1)(m-k)u^m.$$

We have

$$f'(u) = m^2(m-2)(m-k)u^{m-1}$$

and

$$g(x) = m^2(m-1)(m-k)x^{\frac{m-1}{k-1}}, \quad g''(x) = \frac{m^2(m-1)^2(m-k)^2}{(k-1)^2} x^{\frac{1+m-2k}{k-1}}.$$

Since $g''(x) > 0$ for $x > 0$, g is strictly convex on $(0, \infty)$.

Corollary 6. Suppose that $x_1, x_2, \dots, x_n$ ($n \geq 3$) are nonnegative real numbers such that $x_1 \leq x_2 \leq \cdots \leq x_n$, and the sums $x_1^p + x_2^p + \cdots + x_n^p$ and $x_1^q + x_2^q + \cdots + x_n^q$ are fixed, where

$$p, q \in \{1, 2, 3\}, \quad p \neq q.$$

The symmetric sum

$$S_n = \sum_{1 \leq i_1 < i_2 < i_3 \leq n} x_{i_1} x_{i_2} x_{i_3}$$

has the maximum value only for

$$0 \leq x_1 = x_2 = \cdots = x_{n-1} \leq x_n,$$

and the minimum value only for $x_1 = 0$ or $0 < x_1 \leq x_2 = x_3 = \cdots = x_n$.

Proof. Taking into account that

$$6 \sum_{1 \leq i_1 < i_2 < i_3 \leq n} x_{i_1} x_{i_2} x_{i_3} = \left(\sum x_1 \right)^3 - 3 \left(\sum x_1 \right) \left(\sum x_1^2 \right) + 2 \sum x_1^3,$$

Corollary 6 is a consequence of Corollary 5. For $p = 2$ and $q = 3$, according to this identity, the sum $\sum_{1 \leq i_1 < i_2 < i_3 \leq n} x_{i_1} x_{i_2} x_{i_3}$ is maximum/minimum when $\sum x_1$ is maximum/minimum. Therefore,

we need to show that for fixed $x_1^2 + x_2^2 + \cdots + x_n^2$ and $x_1^3 + x_2^3 + \cdots + x_n^3$, the sum $\sum x_1$ has the maximum value when

$$0 \leq x_1 = x_2 = \cdots = x_{n-1} \leq x_n,$$

and the minimum value when either $x_1 = 0$ or $0 < x_1 \leq x_2 = x_3 = \cdots = x_n$. This result follows by replacing $x_1, x_2, \dots, x_n$ with $x_1^2, x_2^2, \dots, x_n^2$ in Corollary 5, case $k = 3/2$ and $m = 1/2$.

Note 2. The EV-Theorem and Corollaries 1-3 can be extended to the cases where:

(a) $x_1, x_2, \dots, x_n \geq m \geq 0$, f is continuous on $[m, \infty)$ and differentiable on (m, ∞) , and $g(x)$ is strictly convex for $x^{\frac{1}{k-1}} > m$; so, the sum

$$S_n = f(x_1) + f(x_2) + \cdots + f(x_n)$$

has the maximum value for $x_1 = x_2 = \cdots = x_{n-1} \leq x_n$, and the minimum value for either $x_1 = m$ or $m < x_1 \leq x_2 = x_3 = \cdots = x_n$;

(b) $0 \leq x_1, x_2, \dots, x_n \leq M$, f is continuous on $[0, M]$ and differentiable on $(0, M)$, and $g(x)$ is strictly convex for $x^{\frac{1}{k-1}} < M$; so, the sum

$$S_n = f(x_1) + f(x_2) + \cdots + f(x_n)$$

has the maximum value for either $x_n = M$ or $x_1 = x_2 = \cdots = x_{n-1} \leq x_n$, and the minimum value for $x_1 \leq x_2 = x_3 = \cdots = x_n$;

Note 3. The EV-Theorem and Corollaries 1-3 can be extended to the cases where:

(a) $x_1, x_2, \dots, x_n > m \geq 0$, f is continuous and differentiable on (m, ∞) , $f(m+) = \pm\infty$, $g(x)$ is strictly convex for $x^{\frac{1}{k-1}} > m$ and the sum S_n has a global maximum (minimum);

(b) $0 \leq x_1, x_2, \dots, x_n < M$, f is continuous and differentiable on $[0, M)$, $f(M-) = \pm\infty$, $g(x)$ is strictly convex for $x^{\frac{1}{k-1}} < M$ and the sum S_n has a global maximum (minimum).

The following theorems are also true (see [25]).

Theorem 1. Let $x_1, x_2, \dots, x_n$ be nonnegative real numbers such that $x_1 \leq x_2 \leq \cdots \leq x_n$, and let m be a positive real number such that $1 < m \leq 3$ for $n = 3$, and $n - 2 \leq m \leq n$ for $n \geq 4$. If

$$\sum_{i=1}^n \frac{1}{(n-m)x_i + m} = 1 \text{ and } x_1 + x_2 + \cdots + x_n \text{ is fixed, then:}$$

(1) the product $P_n = x_1 x_2 \cdots x_n$ has the maximum value only when $x_1 = x_2 = \cdots = x_{n-1} \leq x_n$;

(2) the product $P_n = x_1 x_2 \cdots x_n$ has the minimum value only when $x_1 = 0$ or $0 < x_1 \leq x_2 = x_3 = \cdots = x_n$.

Theorem 2. Let $x_1, x_2, \dots, x_n$ be nonnegative real numbers such that $x_1 \leq x_2 \leq \cdots \leq x_n$, and let m be a positive real number such that $0 < m < 1$ for $n = 3$, and $0 < m \leq 1$ for $n \geq 4$. If

$$\sum_{i=1}^n \frac{1}{(n-m)x_i + m} = 1 \text{ and } x_1 + x_2 + \cdots + x_n \text{ is fixed, then:}$$

(1) the product $P_n = x_1 x_2 \cdots x_n$ has the maximum value only when $x_1 \leq x_2 = x_3 = \cdots = x_n$;

(2) the product $P_n = x_1 x_2 \cdots x_n$ has the minimum value only when $x_1 = x_2 = \cdots = x_{n-1} \leq x_n$.

5.2 Applications

5.1. If a, b, c, d are nonnegative real numbers so that

$$a + b + c + d = a^3 + b^3 + c^3 + d^3 = 2,$$

then

$$\frac{7}{4} \leq a^2 + b^2 + c^2 + d^2 \leq 2.$$

5.2. If $a_1, a_2, \dots, a_9$ are nonnegative real numbers so that

$$a_1 + a_2 + \dots + a_9 = a_1^2 + a_2^2 + \dots + a_9^2 = 3,$$

then

$$3 \leq a_1^3 + a_2^3 + \dots + a_9^3 \leq \frac{14}{3}.$$

5.3. If a, b, c, d are nonnegative real numbers so that

$$a + b + c + d = a^2 + b^2 + c^2 + d^2 = \frac{27}{7},$$

then

$$\frac{5427}{1372} \leq a^3 + b^3 + c^3 + d^3 \leq \frac{1377}{343}.$$

5.4. If a, b, c are positive real numbers so that $abc = 1$, then

$$a^5 + b^5 + c^5 \geq \sqrt{3(a^7 + b^7 + c^7)}.$$

5.5. If a, b, c, d are positive real numbers so that $abcd = 1$, then

$$a^3 + b^3 + c^3 + d^3 \geq \sqrt{4(a^4 + b^4 + c^4 + d^4)}.$$

5.6. If a, b, c, d are nonnegative real numbers so that $a + b + c + d = 4$, then

$$\frac{bcd}{11a + 16} + \frac{cda}{11b + 16} + \frac{dab}{11c + 16} + \frac{abc}{11d + 16} \leq \frac{4}{27}.$$

5.7. If a, b, c are real numbers, then

$$\frac{bc}{3a^2 + b^2 + c^2} + \frac{ca}{3b^2 + c^2 + a^2} + \frac{ab}{3c^2 + a^2 + b^2} \leq \frac{3}{5}.$$

5.8. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$\begin{aligned} \text{(a)} \quad & \frac{bc}{a^2 + 2} + \frac{ca}{b^2 + 2} + \frac{ab}{c^2 + 2} \leq \frac{9}{8}; \\ \text{(b)} \quad & \frac{bc}{a^2 + 3} + \frac{ca}{b^2 + 3} + \frac{ab}{c^2 + 3} \leq \frac{11\sqrt{33} - 45}{24}; \\ \text{(c)} \quad & \frac{bc}{a^2 + 4} + \frac{ca}{b^2 + 4} + \frac{ab}{c^2 + 4} \leq \frac{3}{5}. \end{aligned}$$

5.9. If a, b, c, d are nonnegative real numbers so that

$$(3a + 1)(3b + 1)(3c + 1)(3d + 1) = 64,$$

then

$$abc + bcd + cda + dab \leq 1.$$

5.10. If $a_1, a_2, \dots, a_n$ and p, q are nonnegative real numbers so that

$$a_1 + a_2 + \dots + a_n = p + q, \quad a_1^3 + a_2^3 + \dots + a_n^3 = p^3 + q^3,$$

then

$$a_1^2 + a_2^2 + \dots + a_n^2 \leq p^2 + q^2.$$

5.11. If a, b, c are nonnegative real numbers, then

$$a\sqrt{a^2 + 4b^2 + 4c^2} + b\sqrt{b^2 + 4c^2 + 4a^2} + c\sqrt{c^2 + 4a^2 + 4b^2} \geq (a + b + c)^2.$$

5.12. If a, b, c are nonnegative real numbers so that $ab + bc + ca = 3$, then

$$\frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} \leq \frac{3}{2(a + b + c)} + \frac{a + b + c}{3}.$$

5.13. If a, b, c are nonnegative real numbers so that $ab + bc + ca = 3$, then

$$\frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} \geq \frac{3}{a + b + c} + \frac{a + b + c}{6}.$$

5.14. Let a, b, c be nonnegative real numbers, no two of which are zero. If

$$a^2 + b^2 + c^2 = 3,$$

then

$$\frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} + \frac{a + b + c}{9} \geq \frac{11}{2(a + b + c)}.$$

5.15. Let a, b, c be nonnegative real numbers, no two of which are zero. If

$$a + b + c = 4,$$

then

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{15}{8+ab+bc+ca}.$$

5.16. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{1}{a+b+c} + \frac{2}{\sqrt{ab+bc+ca}}.$$

5.17. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{3-\sqrt{3}}{a+b+c} + \frac{2+\sqrt{3}}{2\sqrt{ab+bc+ca}}.$$

5.18. Let a, b, c be nonnegative real numbers, no two of which are zero, so that

$$ab + bc + ca = 3.$$

If

$$0 \leq k \leq \frac{9+5\sqrt{3}}{6} \approx 2.943,$$

then

$$\frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{c+a} \geq \frac{9(1+k)}{a+b+c+3k}.$$

5.19. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{20}{a+b+c+6\sqrt{ab+bc+ca}}.$$

5.20. If a, b, c are positive real numbers so that

$$7(a^2 + b^2 + c^2) = 11(ab + bc + ca),$$

then

$$\frac{51}{28} \leq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \leq 2.$$

5.21. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that

$$\frac{a_1^2 + a_2^2 + \dots + a_n^2}{n+3} = \left(\frac{a_1 + a_2 + \dots + a_n}{n+1} \right)^2,$$

then

$$\frac{(n+1)(2n-1)}{2} \leq (a_1 + a_2 + \dots + a_n) \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right) \leq \frac{3n^2(n+1)}{2(n+2)}.$$

5.22. If a, b, c, d are nonnegative real numbers so that $a + b + c + d = 3$, then

$$abc + bcd + cda + dab \leq 1 + \frac{176}{81} abcd.$$

5.23. If a, b, c, d are nonnegative real numbers so that $a + b + c + d = 3$, then

$$a^2b^2c^2 + b^2c^2d^2 + c^2d^2a^2 + d^2a^2b^2 + \frac{3}{4}abcd \leq 1.$$

5.24. If a, b, c, d are nonnegative real numbers so that $a + b + c + d = 3$, then

$$a^2b^2c^2 + b^2c^2d^2 + c^2d^2a^2 + d^2a^2b^2 + \frac{4}{3}(abcd)^{3/2} \leq 1.$$

5.25. If a, b, c, d are nonnegative real numbers so that $a + b + c + d = 4$, then

$$a^2b^2c^2 + b^2c^2d^2 + c^2d^2a^2 + d^2a^2b^2 + 2(abcd)^{3/2} \leq 6.$$

5.26. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$11(ab + bc + ca) + 4(a^2b^2 + b^2c^2 + c^2a^2) \leq 45.$$

5.27. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$a^2b^2 + b^2c^2 + c^2a^2 + a^3b^3 + b^3c^3 + c^3a^3 \geq 6abc.$$

5.28. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$2(a^2 + b^2 + c^2) + 5(\sqrt{a} + \sqrt{b} + \sqrt{c}) \geq 21.$$

5.29. If a, b, c are nonnegative real numbers so that $ab + bc + ca = 3$, then

$$\sqrt{\frac{1+2a}{3}} + \sqrt{\frac{1+2b}{3}} + \sqrt{\frac{1+2c}{3}} \geq 3.$$

5.30. Let a, b, c be nonnegative real numbers, no two of which are zero. If

$$0 \leq k \leq 15,$$

then

$$\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} + \frac{k}{(a+b+c)^2} \geq \frac{9+k}{4(ab+bc+ca)}.$$

5.31. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} + \frac{24}{(a+b+c)^2} \geq \frac{8}{ab+bc+ca}.$$

5.32. If a, b, c are nonnegative real numbers, no two of which are zero, so that

$$k(a^2 + b^2 + c^2) + (2k+3)(ab+bc+ca) = 9(k+1), \quad 0 \leq k \leq 6,$$

then

$$\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} + \frac{9k}{(a+b+c)^2} \geq \frac{3}{4} + k.$$

5.33. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$(a) \quad \frac{2}{a^2+b^2} + \frac{2}{b^2+c^2} + \frac{2}{c^2+a^2} \geq \frac{8}{a^2+b^2+c^2} + \frac{1}{ab+bc+ca};$$

$$(b) \quad \frac{2}{a^2+b^2} + \frac{2}{b^2+c^2} + \frac{2}{c^2+a^2} \geq \frac{7}{a^2+b^2+c^2} + \frac{6}{(a+b+c)^2};$$

$$(c) \quad \frac{2}{a^2+b^2} + \frac{2}{b^2+c^2} + \frac{2}{c^2+a^2} \geq \frac{45}{4(a^2+b^2+c^2) + ab+bc+ca}.$$

5.34. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{1}{a^2+b^2} + \frac{1}{b^2+c^2} + \frac{1}{c^2+a^2} + \frac{3}{a^2+b^2+c^2} \geq \frac{4}{ab+bc+ca}.$$

5.35. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\begin{aligned} \text{(a)} \quad & \frac{3}{a^2 + ab + b^2} + \frac{3}{b^2 + bc + c^2} + \frac{3}{c^2 + ca + a^2} \geq \frac{5}{ab + bc + ca} + \frac{4}{a^2 + b^2 + c^2}; \\ \text{(b)} \quad & \frac{3}{a^2 + ab + b^2} + \frac{3}{b^2 + bc + c^2} + \frac{3}{c^2 + ca + a^2} \geq \frac{1}{ab + bc + ca} + \frac{24}{(a + b + c)^2}; \\ \text{(c)} \quad & \frac{1}{a^2 + ab + b^2} + \frac{1}{b^2 + bc + c^2} + \frac{1}{c^2 + ca + a^2} \geq \frac{21}{2(a^2 + b^2 + c^2) + 5(ab + bc + ca)}. \end{aligned}$$

5.36. Let f be a real-valued function, continuous on $[0, \infty)$ and differentiable on $(0, \infty)$, so that $f'''(u) \geq 0$ for $u \in (0, \infty)$. If $a, b, c \geq 0$, then

$$f(a^2 + 2bc) + f(b^2 + 2ca) + f(c^2 + 2ab) \leq f(a^2 + b^2 + c^2) + 2f(ab + bc + ca).$$

5.37. If a, b, c are the lengths of the side of a triangle, then

$$\frac{1}{(a + b)^2} + \frac{1}{(b + c)^2} + \frac{1}{(c + a)^2} \leq \frac{85}{36(ab + bc + ca)}.$$

5.38. If a, b, c are the lengths of the side of a triangle so that $a + b + c = 3$, then

$$\frac{1}{(a + b)^2} + \frac{1}{(b + c)^2} + \frac{1}{(c + a)^2} \leq \frac{3(a^2 + b^2 + c^2)}{4(ab + bc + ca)}.$$

5.39. Let $a, b, c \geq \frac{2}{5}$ so that $a + b + c = 3$. Then,

$$\frac{1}{3 + 2(a^2 + b^2)} + \frac{1}{3 + 2(b^2 + c^2)} + \frac{1}{3 + 2(c^2 + a^2)} \leq \frac{3}{7}.$$

5.40. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$\frac{2}{2 + a^2 + b^2} + \frac{2}{2 + b^2 + c^2} + \frac{2}{2 + c^2 + a^2} \leq \frac{99}{63 + a^2 + b^2 + c^2}.$$

5.41. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$\frac{1}{3 + a^2 + b^2} + \frac{1}{3 + b^2 + c^2} + \frac{1}{3 + c^2 + a^2} \leq \frac{18}{27 + a^2 + b^2 + c^2}.$$

5.42. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$\frac{5}{3 + a^2 + b^2} + \frac{5}{3 + b^2 + c^2} + \frac{5}{3 + c^2 + a^2} \geq \frac{27}{6 + a^2 + b^2 + c^2}.$$

5.43. If a, b, c, d are nonnegative real numbers so that $a + b + c + d = 4$, then

$$\sum \frac{3}{3 + 2(a^2 + b^2 + c^2)} \leq \frac{296}{218 + a^2 + b^2 + c^2 + d^2}.$$

5.44. If a, b, c are nonnegative real numbers so that $ab + bc + ca = 3$, then

$$\frac{4}{2 + a^2 + b^2} + \frac{4}{2 + b^2 + c^2} + \frac{4}{2 + c^2 + a^2} \geq \frac{21}{4 + a^2 + b^2 + c^2}.$$

5.45. If a, b, c are nonnegative real numbers so that $a^2 + b^2 + c^2 = 3$, then

$$\frac{1}{10 - (a + b)^2} + \frac{1}{10 - (b + c)^2} + \frac{1}{10 - (c + a)^2} \leq \frac{1}{2}.$$

5.46. If a, b, c are nonnegative real numbers, no two of which are zero, so that $a^4 + b^4 + c^4 = 3$, then

$$\frac{1}{a^5 + b^5} + \frac{1}{b^5 + c^5} + \frac{1}{c^5 + a^5} \geq \frac{3}{2}.$$

5.47. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$, then

$$\sqrt{a_1^2 + 1} + \sqrt{a_2^2 + 1} + \dots + \sqrt{a_n^2 + 1} \geq \sqrt{2 \left(1 - \frac{1}{n}\right) (a_1^2 + a_2^2 + \dots + a_n^2) + 2(n^2 - n + 1)}.$$

5.48. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$, then

$$\sum \sqrt{(3n - 4)a_1^2 + n} \geq \sqrt{(3n - 4)(a_1^2 + a_2^2 + \dots + a_n^2) + n(4n^2 - 7n + 4)}.$$

5.49. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$\sqrt{a^2 + 4} + \sqrt{b^2 + 4} + \sqrt{c^2 + 4} \leq \sqrt{\frac{8}{3}(a^2 + b^2 + c^2) + 37}.$$

5.50. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$\sqrt{32a^2 + 3} + \sqrt{32b^2 + 3} + \sqrt{32c^2 + 3} \leq \sqrt{32(a^2 + b^2 + c^2) + 219}.$$

5.51. If $a_1, a_2, \dots, a_n$ are positive real numbers so that $a_1 + a_2 + \dots + a_n = n$, then

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} + \frac{2n\sqrt{n-1}}{a_1^2 + a_2^2 + \dots + a_n^2} \geq n + 2\sqrt{n-1}.$$

5.52. If $a, b, c \in [0, 1]$, then

$$(1 + 3a^2)(1 + 3b^2)(1 + 3c^2) \geq (1 + ab + bc + ca)^3.$$

5.53. If a, b, c are nonnegative real numbers so that $a + b + c = ab + bc + ca$, then

$$\frac{1}{4 + 5a^2} + \frac{1}{4 + 5b^2} + \frac{1}{4 + 5c^2} \geq \frac{1}{3}.$$

5.54. If a, b, c, d are positive real numbers so that $a + b + c + d = 4abcd$, then

$$\frac{1}{1 + 3a} + \frac{1}{1 + 3b} + \frac{1}{1 + 3c} + \frac{1}{1 + 3d} \geq 1.$$

5.55. If $a_1, a_2, \dots, a_n$ are positive real numbers so that

$$a_1 + a_2 + \dots + a_n = \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n},$$

then

$$\frac{1}{1 + (n-1)a_1} + \frac{1}{1 + (n-1)a_2} + \dots + \frac{1}{1 + (n-1)a_n} \geq 1.$$

5.56. If a, b, c, d, e are nonnegative real numbers so that $a^4 + b^4 + c^4 + d^4 + e^4 = 5$, then

$$7(a^2 + b^2 + c^2 + d^2 + e^2) \geq (a + b + c + d + e)^2 + 10.$$

5.57. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$, then

$$(a_1^2 + a_2^2 + \dots + a_n^2)^2 - n^2 \geq \frac{n(n-1)}{n^2 - n + 1} (a_1^4 + a_2^4 + \dots + a_n^4 - n).$$

5.58. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that $a_1^2 + a_2^2 + \dots + a_n^2 = n$, then

$$a_1^3 + a_2^3 + \dots + a_n^3 \geq \sqrt{n^2 - n + 1 + \left(1 - \frac{1}{n}\right)(a_1^6 + a_2^6 + \dots + a_n^6)}.$$

5.59. If a, b, c are positive real numbers so that $abc = 1$, then

$$4\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + \frac{50}{a+b+c} \geq 27.$$

5.60. If a, b, c are positive real numbers so that $abc = 1$, then

$$a^3 + b^3 + c^3 + 15 \geq 6\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

5.61. Let $a_1, a_2, \dots, a_n$ be positive numbers so that $a_1 a_2 \dots a_n = 1$. If $k \geq n - 1$, then

$$a_1^k + a_2^k + \dots + a_n^k + (2k - n)n \geq (2k - n + 1)\left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}\right).$$

5.62. Let $a_1, a_2, \dots, a_n$ ($n \geq 3$) be nonnegative numbers so that $a_1 + a_2 + \dots + a_n = n$, and let k be an integer satisfying $2 \leq k \leq n + 2$. If

$$m = \left(\frac{n}{n-1}\right)^{k-1} - 1,$$

then

$$\frac{a_1^k + a_2^k + \dots + a_n^k}{n} - 1 \geq m(1 - a_1 a_2 \dots a_n).$$

5.63. If a, b, c are positive real numbers so that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 3$, then

$$4(a^2 + b^2 + c^2) + 9 \geq 21abc.$$

5.64. If $a_1, a_2, \dots, a_n$ are positive real numbers so that $\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} = n$, then,

$$a_1 + a_2 + \dots + a_n - n \leq e_{n-1}(a_1 a_2 \dots a_n - 1),$$

where

$$e_{n-1} = \left(1 + \frac{1}{n-1}\right)^{n-1}.$$

5.65. If $a_1, a_2, \dots, a_n$ are positive real numbers, then

$$\frac{a_1^n + a_2^n + \dots + a_n^n}{a_1 a_2 \dots a_n} + n(n-1) \geq (a_1 + a_2 + \dots + a_n) \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right).$$

5.66. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers, then

$$(n-1)(a_1^n + a_2^n + \dots + a_n^n) + n a_1 a_2 \dots a_n \geq (a_1 + a_2 + \dots + a_n)(a_1^{n-1} + a_2^{n-1} + \dots + a_n^{n-1}).$$

5.67. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers, then

$$(n-1)(a_1^{n+1} + a_2^{n+1} + \dots + a_n^{n+1}) \geq (a_1 + a_2 + \dots + a_n)(a_1^n + a_2^n + \dots + a_n^n - a_1 a_2 \dots a_n).$$

5.68. If $a_1, a_2, \dots, a_n$ are positive real numbers, then

$$(a_1 + a_2 + \dots + a_n - n) \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} - n \right) + a_1 a_2 \dots a_n + \frac{1}{a_1 a_2 \dots a_n} \geq 2.$$

5.69. If $a_1, a_2, \dots, a_n$ are positive real numbers so that $a_1 a_2 \dots a_n = 1$, then

$$\left| \frac{1}{\sqrt{a_1 + a_2 + \dots + a_n - n}} - \frac{1}{\sqrt{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} - n}} \right| < 1.$$

5.70. If $a_1, a_2, \dots, a_n$ are positive real numbers so that $a_1 a_2 \dots a_n = 1$, then

$$a_1^{n-1} + a_2^{n-1} + \dots + a_n^{n-1} + \frac{n^2(n-2)}{a_1 + a_2 + \dots + a_n} \geq (n-1) \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right).$$

5.71. If a, b, c are nonnegative real numbers, then

$$(a + b + c - 3)^2 \geq \frac{abc - 1}{abc + 1} (a^2 + b^2 + c^2 - 3).$$

5.72. If $a_1, a_2, \dots, a_n$ are positive real numbers so that $a_1 + a_2 + \dots + a_n = n$, then

$$(a_1 a_2 \dots a_n)^{\frac{1}{\sqrt{n-1}}} (a_1^2 + a_2^2 + \dots + a_n^2) \leq n.$$

5.73. If $a_1, a_2, \dots, a_n$ are positive real numbers such that $a_1 + a_2 + \dots + a_n = n - 1$, then

$$\sqrt[n]{\frac{n-1}{a_1 a_2 \cdots a_n}} \geq 4 \sqrt{\frac{a_1^2 + a_2^2 + \dots + a_n^2}{n(n-1)}}.$$

5.74. If $a_1, a_2, \dots, a_n$ are positive real numbers so that $a_1^3 + a_2^3 + \dots + a_n^3 = n$, then

$$a_1 + a_2 + \dots + a_n \geq n \sqrt[n+1]{a_1 a_2 \cdots a_n}.$$

5.75. Let a, b, c be nonnegative real numbers so that $ab + bc + ca = 3$. If

$$k \geq 2 - \frac{\ln 4}{\ln 3} \approx 0.738,$$

then

$$a^k + b^k + c^k \geq 3.$$

5.76. Let a, b, c be nonnegative real numbers so that $a + b + c = 3$. If

$$k \geq \frac{\ln 9 - \ln 8}{\ln 3 - \ln 2} \approx 0.29,$$

then

$$a^k + b^k + c^k \geq ab + bc + ca.$$

5.77. If $a_1, a_2, \dots, a_n$ ($n \geq 4$) are nonnegative numbers so that $a_1 + a_2 + \dots + a_n = n$, then

$$\frac{1}{n+1-a_2 a_3 \cdots a_n} + \frac{1}{n+1-a_3 a_4 \cdots a_1} + \dots + \frac{1}{n+1-a_1 a_2 \cdots a_{n-1}} \leq 1.$$

5.78. If a, b, c are nonnegative real numbers so that

$$a + b + c \geq 2, \quad ab + bc + ca \geq 1,$$

then

$$\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} \geq 2.$$

5.79. If a, b, c, d are positive real numbers so that $abcd = 1$, then

$$(a + b + c + d)^4 \geq 36\sqrt{3} (a^2 + b^2 + c^2 + d^2).$$

5.80. If a, b, c are nonnegative real numbers so that $ab + bc + ca = 1$, then

$$\sqrt{33a^2 + 16} + \sqrt{33b^2 + 16} + \sqrt{33c^2 + 16} \leq 9(a + b + c).$$

5.81. If a, b, c are positive real numbers so that $a + b + c = 3$, then

$$a^2b^2 + b^2c^2 + c^2a^2 \leq \frac{3}{\sqrt[3]{abc}}.$$

5.82. If $a_1, a_2, \dots, a_n$ ($n \leq 81$) are nonnegative real numbers so that

$$a_1^2 + a_2^2 + \dots + a_n^2 = a_1^5 + a_2^5 + \dots + a_n^5,$$

then

$$a_1^6 + a_2^6 + \dots + a_n^6 \leq n.$$

5.83. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$1 + \sqrt{1 + a^3 + b^3 + c^3} \geq \sqrt{3(a^2 + b^2 + c^2)}.$$

5.84. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$\sqrt{a+b} + \sqrt{b+c} + \sqrt{c+a} \leq \sqrt{16 + \frac{2}{3}(ab + bc + ca)}.$$

5.85. If $a, b, c \in [0, 4]$ and $ab + bc + ca = 4$, then

$$\sqrt{a+b} + \sqrt{b+c} + \sqrt{c+a} \leq 3 + \sqrt{5}.$$

5.86. If a, b, c are positive real numbers so that $abc = 1$, then

$$(a) \quad \frac{a+b+c}{3} \geq \sqrt[3]{\frac{2+a^2+b^2+c^2}{5}};$$

$$(b) \quad a^3 + b^3 + c^3 \geq \sqrt{3(a^4 + b^4 + c^4)}.$$

5.87. If a, b, c, d are nonnegative real numbers so that $a + b + c + d = 4$, then

$$(a^2 + b^2 + c^2 + d^2 - 4)(a^2 + b^2 + c^2 + d^2 + 18) \leq 10(a^3 + b^3 + c^3 + d^3 - 4).$$

5.88. If a, b, c, d are nonnegative real numbers such that

$$a + b + c + d = 4,$$

then

$$(a^4 + b^4 + c^4 + d^4)^2 \geq (a^2 + b^2 + c^2 + d^2)(a^5 + b^5 + c^5 + d^5).$$

5.89. If a, b, c, d are nonnegative real numbers such that

$$a + b + c + d = 4,$$

then

$$13(a^2 + b^2 + c^2 + d^2)^2 \geq 12(a^4 + b^4 + c^4 + d^4) + 160.$$

5.90. If $a_1, a_2, \dots, a_8$ are nonnegative real numbers, then

$$19(a_1^2 + a_2^2 + \dots + a_8^2)^2 \geq 12(a_1 + a_2 + \dots + a_8)(a_1^3 + a_2^3 + \dots + a_8^3).$$

5.91. If a, b, c are nonnegative real numbers so that

$$5(a^2 + b^2 + c^2) = 17(ab + bc + ca),$$

then

$$3\sqrt{\frac{3}{5}} \leq \sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} \leq \frac{1+\sqrt{7}}{\sqrt{2}}.$$

5.92. If a, b, c are nonnegative real numbers so that

$$8(a^2 + b^2 + c^2) = 9(ab + bc + ca),$$

then

$$\frac{19}{12} \leq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \leq \frac{141}{88}.$$

5.93. If $a, b, c \in (0, 2]$ such that $a + b + c = 3$, then

$$\sqrt{\frac{2(b+c)}{a}} - 1 + \sqrt{\frac{2(c+a)}{b}} - 1 + \sqrt{\frac{2(a+b)}{c}} - 1 \geq \frac{9}{\sqrt{ab+bc+ca}}.$$

5.94. Let a, b, c and x, y, z be nonnegative real numbers such that

$$x^3 + y^3 + z^3 = a^3 + b^3 + c^3.$$

Then,

$$\frac{(a+b+c)(x+y+z)}{ab+bc+ca+xy+yz+zx} \geq \sqrt[3]{3}.$$

5.95. If a, b, c, d are positive numbers such that

$$a+b+c+d = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d},$$

then

$$ab+ac+ad+bc+bd+cd+3abcd \geq 9.$$

5.96. If a_1, a_2, a_3, a_4, a_5 are nonnegative real numbers, then

$$\frac{(a_1^3 + a_2^3 + a_3^3 + a_4^3 + a_5^3)^2}{a_1^4 + a_2^4 + a_3^4 + a_4^4 + a_5^4} \geq \frac{1}{2} \sum_{i < j} a_i a_j.$$

5.97. If $a_1, a_2, \dots, a_n \geq 0$ such that

$$a_1 + a_2 + \dots + a_n = n,$$

then

$$\sqrt{a_1} + \sqrt{a_2} + \dots + \sqrt{a_n} \leq \sqrt{2n-1 + 2 \left(1 - \frac{1}{n}\right) \sum_{i < j} a_i a_j}.$$

5.98. If $a_1, a_2, \dots, a_n \geq 0$ such that

$$a_1 + a_2 + \dots + a_n = \sum_{i < j} a_i a_j > 0,$$

then

$$\frac{(n-1)(n-2)}{2} (a_1 + a_2 + \dots + a_n) + \sum_{i < j} \sqrt{a_i a_j} \geq n(n-1).$$

5.99. Let

$$F(a_1, a_2, \dots, a_n) = n(a_1^2 + a_2^2 + \dots + a_n^2) - (a_1 + a_2 + \dots + a_n)^2,$$

where $a_1, a_2, \dots, a_n$ are positive real numbers such that $a_1 = \min\{a_1, a_2, \dots, a_n\}$ and

$$a_1^2(a_2^2 + a_3^2 + \dots + a_n^2) \geq n-1.$$

Then,

$$F(a_1, a_2, \dots, a_n) \geq F\left(\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_n}\right).$$

5.100. Let

$$F(a_1, a_2, \dots, a_n) = a_1 + a_2 + \dots + a_n - n \sqrt[n]{a_1 a_2 \dots a_n},$$

where $a_1, a_2, \dots, a_n$ are positive real numbers such that $a_1 = \min\{a_1, a_2, \dots, a_n\}$ and

$$a_1(a_2 + a_3 + \dots + a_n) \geq n - 1.$$

Then,

$$F(a_1, a_2, \dots, a_n) \geq F\left(\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_n}\right).$$

5.101. Let

$$F(a_1, a_2, \dots, a_n) = \sqrt{\frac{a_1^2 + a_2^2 + \dots + a_n^2}{n}} - \frac{a_1 + a_2 + \dots + a_n}{n},$$

where $a_1, a_2, \dots, a_n$ are positive real numbers such that $a_1 = \min\{a_1, a_2, \dots, a_n\}$ and

$$a_1^{n-1}(a_2 + a_3 + \dots + a_n) \geq n - 1.$$

Then,

$$F(a_1, a_2, \dots, a_n) \geq F\left(\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_n}\right).$$

5.102. If $a_1, a_2, \dots, a_n$ ($n \geq 4$) are positive real numbers such that

$$a_1 + a_2 + \dots + a_n = n, \quad a_n = \max\{a_1, a_2, \dots, a_n\},$$

then

$$n \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_{n-1}} \right) \geq 4(a_1^2 + a_2^2 + \dots + a_n^2) + n(n-5).$$

5.103. If $a_1, a_2, \dots, a_n$ are positive real numbers such that

$$a_1 + a_2 + \dots + a_n = n,$$

then

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} + n - 2 \geq \frac{n(n-1)^2}{\sum_{1 \leq i < j \leq n} a_i a_j}.$$

5.104. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers such that

$$a_1 + a_2 + \dots + a_n = n,$$

then

$$\sum_{i=1}^n \frac{1}{n - a_i} \leq \frac{n-2}{n-1} + \frac{n}{\sum_{1 \leq i < j \leq n} a_i a_j}.$$

5.105. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers such that

$$a_1 + a_2 + \dots + a_n = n,$$

then

$$\sum_{i=1}^n \frac{1}{n - a_i} \geq 1 + \frac{n}{2 \sum_{1 \leq i < j \leq n} a_i a_j}.$$

5.106. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers such that

$$\sum_{i=1}^n \frac{1}{2a_i + n - 2} \leq 1,$$

then

$$2 \sum_{1 \leq i < j \leq n} a_i a_j \geq (n - 1)(a_1 + a_2 + \dots + a_n).$$

5.107. If $a_1, a_2, \dots, a_n$ are positive real numbers such that

$$\frac{1}{2a_1 + n - 2} + \frac{1}{2a_2 + n - 2} + \dots + \frac{1}{2a_n + n - 2} = 1,$$

then

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} + \frac{n(n - 2)}{a_1 + a_2 + \dots + a_n} \geq 2(n - 1).$$

5.108. If $a_1, a_2, \dots, a_n$ are positive real numbers such that

$$\frac{1}{a_1 + n - 1} + \frac{1}{a_2 + n - 1} + \dots + \frac{1}{a_n + n - 1} = 1,$$

then

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} + \frac{n(3n - 4)}{a_1 + a_2 + \dots + a_n} \geq 4(n - 1).$$

5.109. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers such that

$$\sum_{i=1}^n \frac{1}{a_i + 1} \geq n - 1,$$

then

$$(n - 2) \sum_{i=1}^n a_i + \frac{n}{2 \sum_{1 \leq i < j \leq n} a_i a_j} \geq \frac{2n^2 - 4n + 1}{n - 1}.$$

5.110. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers such that

$$\sum_{i=1}^n \frac{1}{a_i + n - 1} = 1,$$

then

$$2 \sum_{1 \leq i < j \leq n} a_i a_j \geq \frac{n^2(n-1)}{a_1 + a_2 + \dots + a_n}.$$

5.111. If $a_1, a_2, \dots, a_n$ ($n \geq 3$) are nonnegative real numbers such that

$$\sum_{i=1}^n \frac{1}{a_i + n - 1} \leq 1,$$

then

$$\frac{n}{a_1 + a_2 + \dots + a_n} - 1 + k \left[\frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} a_i a_j - 1 \right] \geq 0,$$

where

$$k = \frac{4(n-1)^2}{(n-2)(3n-4)}.$$

5.112. If $a_1, a_2, \dots, a_n$ are positive real numbers such that $\sum_{1 \leq i < j \leq n} a_i a_j = \frac{n(n-1)}{2}$, then

$$n \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right) + n(n-2) \geq 2(n-1)(a_1 + a_2 + \dots + a_n).$$

5.113. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers such that $\sum_{1 \leq i < j \leq n} a_i a_j = \frac{n(n-1)}{2}$, then

$$\frac{1}{a_1 + 1} + \frac{1}{a_2 + 1} + \dots + \frac{1}{a_n + 1} - \frac{n}{2} + k \left(\frac{n}{a_1 + a_2 + \dots + a_n} - 1 \right) \leq 0,$$

where

$$k = \frac{1}{2}(n-1)\sqrt{n(n-2)}.$$

5.114. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers such that $\sum_{1 \leq i < j \leq n} a_i a_j = \frac{n(n-1)}{2}$, then

$$\frac{1}{a_1 + 1} + \frac{1}{a_2 + 1} + \dots + \frac{1}{a_n + 1} - \frac{n}{2} + k \left(\frac{n}{a_1 + a_2 + \dots + a_n} - 1 \right) \geq 0,$$

where

$$k = \frac{n(n-2)}{2(n-1)^2} \left(\sqrt{n^2 - 2n + 2} - 1 \right).$$

5.115. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers such that $\sum_{1 \leq i < j \leq n} a_i a_j = \frac{n(n-1)}{2}$, then

$$(n-2)(a_1 + a_2 + \dots + a_n) \geq 2 \left(\frac{1}{a_1+1} + \frac{1}{a_2+1} + \dots + \frac{1}{a_n+1} \right) + n(n-3).$$

5.116. If $a_1, a_2, \dots, a_n$ are positive real numbers such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} = n,$$

then

$$2(n+1) \left(\frac{1}{a_1+1} + \frac{1}{a_2+1} + \dots + \frac{1}{a_n+1} \right) \geq \frac{n^2(n-1)}{a_1+a_2+\dots+a_n} + 2n.$$

5.117. If $a_1, a_2, \dots, a_n$ are positive real numbers such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} = n,$$

then

$$a_1^2 + a_2^2 + \dots + a_n^2 + \frac{2n^2}{a_1 + a_2 + \dots + a_n} \geq 3n.$$

5.118. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers such that $a_1 + a_2 + \dots + a_n = n$, then

$$\sum \frac{1}{a_1^2 + (n^2 - n + 1)(a_2^2 + \dots + a_n^2)} \leq \frac{1}{n^2 - 2n + 2}.$$

5.119. If a, b, c are nonnegative real numbers such that $ab + bc + ca = 3$, then

$$\frac{1}{(5a + 2b + 2c)^2} + \frac{1}{(2a + 5b + 2c)^2} + \frac{1}{(2a + 2b + 5c)^2} \leq \frac{1}{27}.$$

5.120. If a, b, c are nonnegative real numbers such that $ab + bc + ca = 3$, then

$$\frac{1}{(3a + b + c)^2} + \frac{1}{(a + 3b + c)^2} + \frac{1}{(a + b + 3c)^2} \leq \frac{1}{8}.$$

5.121. If a, b, c are nonnegative real numbers such that $ab + bc + ca = 3$, then

$$\frac{1}{(a + 4b + 4c)^2} + \frac{1}{(4a + b + 4c)^2} + \frac{1}{(4a + 4b + c)^2} \leq \frac{1}{27}.$$

5.122. If $n \geq 3$, $1 < k \leq n$ and $a_1, a_2, \dots, a_n$ are nonnegative real numbers, then

$$(n-1)^{k-1}(a_1^k + \dots + a_n^k) + n[n^{k-1} - (n-1)^{k-1}](a_1 \cdots a_n)^{k/n} \geq (a_1 + a_2 + \dots + a_n)^k.$$

5.123. If $a_1, a_2, \dots, a_n$ ($n \geq 3$) are nonnegative real numbers such that

$$\sum_{i=1}^n \frac{1}{(n-1)a_i + 1} = 1.$$

then

$$a_1 + a_2 + \dots + a_n \geq n \sqrt[n-2]{a_1 a_2 \cdots a_n}.$$

5.124. If $a_1, a_2, \dots, a_n$ ($n \geq 3$) are nonnegative real numbers such that

$$\sum_{i=1}^n \frac{1}{(n-1)a_i + 1} = 1,$$

then

$$a_1 + a_2 + \dots + a_n - n \leq k(a_1 a_2 \cdots a_n - 1),$$

where $k = \left(\frac{n-1}{n-2}\right)^{n-1}$.

5.125. If $a_1, a_2, \dots, a_n$ ($n \geq 3$) are nonnegative real numbers such that

$$\sum_{i=1}^n \frac{1}{(n-1)a_i + 1} = 1,$$

then

$$(n-2)(a_1 + a_2 + \dots + a_n) + a_1 a_2 \cdots a_n \geq (n-1)^2.$$

5.126. If $a_1, a_2, \dots, a_n$ ($n \geq 3$) are nonnegative real numbers such that

$$\sum_{i=1}^n \frac{1}{2a_i + n-2} = 1,$$

then

$$a_1 + a_2 + \dots + a_n - n \geq 2^{n-1}(a_1 a_2 \cdots a_n - 1).$$

5.127. Let $n \geq 3$, and let $a_1 \geq a_2 \geq \dots \geq a_n > 0$ such that $\sum_{1 \leq i < j \leq n} a_i a_j = \frac{n(n-1)}{2}$. Then

$$(a) \quad \frac{1}{a_1^2} + \frac{1}{a_2^2} + \dots + \frac{1}{a_n^2} - n \geq \left(3 - \frac{4}{n}\right) \left(\frac{a_{n-1}}{a_n} + \frac{a_n}{a_{n-1}} - 2\right);$$

$$(b) \quad \frac{1}{a_1^2} + \frac{1}{a_2^2} + \dots + \frac{1}{a_n^2} - n \geq \left(3 - \frac{4}{n}\right) (a_{n-1} - a_n)^2.$$

5.3 Solutions

P 5.1. If a, b, c, d are nonnegative real numbers so that

$$a + b + c + d = a^3 + b^3 + c^3 + d^3 = 2,$$

then

$$\frac{7}{4} \leq a^2 + b^2 + c^2 + d^2 \leq 2.$$

(Vasile C., 2010)

Solution. The right inequality follows from the Cauchy-Schwarz inequality

$$(a^2 + b^2 + c^2 + d^2)^2 \leq (a + b + c + d)(a^3 + b^3 + c^3 + d^3).$$

The equality holds for $a = b = 0$ and $c = d = 1$ (or any permutation).

To prove the left inequality, assume that $a \leq b \leq c \leq d$, then apply Corollary 5 for $k = 3$ and $m = 2$:

- If a, b, c, d are nonnegative real numbers so that

$$a + b + c + d = 2, \quad a^3 + b^3 + c^3 + d^3 = 2, \quad a \leq b \leq c \leq d,$$

then

$$S_4 = a^2 + b^2 + c^2 + d^2$$

has the minimum value for $a = b = c$.

So, we only need to prove that the equations

$$3a + d = 3a^3 + d^3 = 2, \quad a, d \geq 0,$$

imply

$$\frac{7}{4} \leq 3a^2 + d^2.$$

Indeed, from $3a + d = 3a^3 + d^3 = 2$, we get $a = 1/4$ and $d = 5/4$, when

$$3a^2 + d^2 = \frac{7}{4}.$$

The left inequality is an equality for

$$a = b = c = \frac{1}{4}, \quad d = \frac{5}{4}$$

(or any cyclic permutation).

□

P 5.2. If $a_1, a_2, \dots, a_9$ are nonnegative real numbers so that

$$a_1 + a_2 + \dots + a_9 = a_1^2 + a_2^2 + \dots + a_9^2 = 3,$$

then

$$3 \leq a_1^3 + a_2^3 + \dots + a_9^3 \leq \frac{14}{3}.$$

(Vasile C., 2010)

Solution. The left inequality follows from the Cauchy-Schwarz inequality

$$(a_1 + a_2 + \dots + a_9)(a_1^3 + a_2^3 + \dots + a_9^3) \geq (a_1^2 + a_2^2 + \dots + a_9^2)^2.$$

The equality holds for $a_1 = a_2 = \dots = a_6 = 0$ and $a_7 = a_8 = a_9 = 1$ (or any permutation).

To prove the right inequality, assume that

$$a_1 \leq a_2 \leq \dots \leq a_9,$$

then apply Corollary 5 for $k = 2$ and $m = 3$:

- If $a_1, a_2, \dots, a_9$ are nonnegative real numbers so that

$$a_1 + a_2 + \dots + a_9 = 3, \quad a_1^2 + a_2^2 + \dots + a_9^2 = 3, \quad a_1 \leq a_2 \leq \dots \leq a_9,$$

then

$$S_9 = a_1^3 + a_2^3 + \dots + a_9^3$$

has the maximum value for $a_1 = a_2 = \dots = a_8 \leq a_9$.

Thus, we only need to prove that the equations

$$8a + b = 3, \quad 8a^2 + b^2 = 3, \quad a, b \geq 0,$$

involve

$$8a^3 + b^3 \leq \frac{14}{3}.$$

Indeed, from the equations above, we get $a = 1/6$ and $b = 5/3$, when

$$8a^3 + b^3 = \frac{1}{27} + \frac{125}{27} = \frac{14}{3}.$$

The equality holds for

$$a_1 = a_2 = \dots = a_8 = \frac{1}{6}, \quad a_9 = \frac{5}{3}$$

(or any cyclic permutation).

□

P 5.3. If a, b, c, d are nonnegative real numbers so that

$$a + b + c + d = a^2 + b^2 + c^2 + d^2 = \frac{27}{7},$$

then

$$\frac{5427}{1372} \leq a^3 + b^3 + c^3 + d^3 \leq \frac{1377}{343}.$$

(Vasile C., 2014)

Solution. Assume that $a \leq b \leq c \leq d$.

(a) To prove the right inequality, we apply Corollary 5 for $k = 2$ and $m = 3$:

- If a, b, c, d are nonnegative real numbers so that

$$a + b + c + d = \frac{27}{7}, \quad a^2 + b^2 + c^2 + d^2 = \frac{27}{7}, \quad a \leq b \leq c \leq d,$$

then

$$S_4 = a^3 + b^3 + c^3 + d^3$$

has the maximum value for $a = b = c \leq d$

Thus, we only need to prove that the equations

$$3a + d = \frac{27}{7}, \quad 3a^2 + d^2 = \frac{27}{7}, \quad a, d \geq 0,$$

involve

$$3a^3 + d^3 \leq \frac{1377}{343}.$$

Indeed, from the equations above, we get $a = 6/7$ and $d = 9/7$, when

$$3a^3 + d^3 = 3 \left(\frac{6}{7} \right)^3 + \left(\frac{9}{7} \right)^3 = \frac{1377}{343}.$$

The equality holds for

$$a = b = c = \frac{6}{7}, \quad d = \frac{9}{7}$$

(or any cyclic permutation).

(b) To prove the left inequality, we apply Corollary 5 for $k = 2$ and $m = 3$:

- If a, b, c, d are nonnegative real numbers so that

$$a + b + c + d = \frac{27}{7}, \quad a^2 + b^2 + c^2 + d^2 = \frac{27}{7}, \quad a \leq b \leq c \leq d,$$

then

$$S_4 = a^3 + b^3 + c^3 + d^3$$

has the minimum value for either $a = 0$ or $a \leq b = c = d$.

The case $a = 0$ is not possible because from

$$b + c + d = \frac{27}{7}, \quad b^2 + c^2 + d^2 = \frac{27}{7},$$

we get

$$3(b^2 + c^2 + d^2) - (b + c + d)^2 = \frac{27}{7} \left(3 - \frac{27}{7} \right) < 0,$$

which contradicts the known inequality

$$3(b^2 + c^2 + d^2) \geq (b + c + d)^2.$$

For $a \leq b = c = d$, we need to prove that the equations

$$a + 3d = \frac{27}{7}, \quad a^2 + 3d^2 = \frac{27}{7}, \quad a, d \geq 0,$$

involve

$$a^3 + 3d^3 \geq \frac{5427}{1372}.$$

Indeed, from the equations above, we get $a = 9/14$ and $d = 15/14$, when

$$a^3 + 3d^3 = \left(\frac{9}{14}\right)^3 + 3\left(\frac{15}{14}\right)^3 = \frac{5427}{1372}.$$

The equality holds for

$$a = \frac{9}{14}, \quad b = c = d = \frac{15}{14}$$

(or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization:

• Let k be a positive real number ($k > 2$), and let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that

$$a_1 + a_2 + \dots + a_n = a_1^2 + a_2^2 + \dots + a_n^2 = \frac{(n-1)^3}{n^2 - 3n + 3}.$$

The sum

$$S_n = a_1^k + a_2^k + \dots + a_n^k$$

has the maximum value for

$$a_1 = \dots = a_{n-1} = \frac{(n-1)(n-2)}{n^2 - 3n + 3}, \quad a_n = \frac{(n-1)^2}{n^2 - 3n + 3},$$

and the minimum value for

$$a_1 = \frac{(n-1)^2(n-2)}{n(n^2 - 3n + 3)}, \quad a_2 = \dots = a_n = \frac{(n-1)(n^2 - 2n + 2)}{n(n^2 - 3n + 3)}.$$

□

P 5.4. If a, b, c are positive real numbers so that $abc = 1$, then

$$a^5 + b^5 + c^5 \geq \sqrt{3(a^7 + b^7 + c^7)}.$$

(Vasile C., 2014)

Solution. Substituting

$$a = x^{1/5}, \quad b = y^{1/5}, \quad c = z^{1/5},$$

we need to show that $xyz = 1$ involves

$$x + y + z \geq \sqrt{3(x^{7/5} + y^{7/5} + z^{7/5})}.$$

Assume that $x \leq y \leq z$, then apply Corollary 5 for $k = 0$ and $m = 7/5$:

- If $x \leq y \leq z > 0$ such that $x + y + z$ is fixed and $xyz = 1$, then

$$S_3 = x^{7/5} + y^{7/5} + z^{7/5}$$

has the maximum value for $x = y$.

So, it suffices to prove the original inequality for $a = b$. Write this inequality in the homogeneous form

$$(a^5 + b^5 + c^5)^2 \geq 3abc(a^7 + b^7 + c^7).$$

We only need to prove this inequality for $a = b = 1$; that is, to show that $f(c) \geq 0$, where

$$f(c) = (c^5 + 2)^2 - 3c(c^7 + 2), \quad c > 0.$$

We have

$$\begin{aligned} f'(c) &= 10c^4(c^5 + 2) - 24c^7 - 6, \\ f''(c) &= 2c^3g(t), \quad g(t) = 45c^5 - 84c^3 + 40. \end{aligned}$$

By the AM-GM inequality, we get

$$\begin{aligned} g(t) &= 15c^5 + 15c^5 + 15c^5 + 20 + 20 - 84c^3 \geq 5\sqrt[5]{(15c^5)^3 \cdot 20^2} - 84c^3 \\ &= \sqrt[5]{27 \cdot 16} \left(25 - 14\sqrt[5]{18} \right) c^3 > 0, \end{aligned}$$

hence $f''(c) > 0$, $f'(c)$ is increasing. Since $f'(0) = 1$, it follows that $f'(c) \leq 0$ for $c \leq 1$, $f'(c) \geq 0$ for $c \geq 1$, therefore f is decreasing on $(0, 1]$ and increasing on $[1, \infty)$; consequently, $f(c) \geq f(1) = 0$. The equality occurs for $a = b = c = 1$.

□

P 5.5. If a, b, c, d are positive real numbers so that $abcd = 1$, then

$$a^3 + b^3 + c^3 + d^3 \geq \sqrt{4(a^4 + b^4 + c^4 + d^4)}.$$

(Vasile C., 2014)

Solution. Substituting

$$a = x^{1/3}, \quad b = y^{1/3}, \quad c = z^{1/3}, \quad d = t^{1/3},$$

we need to show that $xyzt = 1$ involves

$$x + y + z + t \geq \sqrt{4(x^{4/3} + y^{4/3} + z^{4/3} + t^{4/3})}.$$

Apply Corollary 5, case $k = 0$ and $m = 4/3$:

- If $x \geq y \geq z \geq t > 0$ such that $x + y + z + t$ is fixed and $xyzt = 1$, then

$$S_4 = x^{4/3} + y^{4/3} + z^{4/3} + t^{4/3}$$

has the maximum value for $x = y = z$.

Therefore, it suffices to prove the original inequality for $a = b = c$. Write the original inequality in the homogeneous form

$$(a^3 + b^3 + c^3 + d^3)^2 \geq 4\sqrt{abcd} (a^4 + b^4 + c^4 + d^4).$$

We only need to prove this inequality for $a = b = c = 1$; that is, to show that

$$(d^3 + 3)^2 \geq 4\sqrt{d} (d^4 + 3).$$

Putting $u = \sqrt{d}$, we have

$$\begin{aligned} (d^3 + 3)^2 - 4\sqrt{d} (d^4 + 3) &= (u^6 + 3)^2 - 4u(u^8 + 3) \\ &= (u^3 - 1)^4 + 4(u + 2)(u - 1)^2 \geq 0. \end{aligned}$$

The equality holds for $a = b = c = d = 1$.

□

P 5.6. If a, b, c, d are nonnegative real numbers so that $a + b + c + d = 4$, then

$$\frac{bcd}{11a + 16} + \frac{cda}{11b + 16} + \frac{dab}{11c + 16} + \frac{abc}{11d + 16} \leq \frac{4}{27}.$$

(Vasile C., 2008)

Solution. For $a = 0$, the inequality becomes

$$bcd \leq \frac{64}{27},$$

where $b, c, d \geq 0$, $b + c + d = 4$. By the AM-GM inequality, we have

$$bcd \leq \left(\frac{b + c + d}{3} \right)^3 = \left(\frac{4}{3} \right)^3 = \frac{64}{27}.$$

For $abcd \neq 0$, we write the inequality in the form

$$f(a) + f(b) + f(c) + f(d) + \frac{4}{(1 + k)abcd} \geq 0,$$

where

$$f(u) = \frac{-1}{u(u + k)}, \quad k = \frac{16}{11}, \quad u > 0.$$

We have $f(0+) = -\infty$ and

$$\begin{aligned} f'(u) &= \frac{2u + k}{(u^2 + ku)^2}, \\ g(x) &= f'(1/x) = \frac{kx^4 + 2x^3}{(kx + 1)^2}, \\ g''(x) &= \frac{2x(k^3x^3 + 4k^2x^2 + 6kx + 6)}{(kx + 1)^4}. \end{aligned}$$

Since $g''(x) > 0$ for $x > 0$, g is strictly convex on $(0, \infty)$. By Corollary 3 and Note 1, if $0 < a \leq b \leq c \leq d$ such that $a + b + c + d = 4$ and $abcd$ is fixed, then the sum

$$S_4 = f(a) + f(b) + f(c) + f(d)$$

has the maximum value for $b = c = d$. Thus, we only need to prove that

$$\frac{b^3}{11a + 16} + \frac{3ab^2}{11b + 16} \leq \frac{4}{27}$$

for $a + 3b = 4$. The inequality is equivalent to

$$\frac{b^3}{3(20 - 11b)} + \frac{3b^2(4 - 3b)}{11b + 16} \leq \frac{4}{21},$$

$$(b - 1)^2(4 - 3b)(231b + 80) \geq 0,$$

$$(b - 1)^2a(231b + 80) \geq 0.$$

The equality holds for $a = b = c = d = 1$, and also for

$$a = 0, \quad b = c = d = \frac{4}{3}$$

(or any cyclic permutation).

□

P 5.7. If a, b, c are real numbers, then

$$\frac{bc}{3a^2 + b^2 + c^2} + \frac{ca}{3b^2 + c^2 + a^2} + \frac{ab}{3c^2 + a^2 + b^2} \leq \frac{3}{5}.$$

(Vasile Cîrtoaje and Pham Kim Hung, 2005)

Solution. For $a = 0$, the inequality is true because

$$\frac{bc}{b^2 + c^2} \leq \frac{1}{2} < \frac{3}{5}.$$

Consider further that a, b, c are different from zero. The inequality remains unchanged by replacing a, b, c with $-a, -b, -c$, respectively. Thus, we only need to consider the case $a < 0, b, c > 0$, and the case $a, b, c > 0$. In the first case, it suffices to show that

$$\frac{bc}{3a^2 + b^2 + c^2} \leq \frac{3}{5}.$$

Indeed, we have

$$\frac{bc}{3a^2 + b^2 + c^2} < \frac{bc}{b^2 + c^2} \leq \frac{1}{2} < \frac{3}{5}.$$

Consider now the case $a, b, c > 0$. Replacing a, b, c with $\sqrt{a}, \sqrt{b}, \sqrt{c}$, the inequality becomes

$$\frac{1}{\sqrt{a}(3a + b + c)} + \frac{1}{\sqrt{b}(3b + c + a)} + \frac{1}{\sqrt{c}(3c + a + b)} \leq \frac{3}{5\sqrt{abc}}.$$

Due to homogeneity, we may consider that $a + b + c = 2$. So, we need to show that

$$f(a) + f(b) + f(c) + \frac{6}{5\sqrt{abc}} \geq 0,$$

where

$$f(u) = \frac{-1}{\sqrt{u}(u+1)}, \quad u > 0.$$

We have $f(0+) = -\infty$ and

$$\begin{aligned} f'(u) &= \frac{3u+1}{2u\sqrt{u}(u+1)^2}, \\ g(x) &= f'(1/x) = \frac{x^2\sqrt{x}(x+3)}{2(x+1)^2}, \\ g''(x) &= \frac{\sqrt{x}(3x^3+11x^2+5x+45)}{8(x+1)^4}. \end{aligned}$$

Since $g''(x) > 0$ for $x > 0$, g is strictly convex on $(0, \infty)$. By Corollary 3 and Note 1, if

$$a + b + c = 2, \quad abc = \text{fixed}, \quad 0 < a \leq b \leq c,$$

then the sum

$$S_3 = f(a) + f(b) + f(c)$$

is minimum for $b = c$. Thus, we only need to prove the original homogeneous inequality for $b = c = 1$; that is,

$$\begin{aligned} \frac{1}{3a^2+2} + \frac{2a}{a^2+4} &\leq \frac{3}{5}, \\ 9a^4 - 30a^3 + 37a^2 - 20a + 4 &\geq 0, \\ (a-1)^2(3a-2)^2 &\geq 0. \end{aligned}$$

The equality holds for $a = b = c$, and also for

$$3a = 2b = 2c$$

(or any cyclic permutation).

□

P 5.8. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$\begin{aligned} (a) \quad & \frac{bc}{a^2+2} + \frac{ca}{b^2+2} + \frac{ab}{c^2+2} \leq \frac{9}{8}; \\ (b) \quad & \frac{bc}{a^2+3} + \frac{ca}{b^2+3} + \frac{ab}{c^2+3} \leq \frac{11\sqrt{33}-45}{24}; \\ (c) \quad & \frac{bc}{a^2+4} + \frac{ca}{b^2+4} + \frac{ab}{c^2+4} \leq \frac{3}{5}. \end{aligned}$$

(Vasile C., 2008)

Solution. For the nontrivial case $abc \neq 0$, we can write the desired inequalities in the form

$$f(a) + f(b) + f(c) + \frac{m}{abc} \geq 0,$$

where

$$f(u) = \frac{-1}{u(u^2 + k)}, \quad k \in \{2, 3, 4\}, \quad u > 0.$$

We have $f(0+) = -\infty$ and

$$\begin{aligned} f'(u) &= \frac{3u^2 + k}{u^2(u^2 + k)^2}, \\ g(x) &= f'(1/x) = \frac{kx^6 + 3x^4}{(kx^2 + 1)^2}, \\ g''(x) &= \frac{2x^2(k^3x^6 + 4k^2x^4 - 3kx^2 + 18)}{(kx^2 + 1)^4}. \end{aligned}$$

Since

$$k^3x^6 + 4k^2x^4 - 3kx^2 + 18 > 4k^2x^4 - 3kx^2 + 18 > 0,$$

we have $g''(x) > 0$, hence g is strictly convex on $(0, \infty)$. According to Corollary 3 and Note 1, if

$$a + b + c = 3, \quad abc = \text{fixed}, \quad 0 < a \leq b \leq c,$$

then the sum

$$S_3 = f(a) + f(b) + f(c)$$

is minimum for $b = c$. Thus, we only need to prove the original inequalities for $b = c$.

(a) We only need to prove the homogeneous inequality

$$\frac{bc}{9a^2 + 2(a + b + c)^2} + \frac{ca}{9b^2 + 2(a + b + c)^2} + \frac{ab}{9c^2 + 2(a + b + c)^2} \leq \frac{1}{8}$$

for $b = c = 1$, that is

$$\begin{aligned} \frac{1}{11a^2 + 8a + 8} + \frac{2a}{2a^2 + 8a + 17} &\leq \frac{1}{8}, \\ \frac{2a}{2a^2 + 8a + 17} &\leq \frac{a(11a + 8)}{8(11a^2 + 8a + 8)}, \\ a(22a^3 - 72a^2 + 123a + 8) &\geq 0. \end{aligned}$$

Since

$$22a^3 - 72a^2 + 123a + 8 > 20a^3 - 80a^2 + 80a = 20a(a - 2)^2 \geq 0,$$

the conclusion follows. The equality holds for $a = 0$ and $b = c = 3/2$ (or any cyclic permutation).

(b) Let

$$m = \frac{11\sqrt{33} - 45}{72} \approx 0.253, \quad r = \frac{\sqrt{33} - 5}{4} \approx 0.186.$$

We only need to prove the homogeneous inequality

$$\frac{bc}{3a^2 + (a + b + c)^2} + \frac{ca}{3b^2 + (a + b + c)^2} + \frac{ab}{3c^2 + (a + b + c)^2} \leq m$$

for $b = c = 1$; that is, to show that $f(a) \leq m$, where

$$f(a) = \frac{1}{4(a^2 + a + 1)} + \frac{2a}{a^2 + 4a + 7}.$$

We have

$$\begin{aligned} f'(a) &= \frac{-8a^6 - 18a^5 + 15a^4 + 28a^3 + 18a^2 - 42a + 7}{4(a^2 + a + 1)^2(a^2 + 4a + 7)^2} \\ &= \frac{(1-a)^2(7+7a+4a^2)(1-5a-2a^2)}{4(a^2 + a + 1)^2(a^2 + 4a + 7)^2}. \end{aligned}$$

Since $f'(a) \geq 0$ for $a \in [0, r]$, and $f'(a) \leq 0$ for $a \in [r, \infty)$, f is increasing on $[0, r]$ and decreasing on $[r, \infty)$; therefore,

$$f(a) \geq f(r) = m.$$

The equality holds for

$$a/r = b = c$$

(or any cyclic permutation).

(c) We only need to prove the homogeneous inequality

$$\frac{bc}{9a^2 + 4(a+b+c)^2} + \frac{ca}{9b^2 + 4(a+b+c)^2} + \frac{ab}{9c^2 + 4(a+b+c)^2} \leq \frac{1}{15}$$

for $b = c = 1$, that is

$$\begin{aligned} \frac{1}{13a^2 + 16a + 16} + \frac{2a}{4a^2 + 16a + 25} &\leq \frac{1}{15}, \\ 52a^4 - 118a^3 + 105a^2 - 64a + 25 &\geq 0, \\ (a-1)^2(52a^2 - 14a + 25) &\geq 0. \end{aligned}$$

Since

$$52a^2 - 14a + 25 > 7a^2 - 14a + 7 = 7(a-1)^2 \geq 0,$$

the conclusion follows. The equality holds for $a = b = c = 1$.

□

P 5.9. If a, b, c, d are nonnegative real numbers so that

$$(3a+1)(3b+1)(3c+1)(3d+1) = 64,$$

then

$$abc + bcd + cda + dab \leq 1.$$

(Vasile C., 2014)

Solution. For $d = 0$, we need to show that

$$(3a + 1)(3b + 1)(3c + 1) = 64$$

involves $abc \leq 1$. Indeed, by the AM-GM inequality, we have

$$64 = (3a + 1)(3b + 1)(3c + 1) \geq \left(4\sqrt[4]{a^3}\right) \left(4\sqrt[4]{b^3}\right) \left(4\sqrt[4]{c^3}\right) = 64\sqrt[4]{(abc)^3},$$

hence $abc \leq 1$. Consider further that $a, b, c, d > 0$ and use the contradiction method. Assume that

$$abc + bcd + cda + dab > 1,$$

and prove that

$$(3a + 1)(3b + 1)(3c + 1) > 64.$$

It suffices to show that

$$abc + bcd + cda + dab \geq 1$$

involves

$$(3a + 1)(3b + 1)(3c + 1) \geq 64.$$

Replacing a, b, c, d by $1/a, 1/b, 1/c, 1/d$, we need to show that

$$a + b + c + d = abcd$$

involves

$$\left(\frac{3}{a} + 1\right) \left(\frac{3}{b} + 1\right) \left(\frac{3}{c} + 1\right) \left(\frac{3}{d} + 1\right) \geq 64,$$

which is equivalent to

$$f(a) + f(b) + f(c) + f(d) \leq -6 \ln 2,$$

where

$$f(u) = -\ln \left(\frac{3}{u} + 1\right), \quad u > 0.$$

We have $f(0+) = -\infty$ and

$$g(x) = f'(1/x) = \frac{3x^2}{3x + 1}, \quad g''(x) = \frac{6}{(3x + 1)^3} > 0,$$

hence g is strictly convex on $(0, \infty)$. By Corollary 3 and Note 1, if a, b, c, d are positive real numbers so that

$$a + b + c + d = \text{fixed}, \quad abcd = \text{fixed}, \quad a \leq b \leq c \leq d,$$

then

$$S_4 = f(a) + f(b) + f(c) + f(d)$$

is maximum for $a = b = c$.

Thus, we only need to prove that

$$\left(\frac{3}{a} + 1\right)^3 \left(\frac{3}{d} + 1\right) \geq 64$$

for $3a + d = a^3d$, that is

$$\frac{3}{d} = \frac{a^3 - 1}{a}, \quad 1 < a \leq d.$$

Write this inequality as

$$\begin{aligned} (3 + a)^3(3 + d) &\geq 64a^3d, \\ (3 + a)^4(3 + d) &\geq 64a^3d(3 + a), \\ 4 \left(1 + \frac{a - 1}{4}\right)^4 (3 + d) &\geq a^3d(3 + a). \end{aligned}$$

By Bernoulli's inequality, we have

$$\left(1 + \frac{a - 1}{4}\right)^4 \geq 1 + 4 \cdot \frac{a - 1}{4} = a.$$

Thus, it suffices to show that

$$4(3 + d) \geq a^2d(3 + a),$$

which is equivalent to

$$\begin{aligned} \frac{12}{d} &\geq a^3 + 3a^2 - 4, \\ \frac{4(a^3 - 1)}{a} &\geq a^3 + 3a^2 - 4, \\ a^4 - a^3 - 4a + 4 &\leq 0, \\ (a - 1)(a^3 - 4) &\leq 0. \end{aligned}$$

This is true if $a^3 \leq 4$. Indeed, we have

$$0 \leq \frac{3}{a} - \frac{3}{d} = \frac{3}{a} - \frac{a^3 - 1}{a} = \frac{4 - a^3}{a}.$$

The proof is completed. The original inequality is an equality for

$$a = b = c = 1, \quad d = 0$$

(or any cyclic permutation).

□

P 5.10. If $a_1, a_2, \dots, a_n$ and p, q are nonnegative real numbers so that

$$a_1 + a_2 + \dots + a_n = p + q, \quad a_1^3 + a_2^3 + \dots + a_n^3 = p^3 + q^3,$$

then

$$a_1^2 + a_2^2 + \dots + a_n^2 \leq p^2 + q^2.$$

(Vasile C., 2013)

Solution. For $n = 2$, the inequality is an equality. Consider now that $n \geq 3$ and $a_1 \leq a_2 \leq \dots \leq a_n$. We will apply Corollary 5 for $k = 3$ and $m = 2$:

- If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that $a_1 \leq a_2 \leq \dots \leq a_n$ and

$$a_1 + a_2 + \dots + a_n = p + q, \quad a_1^3 + a_2^3 + \dots + a_n^3 = p^3 + q^3,$$

then

$$S_n = a_1^2 + a_2^2 + \dots + a_n^2$$

is maximum for either $a_1 = 0$ or $a_2 = a_3 = \dots = a_n$.

In the first case $a_1 = 0$, the conclusion follows by induction method. In the second case, for

$$a_1 = a, \quad a_2 = a_3 = \dots = a_n = b,$$

we need to show that

$$a^2 + (n-1)b^2 \leq p^2 + q^2$$

for

$$a + (n-1)b = p + q, \quad a^3 + (n-1)b^3 = p^3 + q^3.$$

Since

$$3(p^2 + q^2) = (p + q)^2 + \frac{2(p^3 + q^3)}{p + q},$$

the inequality can be written as

$$3a^2 + 3(n-1)b^2 \leq [a + (n-1)b]^2 + \frac{2[a^3 + (n-1)b^3]}{a + (n-1)b},$$

which is equivalent to

$$(n-1)(n-2)b^2[3a + (n-3)b] \geq 0.$$

The equality holds when $n-2$ of $a_1, a_2, \dots, a_n$ are equal to zero.

□

P 5.11. If a, b, c are nonnegative real numbers, then

$$a\sqrt{a^2 + 4b^2 + 4c^2} + b\sqrt{b^2 + 4c^2 + 4a^2} + c\sqrt{c^2 + 4a^2 + 4b^2} \geq (a + b + c)^2.$$

(Vasile C., 2010)

Solution. Due to homogeneity and symmetry, we may assume that

$$a^2 + b^2 + c^2 = 3, \quad 0 \leq a \leq b \leq c \leq \sqrt{3}.$$

Under this assumption, we write the desired inequality as

$$f(a) + f(b) + f(c) + \frac{1}{\sqrt{3}}(a + b + c)^2 \leq 0,$$

where

$$f(u) = -u\sqrt{4 - u^2}, \quad 0 \leq u \leq \sqrt{3}.$$

We have

$$g(x) = f'(x) = \frac{2(x^2 - 2)}{\sqrt{4 - x^2}},$$

$$g''(x) = \frac{48}{(4 - x^2)^{5/2}}.$$

Since $g''(x) > 0$ for $x \in (0, 2)$, g is strictly convex on $[0, \sqrt{3}]$. According to Corollary 1, if

$$a + b + c = \text{fixed}, \quad a^2 + b^2 + c^2 = 3, \quad 0 \leq a \leq b \leq c,$$

then the sum

$$S_3 = f(a) + f(b) + f(c)$$

is maximum for $a = b \leq c$. Thus, we only need to prove the original inequality for $a = b$. Since the inequality is an identity for $a = b = 0$, we may consider $a = b = 1$ and $c \geq 1$. We need to prove that

$$2\sqrt{4c^2 + 5} + c\sqrt{c^2 + 8} \geq (c + 2)^2.$$

By squaring, the inequality becomes

$$c\sqrt{(4c^2 + 5)(c^2 + 8)} \geq 2c^3 + 8c - 1.$$

This is true if

$$c^2(4c^2 + 5)(c^2 + 8) \geq (2c^3 + 8c - 1)^2,$$

which is equivalent to

$$5c^4 + 4c^3 - 24c^2 + 16c - 1 \geq 0,$$

$$(c - 1)^2(5c^2 + 14c - 1) \geq 0.$$

The equality holds for $a = b = c$, and also for $a = b = 0$ (or any cyclic permutation).

□

P 5.12. If a, b, c are nonnegative real numbers so that $ab + bc + ca = 3$, then

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \leq \frac{3}{2(a+b+c)} + \frac{a+b+c}{3}.$$

(Vasile C., 2010)

Solution. Write the inequality in the homogeneous form

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \leq \frac{3}{2(a+b+c)} + \frac{a+b+c}{ab+bc+ca}.$$

Due to homogeneity and symmetry, we may assume that

$$a + b + c = 1, \quad 0 \leq a \leq b \leq c, \quad ab + bc + ca > 0.$$

Under this assumption, we write the desired inequality as

$$f(a) + f(b) + f(c) \leq \frac{3}{2} + \frac{1}{ab+bc+ca},$$

where

$$f(u) = \frac{1}{1-u}, \quad 0 \leq u < 1.$$

We will apply Corollary 1 to the function f , which satisfies $f(1-) = \infty$ and

$$g(x) = f'(x) = \frac{1}{(1-x)^2},$$

$$g''(x) = \frac{6}{(1-x)^4}.$$

Since $g''(x) > 0$, g is strictly convex on $[0, 1)$. According to Corollary 1 and Note 3, if

$$a + b + c = 1, \quad ab + bc + ca = \text{fixed}, \quad 0 \leq a \leq b \leq c,$$

then the sum

$$S_3 = f(a) + f(b) + f(c)$$

is maximum for $a = b \leq c$. Thus, we only need to prove the homogeneous inequality for $a = b = 1$ and $c \geq 1$; that is,

$$1 + \frac{4}{c+1} \leq \frac{3}{c+2} + \frac{2(c+2)}{2c+1},$$

which reduces to

$$(c-1)^2 \geq 0.$$

The original inequality is an equality for $a = b = c = 1$.

□

P 5.13. If a, b, c are nonnegative real numbers so that $ab + bc + ca = 3$, then

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{3}{a+b+c} + \frac{a+b+c}{6}.$$

(Vasile C., 2010)

Solution. Proceeding in the same manner as in the proof of the preceding P 5.12, we only need to prove the homogeneous inequality

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{3}{a+b+c} + \frac{a+b+c}{2(ab+bc+ca)}$$

for $a = 0$ and for $a \leq b = c = 1$.

Case 1: $a = 0$. The homogeneous inequality reduces to

$$\frac{1}{b} + \frac{1}{c} \geq \frac{2}{b+c} + \frac{b+c}{2bc},$$

which is equivalent to

$$(b-c)^2 \geq 0.$$

Case 2: $a \leq b = c = 1$. The homogeneous inequality becomes

$$\begin{aligned}\frac{1}{2} + \frac{2}{a+1} &\geq \frac{3}{a+2} + \frac{a+2}{2(2a+1)}, \\ \frac{1}{2} - \frac{a+2}{2(2a+1)} &\geq \frac{3}{a+2} - \frac{2}{a+1}, \\ \frac{a-1}{2(2a+1)} &\geq \frac{a-1}{(a+1)(a+2)}, \\ a(a-1)^2 &\geq 0.\end{aligned}$$

The equality holds for $a = b = c = 1$, and also for

$$a = 0, \quad b = c = \sqrt{3}$$

(or any cyclic permutation).

□

P 5.14. Let a, b, c be nonnegative real numbers, no two of which are zero. If

$$a^2 + b^2 + c^2 = 3,$$

then

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} + \frac{a+b+c}{9} \geq \frac{11}{2(a+b+c)}.$$

(Vasile C., 2010)

Solution. Using the same method as in the proof of P 5.12, we only need to prove the homogeneous inequality

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} + \frac{a+b+c}{3(a^2+b^2+c^2)} \geq \frac{11}{2(a+b+c)}$$

for $a = 0$ and for $a \leq b = c = 1$.

Case 1: $a = 0$. The homogeneous inequality reduces to

$$\begin{aligned}\frac{1}{b} + \frac{1}{c} + \frac{1}{b+c} + \frac{b+c}{3(b^2+c^2)} &\geq \frac{11}{2(b+c)}, \\ \frac{b+c}{bc} + \frac{b+c}{3(b^2+c^2)} &\geq \frac{9}{2(b+c)}, \\ (b+c)^2 \left[\frac{1}{bc} + \frac{1}{3(b^2+c^2)} \right] &\geq \frac{9}{2}.\end{aligned}$$

Using the substitution

$$x = \frac{b^2+c^2}{bc}, \quad x \geq 2,$$

the inequality becomes

$$(x+2)\left(1+\frac{1}{3x}\right) \geq \frac{9}{2},$$

which is equivalent to

$$\begin{aligned} 6x^2 - 13x + 4 &\geq 0, \\ x + 2(x-2)(3x-1) &\geq 0. \end{aligned}$$

Case 2: $a \leq 1 = b = c$. The homogeneous inequality becomes

$$\begin{aligned} \frac{1}{2} + \frac{2}{a+1} + \frac{a+2}{3(a^2+2)} &\geq \frac{11}{2(a+2)}, \\ \frac{a+2}{3(a^2+2)} + \frac{a^2-4a-1}{2(a+1)(a+2)} &\geq 0, \\ 3a^4 - 10a^3 + 13a^2 - 8a + 2 &\geq 0, \\ (a-1)^2(3a^2-4a+2) &\geq 0, \\ (a-1)^2[a^2+2(a-1)^2] &\geq 0. \end{aligned}$$

The equality holds for $a = b = c = 1$.

□

P 5.15. Let a, b, c be nonnegative real numbers, no two of which are zero. If

$$a + b + c = 4,$$

then

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{15}{8+ab+bc+ca}.$$

(Vasile C., 2010)

Solution. Using the same method as in P 5.12, we only need to prove the homogeneous inequality

$$\frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{c+a} \geq \frac{15(a+b+c)}{(a+b+c)^2 + 2(ab+bc+ca)}$$

for $a = 0$ and for $a \leq b = c = 1$.

Case 1: $a = 0$. The homogeneous inequality reduces to

$$\begin{aligned} \frac{2(b+c)}{bc} + \frac{2}{b+c} &\geq \frac{15(b+c)}{(b+c)^2 + 2bc}, \\ \frac{2(b+c)^2}{bc} + 2 &\geq \frac{15(b+c)^2}{(b+c)^2 + 2bc}. \end{aligned}$$

Using the substitution

$$x = \frac{(b+c)^2}{bc}, \quad x \geq 4,$$

the inequality becomes

$$2x + 2 \geq \frac{15x}{x + 2},$$

which is equivalent to

$$\begin{aligned} 2x^2 - 9x + 4 &\geq 0, \\ (x - 4)(2x - 1) &\geq 0. \end{aligned}$$

Case 2: $a \leq 1$, $b = c = 1$. The homogeneous inequality becomes

$$\begin{aligned} 1 + \frac{4}{a + 1} &\geq \frac{15(a + 2)}{(a + 2)^2 + 2(2a + 1)}, \\ \frac{a + 5}{a + 1} &\geq \frac{15(a + 2)}{a^2 + 8a + 6}, \\ a(a - 1)^2 &\geq 0. \end{aligned}$$

The equality holds for $a = b = c = 4/3$, and also for

$$a = 0, \quad b = c = 2$$

(or any cyclic permutation).

□

P 5.16. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} \geq \frac{1}{a + b + c} + \frac{2}{\sqrt{ab + bc + ca}}.$$

(Vasile C., 2010)

Solution. Using the same method as in P 5.12, we only need to prove the desired homogeneous inequality for $a = 0$ and for $0 < a \leq b = c = 1$.

Case 1: $a = 0$. The inequality reduces to the obvious form

$$\frac{1}{b} + \frac{1}{c} \geq \frac{2}{\sqrt{bc}}.$$

Case 2: $0 < a \leq 1 = b = c$. The inequality becomes

$$\begin{aligned} \frac{1}{2} + \frac{2}{a + 1} &\geq \frac{1}{a + 2} + \frac{2}{\sqrt{2a + 1}}, \\ \frac{1}{2} - \frac{1}{a + 2} &\geq \frac{2}{\sqrt{2a + 1}} - \frac{2}{a + 1}, \\ \frac{a}{2(a + 2)} &\geq \frac{2(a + 1 - \sqrt{2a + 1})}{(a + 1)\sqrt{2a + 1}}, \\ \frac{a}{2(a + 2)} &\geq \frac{2a^2}{(a + 1)\sqrt{2a + 1} (a + 1 + \sqrt{2a + 1})}. \end{aligned}$$

Since

$$\sqrt{2a+1} (a+1+\sqrt{2a+1}) \geq \sqrt{2a+1}(\sqrt{2a+1}+\sqrt{2a+1}) = 2(2a+1),$$

it suffices to show that

$$\frac{a}{2(a+2)} \geq \frac{a^2}{(a+1)(2a+1)},$$

which is equivalent to

$$a(1-a) \geq 0.$$

The equality holds for

$$a = 0, \quad b = c$$

(or any cyclic permutation).

□

P 5.17. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{3-\sqrt{3}}{a+b+c} + \frac{2+\sqrt{3}}{2\sqrt{ab+bc+ca}}.$$

(Vasile C., 2010)

Solution. As shown in the proof of P 5.12, it suffices to consider the cases $a = 0$ and $a \leq b = c = 1$.

Case 1: $a = 0$. The inequality reduces to

$$\frac{1}{b} + \frac{1}{c} \geq \frac{2-\sqrt{3}}{b+c} + \frac{2+\sqrt{3}}{2\sqrt{bc}}.$$

It suffices to show that

$$\frac{1}{b} + \frac{1}{c} \geq \frac{2-\sqrt{3}}{2\sqrt{bc}} + \frac{2+\sqrt{3}}{2\sqrt{bc}},$$

which is equivalent to the obvious inequality

$$\frac{1}{b} + \frac{1}{c} \geq \frac{2}{\sqrt{bc}}.$$

Case 2: $a \leq 1 = b = c$. The inequality reduces to

$$\frac{1}{2} + \frac{2}{a+1} \geq \frac{3-\sqrt{3}}{a+2} + \frac{2+\sqrt{3}}{2\sqrt{2a+1}}.$$

Using the substitution

$$2a+1 = 3x^2, \quad x \geq \frac{\sqrt{3}}{3},$$

the inequality becomes

$$\frac{1}{2} + \frac{4}{3x^2+1} \geq \frac{6-2\sqrt{3}}{3(x^2+1)} + \frac{2+\sqrt{3}}{2\sqrt{3}x},$$

$$\begin{aligned}\frac{1}{2} + \frac{4}{3x^2+1} - \frac{2}{x^2+1} - \frac{1}{2x} &\geq \frac{1}{\sqrt{3}x} - \frac{2}{\sqrt{3}(x^2+1)}, \\ \frac{3x^5 - 3x^4 - 4x^2 + 5x - 1}{2x(x^2+1)(3x^2+1)} &\geq \frac{1}{\sqrt{3}} \left(\frac{1}{x} - \frac{2}{x^2+1} \right), \\ \frac{(x-1)^2(3x^3+3x^2+3x-1)}{2x(x^2+1)(3x^2+1)} &\geq \frac{(x-1)^2}{\sqrt{3}x(x^2+1)}.\end{aligned}$$

This is true if

$$\frac{3x^3 + 3x^2 + 3x - 1}{2(3x^2 + 1)} \geq \frac{\sqrt{3}}{3},$$

which is equivalent to

$$\begin{aligned}9x^3 + 3(3 - 2\sqrt{3})x^2 + 9x - 3 - 2\sqrt{3} &\geq 0, \\ (3x - \sqrt{3})[3x^2 + (3 - \sqrt{3})x + 2 + \sqrt{3}] &\geq 0.\end{aligned}$$

The equality holds for $a = b = c$, and also for

$$a = 0, \quad b = c$$

(or any cyclic permutation).

□

P 5.18. Let a, b, c be nonnegative real numbers, no two of which are zero, so that

$$ab + bc + ca = 3.$$

If

$$0 \leq k \leq \frac{9 + 5\sqrt{3}}{6} \approx 2.943,$$

then

$$\frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{c+a} \geq \frac{9(1+k)}{a+b+c+3k}.$$

(Vasile Cîrtoaje and Lorian Săceanu, 2014)

Solution. From

$$(a+b+c)^2 \geq 3(ab+bc+ca),$$

we get

$$a+b+c \geq 3.$$

Let

$$m = \frac{9 + 5\sqrt{3}}{6}, \quad m \geq k.$$

We claim that

$$\frac{1+m}{a+b+c+3m} \geq \frac{1+k}{a+b+c+3k}.$$

Indeed, this inequality is equivalent to the obvious inequality

$$(m-k)(a+b+c-3) \geq 0.$$

Thus, we only need to show that

$$\frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{c+a} \geq \frac{9(1+m)}{a+b+c+3m},$$

which can be rewritten in the homogeneous form

$$\frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{c+a} \geq \frac{9(1+m)}{a+b+c+m\sqrt{3(ab+bc+ca)}}.$$

As shown in the proof of P 5.12, it suffices to prove this homogeneous inequality for $a = 0$ and for $a \leq b = c = 1$.

Case 1: $a = 0$. The inequality reduces to

$$\frac{2}{b} + \frac{2}{c} + \frac{2}{b+c} \geq \frac{9(1+m)}{b+c+m\sqrt{3bc}}.$$

Substituting

$$x = \frac{b+c}{\sqrt{bc}}, \quad x \geq 2,$$

the inequality becomes

$$\begin{aligned} 2x + \frac{2}{x} &\geq \frac{9(1+m)}{x+m\sqrt{3}}, \\ 2x^3 + 2\sqrt{3}mx^2 - (7+9m)x + 2\sqrt{3}m &\geq 0, \\ (x-2)[2x^2 + 2(\sqrt{3}m+2)x - \sqrt{3}m] &\geq 0. \end{aligned}$$

Case 2: $a \leq 1 = b = c$. The inequality has the form

$$1 + \frac{4}{a+1} \geq \frac{9(1+m)}{a+2+m\sqrt{3(2a+1)}}.$$

Using the substitution

$$2a+1 = 3x^2, \quad x \geq \frac{\sqrt{3}}{3},$$

the inequality becomes

$$\begin{aligned} \frac{3x^2+9}{3x^2+1} &\geq \frac{6(1+m)}{x^2+2mx+1}, \\ x^4 + 2mx^3 - 2(3m+1)x^2 + 6mx + 1 - 2m &\geq 0, \\ (x-1)^2[x^2 + 2(m+1)x + 1 - 2m] &\geq 0, \end{aligned}$$

which is true since

$$\begin{aligned} x^2 + 2(m+1)x + 1 - 2m &\geq \frac{1}{3} + \frac{2(m+1)\sqrt{3}}{3} + 1 - 2m \\ &= \frac{2[2 + \sqrt{3} - (3 - \sqrt{3})m]}{3} = 0. \end{aligned}$$

The equality holds for $a = b = c = 1$. If $k = \frac{9+5\sqrt{3}}{6}$, then the equality holds also for

$$a = 0, \quad b = c = \sqrt{3}$$

(or any cyclic permutation).

□

P 5.19. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{20}{a+b+c+6\sqrt{ab+bc+ca}}.$$

(Vasile C., 2010)

Solution. The proof is similar to the one of P 5.12. Finally, we only need to prove the inequality for $a = 0$ and for $a \leq b = c = 1$.

Case 1: $a = 0$. The inequality reduces to

$$\frac{1}{b} + \frac{1}{c} + \frac{1}{b+c} \geq \frac{20}{b+c+6\sqrt{bc}}.$$

Substituting

$$x = \frac{b+c}{\sqrt{bc}}, \quad x \geq 2,$$

the inequality becomes

$$\begin{aligned} x + \frac{1}{x} &\geq \frac{20}{x+6}, \\ x^3 + 6x^2 - 19x + 6 &\geq 0, \\ (x-2)(x^2 + 8x - 3) &\geq 0. \end{aligned}$$

Case 2: $a \leq 1 = b = c$. We need to show that

$$\frac{1}{2} + \frac{2}{a+1} \geq \frac{20}{a+2+6\sqrt{2a+1}}.$$

Using the substitution

$$2a+1 = x^2, \quad x \geq 1,$$

the inequality becomes

$$\begin{aligned} \frac{x^2+9}{2(x^2+1)} &\geq \frac{40}{x^2+12x+3}, \\ x^4 + 12x^3 - 68x^2 + 108x - 53 &\geq 0, \\ (x-1)(x^3 + 13x^2 - 55x + 53) &\geq 0. \end{aligned}$$

It is true since

$$\begin{aligned} x^3 + 13x^2 - 55x + 53 &= (x-1)^3 + 16x^2 - 58x + 54 \\ &= (x-1)^3 + 16\left(x - \frac{29}{16}\right)^2 + \frac{23}{16} > 0. \end{aligned}$$

The equality holds for

$$a = 0, \quad b = c$$

(or any cyclic permutation).

□

P 5.20. If a, b, c are positive real numbers so that

$$7(a^2 + b^2 + c^2) = 11(ab + bc + ca),$$

then

$$\frac{51}{28} \leq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \leq 2.$$

(Vasile C., 2008)

Solution. Due to homogeneity and symmetry, we may consider that

$$a + b + c = 1, \quad 0 < a \leq b \leq c < 1.$$

Thus, we need to show that

$$a + b + c = 1, \quad a^2 + b^2 + c^2 = \frac{11}{25}, \quad 0 < a \leq b \leq c < 1$$

involves

$$\frac{51}{28} \leq \frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c} \leq 2.$$

We apply Corollary 1 to the function

$$f(u) = \frac{u}{1-u}, \quad 0 \leq u < 1.$$

We have $f(1-) = \infty$ and

$$g(x) = f'(x) = \frac{1}{(1-x)^2}, \quad g''(x) = \frac{6}{(1-x)^4}.$$

Since $g''(x) > 0$, g is strictly convex on $[0, 1)$. According to Corollary 1 and Note 3, if

$$a + b + c = 1, \quad a^2 + b^2 + c^2 = \frac{11}{25}, \quad 0 \leq a \leq b \leq c < 1,$$

then the sum

$$S_3 = f(a) + f(b) + f(c)$$

is maximum for $a = b \leq c$, and is minimum for either $a = 0$ or $0 < a \leq b = c$. Note that the case $a = 0$ is not possible because it involves $7(b^2 + c^2) = 11bc$, which is false.

(1) To prove the right original inequality for $a = b \leq c$, let us denote

$$t = \frac{c}{a}, \quad t \geq 1.$$

The hypothesis $7(a^2 + b^2 + c^2) = 11(ab + bc + ca)$ involves $t = 3$, hence

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = \frac{2a}{a+c} + \frac{c}{2a} = \frac{2}{1+t} + \frac{t}{2} = 2.$$

The right inequality is an equality for $a = b = \frac{c}{3}$ (or any cyclic permutation).

(2) To prove the left original inequality for $0 < a \leq b = c$, let us denote

$$t = \frac{a}{b}, \quad 0 < t \leq 1.$$

The hypothesis $7(a^2 + b^2 + c^2) = 11(ab + bc + ca)$ involves $t = \frac{1}{7}$, hence

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = \frac{a}{2b} + \frac{2b}{a+b} = \frac{t}{2} + \frac{2}{t+1} = \frac{51}{28}.$$

The left inequality is an equality for $7a = b = c$ (or any cyclic permutation).

□

P 5.21. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that

$$\frac{a_1^2 + a_2^2 + \dots + a_n^2}{n+3} = \left(\frac{a_1 + a_2 + \dots + a_n}{n+1} \right)^2,$$

then

$$\frac{(n+1)(2n-1)}{2} \leq (a_1 + a_2 + \dots + a_n) \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right) \leq \frac{3n^2(n+1)}{2(n+2)}.$$

(Vasile C., 2008)

Solution. For $n = 2$, both inequalities are identities. For $n \geq 3$, assume that

$$a_1 \leq a_2 \leq \dots \leq a_n.$$

The case $a_1 = 0$ is not possible because the hypothesis involves

$$\frac{a_2^2 + \dots + a_n^2}{(a_2 + \dots + a_n)^2} = \frac{n+3}{(n+1)^2} < \frac{1}{n-1},$$

which contradicts the Cauchy-Schwarz inequality

$$\frac{a_2^2 + \dots + a_n^2}{(a_2 + \dots + a_n)^2} \geq \frac{1}{n-1}.$$

Due to homogeneity and symmetry, we may consider that

$$a_1 + a_2 + \dots + a_n = n+1,$$

which implies

$$a_1^2 + a_2^2 + \dots + a_n^2 = n+3.$$

Thus, we need to show that

$$a_1 + a_2 + \dots + a_n = n+1, \quad a_1^2 + a_2^2 + \dots + a_n^2 = n+3, \quad 0 < a_1 \leq a_2 \leq \dots \leq a_n$$

involves

$$\frac{2n-1}{2} \leq \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \leq \frac{3n^2}{2(n+2)}.$$

We apply Corollary 5 for $k = 2$ and $m = -1$:

- If $a_1, a_2, \dots, a_n$ are positive real numbers so that $0 < a_1 \leq a_2 \leq \dots \leq a_n$ and

$$a_1 + a_2 + \dots + a_n = n + 1, \quad a_1^2 + a_2^2 + \dots + a_n^2 = n + 3,$$

then

$$S_n = \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}$$

is minimum for

$$0 < a_1 = a_2 = \dots = a_{n-1} \leq a_n,$$

and is maximum for

$$a_1 \leq a_2 = a_3 = \dots = a_n.$$

(1) To prove the left original inequality, we only need to consider the case

$$a_1 = a_2 = \dots = a_{n-1} \leq a_n.$$

The hypothesis

$$\frac{a_1^2 + a_2^2 + \dots + a_n^2}{n + 3} = \left(\frac{a_1 + a_2 + \dots + a_n}{n + 1} \right)^2$$

implies

$$\frac{(n-1)a_1^2 + a_n^2}{n+3} = \left[\frac{(n-1)a_1 + a_n}{n+1} \right]^2,$$

$$(2a_1 - a_n)[2a_1 - (n+2)a_n] = 0,$$

$$a_1 = \frac{a_n}{2},$$

hence

$$\begin{aligned} (a_1 + a_2 + \dots + a_n) \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right) &= [(n-1)a_1 + a_n] \left(\frac{n-1}{a_1} + \frac{1}{a_n} \right) \\ &= (n-1)^2 + 1 + (n-1) \left(\frac{a_1}{a_n} + \frac{a_n}{a_1} \right) \\ &= \frac{(n+1)(2n-1)}{2}. \end{aligned}$$

The equality holds for

$$a_1 = a_2 = \dots = a_{n-1} = \frac{a_n}{2}$$

(or any cyclic permutation).

(2) To prove the right original inequality, we only need to consider the case

$$a_1 \leq a_2 = a_3 = \dots = a_n.$$

The hypothesis involves

$$(a_1 - 2a_n)[(n+2)a_1 - 2a_n] = 0,$$

$$a_1 = \frac{2a_n}{n+2},$$

hence

$$\begin{aligned} (a_1 + a_2 + \cdots + a_n) \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right) &= [(n-1)a_1 + a_n] \left(\frac{n-1}{a_1} + \frac{1}{a_n} \right) \\ &= (n-1)^2 + 1 + (n-1) \left(\frac{a_1}{a_n} + \frac{a_n}{a_1} \right) \\ &= \frac{3n^2(n+1)}{2(n+2)}. \end{aligned}$$

The equality holds for

$$a_1 = a_2 = \cdots = a_{n-1} = \frac{2a_n}{n+2}$$

(or any cyclic permutation).

□

P 5.22. If a, b, c, d are nonnegative real numbers so that $a + b + c + d = 3$, then

$$abc + bcd + cda + dab \leq 1 + \frac{176}{81} abcd.$$

(Vasile C., 2005)

Solution. Assume that

$$a \leq b \leq c \leq d.$$

For $a = 0$, we need to show that $b + c + d = 3$ implies

$$bcd \leq 1,$$

which follows immediately from the AM-GM inequality:

$$bcd \leq \left(\frac{b+c+d}{3} \right)^3 = 1.$$

For $a > 0$, rewrite the inequality in the form

$$abcd \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right) \leq 1 + \frac{176}{81} abcd$$

and apply Corollary 5 for $k = 0$ and $m = -1$:

• If

$$a + b + c + d = 3, \quad abcd = \text{fixed}, \quad 0 < a \leq b \leq c \leq d,$$

then

$$S_4 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$$

is maximum for

$$a \leq b = c = d.$$

Thus, we only need to prove the homogeneous inequality

$$27(a+b+c+d)(abc+bcd+cda+dab) \leq (a+b+c+d)^4 + 176abcd$$

for $a \leq b = c = d = 1$. The inequality becomes

$$27(a+3)(3a+1) \leq (a+3)^4 + 176a,$$

$$a^4 + 12a^3 - 27a^2 + 14a \geq 0,$$

$$a(a-1)^2(a+14) \geq 0.$$

The equality holds for $a = b = c = d = 3/4$, and also for

$$a = 0, \quad b = c = d = 1$$

(or any cyclic permutation).

□

P 5.23. If a, b, c, d are nonnegative real numbers so that $a + b + c + d = 3$, then

$$a^2b^2c^2 + b^2c^2d^2 + c^2d^2a^2 + d^2a^2b^2 + \frac{3}{4}abcd \leq 1.$$

(Gabriel Dospinescu and Vasile Cîrtoaje, 2005)

Solution. Assume that

$$a \leq b \leq c \leq d.$$

For $a = 0$, we need to show that

$$b^2c^2d^2 \leq 1,$$

which follows immediately from the AM-GM inequality:

$$bcd \leq \left(\frac{b+c+d}{3} \right)^3 = 1.$$

For $a > 0$, rewrite the inequality in the form

$$a^2b^2c^2d^2 \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} \right) + \frac{3}{4}abcd \leq 1,$$

and apply Corollary 5 for $k = 0$ and $m = -2$:

• If

$$a + b + c + d = 3, \quad abcd = \text{fixed}, \quad 0 < a \leq b \leq c \leq d,$$

then

$$S_4 = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2}$$

is maximum for $a \leq b = c = d$.

Thus, we only need to prove the homogeneous inequality

$$\left(\frac{a+b+c+d}{3}\right)^6 \geq a^2b^2c^2 + b^2c^2d^2 + c^2d^2a^2 + d^2a^2b^2 + \frac{1}{12}abcd(a+b+c+d)^2$$

for $a \leq b = c = d = 1$; that is, to show that $0 < a \leq 1$ implies

$$\left(1 + \frac{a}{3}\right)^6 \geq 1 + 3a^2 + \frac{1}{12}a(a+3)^2.$$

Since

$$\left(1 + \frac{a}{3}\right)^3 = 1 + a + \frac{a^2}{3} + \frac{a^3}{27} > 1 + a + \frac{a^2}{3},$$

it suffices to show that

$$\left(1 + a + \frac{a^2}{3}\right)^2 \geq 1 + 3a^2 + \frac{1}{12}a(a+3)^2,$$

which is equivalent to the obvious inequality

$$4a^4 + 3a(1-a)(15-7a) \geq 0.$$

The equality holds for

$$a = 0, \quad b = c = d = 1$$

(or any cyclic permutation).

□

P 5.24. If a, b, c, d are nonnegative real numbers so that $a + b + c + d = 3$, then

$$a^2b^2c^2 + b^2c^2d^2 + c^2d^2a^2 + d^2a^2b^2 + \frac{4}{3}(abcd)^{3/2} \leq 1.$$

(Vasile C., 2005)

Solution. The proof is similar to the one of the preceding P 5.23. We need to prove that

$$\left(1 + \frac{a}{3}\right)^6 \geq 1 + 3a^2 + \frac{4}{3}a^{3/2}$$

for $0 \leq a \leq 1$. Since

$$2a^{3/2} \leq a^2 + a,$$

it suffices to show that

$$\left(1 + \frac{a}{3}\right)^6 \geq 1 + \frac{2}{3}a + \frac{11}{3}a^2.$$

Since

$$\left(1 + \frac{a}{3}\right)^3 = 1 + a + \frac{a^2}{3} + \frac{a^3}{27} \geq 1 + a + \frac{a^2}{3}$$

and

$$\begin{aligned}\left(1 + a + \frac{a^2}{3}\right)^2 &= 1 + 2a + \frac{5}{3}a^2 + \frac{2}{3}a^3 + \frac{1}{9}a^4 \\ &\geq 1 + 2a + \frac{5}{3}a^2 + \frac{2}{3}a^3,\end{aligned}$$

it suffices to show that

$$1 + 2a + \frac{5}{3}a^2 + \frac{2}{3}a^3 \geq 1 + \frac{2}{3}a + \frac{11}{3}a^2,$$

which is equivalent to the obvious inequality

$$a(1 - a)(2 - a) \geq 0.$$

The equality holds for

$$a = 0, \quad b = c = d = 1$$

(or any cyclic permutation).

□

P 5.25. If a, b, c, d are nonnegative real numbers so that $a + b + c + d = 4$, then

$$a^2b^2c^2 + b^2c^2d^2 + c^2d^2a^2 + d^2a^2b^2 + 2(abcd)^{3/2} \leq 6.$$

(Vasile C., 2005)

Solution. The proof is similar to the one of P 5.23. We need to prove that

$$6\left(\frac{a+3}{4}\right)^6 \geq 1 + 3a^2 + 2a^{3/2}$$

for $0 \leq a \leq 1$. Since

$$2a^{3/2} \leq a^2 + a,$$

it suffices to show that

$$6\left(\frac{a+3}{4}\right)^6 \geq 1 + a + 4a^2.$$

Using the substitution

$$x = \frac{1-a}{4}, \quad 0 \leq x \leq \frac{1}{4},$$

the inequality becomes

$$\begin{aligned}3(1-x)^6 &\geq 3 - 18x + 32x^2, \\ x^2(13 - 60x + 45x^2 - 18x^3 + 3x^4) &\geq 0.\end{aligned}$$

It is true since

$$\begin{aligned}2(13 - 60x + 45x^2 - 18x^3 + 3x^4) &> 25 - 120x + 90x^2 - 40x^3 \\ &= 5(1 - 4x)(5 - 4x + 2x^2) \geq 0.\end{aligned}$$

The equality holds for $a = b = c = d = 1$.

□

P 5.26. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$11(ab + bc + ca) + 4(a^2b^2 + b^2c^2 + c^2a^2) \leq 45.$$

(Vasile C., 2005)

Solution. Assume that $a \leq b \leq c$. For $a = 0$, we need to show that $b + c = 3$ involves

$$11bc + 4b^2c^2 \leq 45.$$

We have

$$bc \leq \left(\frac{b+c}{2}\right)^2 = \frac{9}{4},$$

hence

$$11bc + 4b^2c^2 \leq \frac{99}{4} + \frac{81}{4} = 45.$$

For $a > 0$, rewrite the desired inequality in the form

$$11abc \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + 4a^2b^2c^2 \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) \leq 45.$$

According to Corollary 5 (case $k = 2$ and $m < 0$), if

$$a + b + c = 3, \quad abc = \text{fixed}, \quad 0 < a \leq b \leq c,$$

then the sums $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ and $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$ are maximum for $0 < a \leq b = c$.

Therefore, we only need to prove that $a + 2b = 3$ involves

$$11(2ab + b^2) + 4(2a^2b^2 + b^4) \leq 45,$$

which is equivalent to

$$15 - 22b - 13b^2 + 32b^3 - 12b^4 \geq 0,$$

$$(3 - 2b)(1 - b)^2(5 + 6b) \geq 0,$$

$$a(1 - b)^2(5 + 6b) \geq 0.$$

The equality holds for $a = b = c = 1$, and also for

$$a = 0, \quad b = c = \frac{3}{2}$$

(or any cyclic permutation).

Remark. In the same manner, we can prove the following statement:

- If a, b, c, d are nonnegative real numbers so that $a + b + c + d = 4$, then

$$abc + bcd + cda + dab + a^2b^2c^2 + b^2c^2d^2 + c^2d^2a^2 + d^2a^2b^2 \leq 8,$$

with equality for $a = b = c = d = 1$.

□

P 5.27. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$a^2b^2 + b^2c^2 + c^2a^2 + a^3b^3 + b^3c^3 + c^3a^3 \geq 6abc.$$

(Vasile C., 2005)

Solution. Assume that $a \leq b \leq c$. For $a = 0$, the inequality is trivial. For $a > 0$, rewrite the desired inequality in the form

$$abc \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right) + a^2b^2c^2 \left(\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} \right) \geq 6.$$

According to Corollary 5 (case $k = 0$ and $m < 0$), if

$$a + b + c = 3, \quad abc = \text{fixed}, \quad 0 < a \leq b \leq c,$$

then the sums $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$ and $\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3}$ are maximum for $0 < a \leq b = c$.

Thus, we only need to prove that

$$2a^2b^2 + b^4 + 2a^3b^3 + b^6 \geq 6ab^2$$

for

$$a + 2b = 3, \quad 1 \leq b < 3/2.$$

The inequality is equivalent to

$$b^3(14 - 33b + 24b^2 - 5b^3) \geq 0,$$

$$b^3(1 - b)^2(14 - 5b) \geq 0.$$

The equality holds for $a = b = c = 1$, and also for

$$a = b = 0, \quad c = 3$$

(or any cyclic permutation).

□

P 5.28. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$2(a^2 + b^2 + c^2) + 5(\sqrt{a} + \sqrt{b} + \sqrt{c}) \geq 21.$$

(Vasile C., 2008)

Solution. Apply Corollary 5 for $k = 2$ and $m = 1/2$:

• If

$$a + b + c = 3, \quad a^2 + b^2 + c^2 = \text{fixed}, \quad 0 \leq a \leq b \leq c,$$

then

$$S_3 = \sqrt{a} + \sqrt{b} + \sqrt{c}$$

is minimum for either $a = 0$ or $0 < a \leq b = c$.

Case 1: $a = 0$. We need to show that $b + c = 3$ involves

$$2(b^2 + c^2) + 5(\sqrt{b} + \sqrt{c}) \geq 21,$$

which is equivalent to

$$5\sqrt{3 + 2\sqrt{bc}} \geq 3 + 4bc.$$

Substituting

$$x = \sqrt{bc}, \quad 0 \leq x \leq \frac{b+c}{2} = \frac{3}{2},$$

the inequality becomes

$$\begin{aligned} 5\sqrt{3 + 2x} &\geq 3 + 4x^2, \\ 25(3 + 2x) &\geq (3 + 4x^2)^2. \end{aligned}$$

This inequality is equivalent to $f(x) \geq 0$, where

$$f(x) = \frac{66}{x} + 50 - 24x - 16x^3, \quad 0 < x \leq 3/2.$$

Since f is decreasing, we have

$$f(x) \geq f(3/2) = 4 > 0.$$

Case 2: $0 < a \leq b = c$. We need to show that

$$2(a^2 + 2b^2) + 5(\sqrt{a} + 2\sqrt{b}) \geq 21$$

for

$$a + 2b = 3, \quad 1 \leq b < \frac{3}{2}.$$

Write the inequality as

$$5\sqrt{3 - 2b} + 10\sqrt{b} \geq 3 + 24b - 12b^2.$$

Substituting

$$x = \sqrt{b}, \quad 1 \leq x < \sqrt{\frac{3}{2}},$$

the inequality becomes

$$\begin{aligned} 5\sqrt{3 - 2x^2} &\geq 3 - 10x + 24x^2 - 12x^4, \\ 12(x^2 - 1)^2 &\geq 5(3 - 2x - \sqrt{3 - 2x^2}), \\ 12(x^2 - 1)^2 &\geq \frac{30(x - 1)^2}{3 - 2x + \sqrt{3 - 2x^2}}, \end{aligned}$$

which is true if

$$2(x + 1)^2 \geq \frac{5}{3 - 2x + \sqrt{3 - 2x^2}}.$$

It suffices to show that

$$2(x + 1)^2 \geq \frac{5}{3 - 2x},$$

which is equivalent to

$$1 + 8x - 2x^2 - 4x^3 \geq 0,$$

$$x(5 - 4x) \left(\frac{7}{4} + x \right) + \frac{4 - 3x}{4} \geq 0.$$

Since

$$x < \sqrt{\frac{3}{2}} < \frac{5}{4} < \frac{4}{3},$$

the conclusion follows.

The equality holds for $a = b = c = 1$.

□

P 5.29. If a, b, c are nonnegative real numbers so that $ab + bc + ca = 3$, then

$$\sqrt{\frac{1+2a}{3}} + \sqrt{\frac{1+2b}{3}} + \sqrt{\frac{1+2c}{3}} \geq 3.$$

(Vasile C., 2008)

Solution. Write the hypothesis $ab + bc + ca = 3$ as

$$(a + b + c)^2 = 6 + a^2 + b^2 + c^2,$$

and apply Corollary 1 to

$$f(u) = \sqrt{\frac{1+2u}{3}}, \quad u \geq 0.$$

We have

$$g(x) = f'(x) = \frac{1}{\sqrt{3(1+2x)}},$$

$$g''(x) = \frac{\sqrt{3}}{(1+2x)^{5/2}}.$$

Since $g''(x) > 0$ for $x \geq 0$, g is strictly convex on $[0, \infty)$. According to Corollary 1, if

$$a + b + c = \text{fixed}, \quad a^2 + b^2 + c^2 = \text{fixed}, \quad 0 \leq a \leq b \leq c,$$

then the sum

$$S_3 = f(a) + f(b) + f(c)$$

is minimum for either $a = 0$ or $0 < a \leq b = c$.

Case 1: $a = 0$. We need to show that $bc = 3$ involves

$$\sqrt{1+2b} + \sqrt{1+2c} \geq 3\sqrt{3} - 1.$$

By squaring, the inequality becomes

$$b + c + \sqrt{13 + 2(b+c)} \geq 13 - 3\sqrt{3}.$$

We have $b + c \geq 2\sqrt{bc} = 2\sqrt{3}$, hence

$$b + c + \sqrt{13 + 2(b + c)} \geq 2\sqrt{3} + \sqrt{13 + 4\sqrt{3}} = 4\sqrt{3} + 1 > 13 - 3\sqrt{3}.$$

Case 2: $0 < a \leq b = c$. From $ab + bc + ca = 3$, it follows that

$$a = \frac{3 - b^2}{2b}, \quad 1 \leq b < \sqrt{3}.$$

Thus, the inequality can be written as

$$\sqrt{1 + \frac{3 - b^2}{b}} + 2\sqrt{1 + 2b} \geq 3\sqrt{3}.$$

Substituting

$$t = \sqrt{\frac{1 + 2b}{3}}, \quad 1 \leq t < \sqrt{\frac{1 + 2\sqrt{3}}{3}} < \frac{5}{4},$$

the inequality turns into

$$\sqrt{\frac{3 + 4t^2 - 3t^4}{2(3t^2 - 1)}} \geq 3 - 2t.$$

By squaring, we need to show that

$$7 - 8t - 14t^2 + 24t^3 - 9t^4 \geq 0,$$

which is equivalent to

$$(1 - t)^2(7 + 6t - 9t^2) \geq 0.$$

This is true since

$$7 + 6t - 9t^2 = 8 - (3t - 1)^2 > 8 - \left(\frac{15}{4} - 1\right)^2 = \frac{7}{16} > 0.$$

The equality holds for $a = b = c = 1$.

Remark. The following generalization holds:

- Let a, b, c be nonnegative real numbers such that $ab + bc + ca = 3$. If

$$k \geq k_0 = \frac{14\sqrt{3} - 15}{24} \approx 0.38536,$$

then

$$\sqrt{\frac{a+k}{1+k}} + \sqrt{\frac{b+k}{1+k}} + \sqrt{\frac{c+k}{1+k}} \geq 3,$$

with equality for $a = b = c = 1$. If $k = k_0$, then the equality also occurs for $a = 0$ and $b = c = \sqrt{3}$ (or any cyclic permutation).

□

P 5.30. Let a, b, c be nonnegative real numbers, no two of which are zero. If

$$0 \leq k \leq 15,$$

then

$$\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} + \frac{k}{(a+b+c)^2} \geq \frac{9+k}{4(ab+bc+ca)}.$$

(Vasile C., 2007)

Solution. Due to homogeneity and symmetry, we may consider that

$$a+b+c=1, \quad 0 \leq a \leq b \leq c.$$

On this assumption, the inequality becomes

$$\frac{1}{(1-a)^2} + \frac{1}{(1-b)^2} + \frac{1}{(1-c)^2} + k \geq \frac{9+k}{2(1-a^2-b^2-c^2)}.$$

To prove it, we apply Corollary 1 to the function

$$f(u) = \frac{1}{(1-u)^2}, \quad 0 \leq u < 1.$$

We have $f(1-) = \infty$ and

$$g(x) = f'(x) = \frac{2}{(1-x)^3}, \quad g''(x) = \frac{24}{(1-x)^5}.$$

Since $g''(x) > 0$, g is strictly convex on $[0, 1)$. According to Corollary 1 and Note 3, if

$$a+b+c=1, \quad a^2+b^2+c^2 = \text{fixed}, \quad 0 \leq a \leq b \leq c,$$

then the sum

$$S_3 = f(a) + f(b) + f(c)$$

is minimum for either $a=0$ or $0 < a \leq b=c$.

Case 1: $a=0$. For

$$x = \frac{b}{c} + \frac{c}{b}, \quad x \geq 2,$$

the original inequality becomes

$$\frac{1}{b^2} + \frac{1}{c^2} + \frac{1+k}{(b+c)^2} \geq \frac{9+k}{4bc},$$

$$x + \frac{1+k}{x+2} \geq \frac{9+k}{4},$$

$$(x-2)(4x+7-k) \geq 0.$$

This is true since

$$4x+7-k \geq 15-k \geq 0.$$

Case 2: $0 < a \leq b = c$. The original inequality becomes

$$\frac{2}{(a+b)^2} + \frac{1}{4b^2} + \frac{k}{(a+2b)^2} \geq \frac{9+k}{4b(2a+b)},$$

$$\frac{a(a-b)^2}{2b^2(a+b)^2(2a+b)} + \frac{ka(4b-a)}{4b(a+2b)^2(2a+b)} \geq 0.$$

The equality holds for

$$a = 0, \quad b = c$$

(or any cyclic permutation). If $k = 0$ (Iran 1996 inequality), then the equality holds also for $a = b = c$.

□

P 5.31. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} + \frac{24}{(a+b+c)^2} \geq \frac{8}{ab+bc+ca}.$$

(Vasile C., 2007)

Solution. As shown in the proof of the preceding P 5.30, it suffices to prove the inequality for $a = 0$, and for $0 < a \leq b = c$.

Case 1: $a = 0$. For

$$x = \frac{b}{c} + \frac{c}{b}, \quad x \geq 2,$$

the original inequality becomes

$$\frac{1}{b^2} + \frac{1}{c^2} + \frac{25}{(b+c)^2} \geq \frac{8}{bc},$$

$$x + \frac{25}{x+2} \geq 8,$$

$$(x-3)^2 \geq 0.$$

Case 2: $0 < a \leq b = c$. Due to homogeneity, we only need to prove the homogeneous inequality for $0 < a \leq b = c = 1$; that is,

$$\frac{2}{(a+1)^2} + \frac{1}{4} + \frac{24}{(a+2)^2} \geq \frac{8}{2a+1}.$$

It suffices to show that

$$\frac{2}{(a+1)^2} \geq \frac{8}{2a+1} - \frac{24}{(a+2)^2},$$

which is equivalent to

$$\frac{1}{(1+a)^2} \geq \frac{4(1-a)^2}{(2a+1)(a+2)^2},$$

$$a(2a^2 + 9a + 12) \geq 4a^2(a^2 - 2).$$

This is true since

$$a(2a^2 + 9a + 12) \geq 0 \geq 4a^2(a^2 - 2).$$

The equality holds for

$$a = 0, \quad \frac{b}{c} + \frac{c}{b} = 3$$

(or any cyclic permutation).

Remark. Actually, the following generalization holds:

- Let a, b, c be nonnegative real numbers, no two of which are zero. If $k \geq 15$, then

$$\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} + \frac{k}{(a+b+c)^2} \geq \frac{2(\sqrt{k+1} - 1)}{ab + bc + ca},$$

with equality for

$$a = 0, \quad \frac{b}{c} + \frac{c}{b} = \sqrt{k+1} - 2$$

(or any cyclic permutation).

□

P 5.32. If a, b, c are nonnegative real numbers, no two of which are zero, so that

$$k(a^2 + b^2 + c^2) + (2k + 3)(ab + bc + ca) = 9(k + 1), \quad 0 \leq k \leq 6,$$

then

$$\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} + \frac{9k}{(a+b+c)^2} \geq \frac{3}{4} + k.$$

(Vasile C., 2007)

Solution. Write the inequality in the homogeneous form

$$\frac{4}{(a+b)^2} + \frac{4}{(b+c)^2} + \frac{4}{(c+a)^2} + \frac{36k}{(a+b+c)^2} \geq \frac{9(k+1)(4k+3)}{k(a^2 + b^2 + c^2) + (2k+3)(ab + bc + ca)}.$$

As shown in the proof of P 5.30, it suffices to prove this inequality for $a = 0$, and for $0 < a \leq b = c$.

Case 1: $a = 0$. Let

$$x = \frac{b}{c} + \frac{c}{b}, \quad x \geq 2.$$

The homogeneous inequality becomes

$$4 \left(\frac{1}{b^2} + \frac{1}{c^2} \right) + \frac{36k+4}{(b+c)^2} \geq \frac{9(k+1)(4k+3)}{k(b^2 + c^2) + (2k+3)bc},$$

$$4x + \frac{36k+4}{x+2} \geq \frac{9(k+1)(4k+3)}{kx + 2k + 3},$$

$$4kx^3 + 4(4k+3)x^2 - (43k+3)x - 2(5k+21) \geq 0,$$

$$(x-2)[4kx^2 + 4(6k+3)x + 5k+21] \geq 0.$$

Case 2: $0 < a \leq b = c$. We only need to prove the homogeneous inequality for $b = c = 1$. The inequality becomes

$$\frac{8}{(a+1)^2} + 1 + \frac{36k}{(a+2)^2} \geq \frac{9(k+1)(4k+3)}{ka^2 + (4k+6)a + 4k+3},$$

$$ka^6 + (10k+6)a^5 - (14k-12)a^4 - (10k+18)a^3 + (17k-24)a^2 + (24-4k)a \geq 0,$$

$$a(a-1)^2[ka^3 + 6(2k+1)a^2 + 3(3k+8)a + 4(6-k)] \geq 0.$$

Clearly, the last inequality is true for $0 \leq k \leq 6$.

The equality holds for $a = b = c$, and also for

$$a = 0, \quad b = c$$

(or any cyclic permutation).

□

P 5.33. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$(a) \quad \frac{2}{a^2+b^2} + \frac{2}{b^2+c^2} + \frac{2}{c^2+a^2} \geq \frac{8}{a^2+b^2+c^2} + \frac{1}{ab+bc+ca};$$

$$(b) \quad \frac{2}{a^2+b^2} + \frac{2}{b^2+c^2} + \frac{2}{c^2+a^2} \geq \frac{7}{a^2+b^2+c^2} + \frac{6}{(a+b+c)^2};$$

$$(c) \quad \frac{2}{a^2+b^2} + \frac{2}{b^2+c^2} + \frac{2}{c^2+a^2} \geq \frac{45}{4(a^2+b^2+c^2) + ab+bc+ca}.$$

(Vasile C., 2007)

Solution. (a) Due to homogeneity and symmetry, we may consider that

$$a^2 + b^2 + c^2 = 1, \quad 0 \leq a \leq b \leq c.$$

On this assumption, the inequality can be written as

$$\frac{2}{1-a^2} + \frac{2}{1-b^2} + \frac{2}{1-c^2} \geq 8 + \frac{2}{(a+b+c)^2 - 1}.$$

To prove it, we apply Corollary 1 to the function

$$f(u) = \frac{1}{1-u^2}, \quad 0 \leq u < 1.$$

We have $f(1-) = \infty$ and

$$g(x) = f'(x) = \frac{2x}{(1-x^2)^2}, \quad g''(x) = \frac{24x(1+x^2)}{(1-x^2)^4}.$$

Since $g''(x) > 0$ for $x \in (0, 1)$, g is strictly convex on $[0, 1)$. According to Corollary 1 and Note 3, if

$$a + b + c = \text{fixed}, \quad a^2 + b^2 + c^2 = 1, \quad 0 \leq a \leq b \leq c,$$

then the sum

$$S_3 = f(a) + f(b) + f(c)$$

is minimum for either $a = 0$ or $0 < a \leq b = c$.

Case 1: $a = 0$. For

$$x = \frac{b}{c} + \frac{c}{b}, \quad x \geq 2,$$

the original inequality becomes

$$\frac{2}{b^2} + \frac{2}{c^2} \geq \frac{6}{b^2 + c^2} + \frac{1}{bc},$$

$$2x \geq \frac{6}{x} + 1,$$

$$(x - 2)(2x + 3) \geq 0.$$

Case 2: $0 < a \leq b = c$. Due to homogeneity, it suffices to prove the original inequality for $b = c = 1$. Thus, we need to show that

$$1 + \frac{4}{a^2 + 1} \geq \frac{8}{a^2 + 2} + \frac{1}{2a + 1},$$

which is equivalent to

$$\frac{2a}{2a + 1} \geq \frac{4a^2}{(a^2 + 1)(a^2 + 2)},$$

$$a(a^4 - a^2 - 2a + 2) \geq 0,$$

$$a(a - 1)^2(a^2 + 2a + 2) \geq 0.$$

The equality holds for $a = b = c$, and also for $a = 0, b = c$ (or any cyclic permutation).

(b) The proof is similar to the one of the inequality in (a). For $a = 0$ and

$$x = \frac{b}{c} + \frac{c}{b}, \quad x \geq 2,$$

the original inequality becomes

$$\frac{2}{b^2} + \frac{2}{c^2} \geq \frac{5}{b^2 + c^2} + \frac{6}{(b + c)^2},$$

$$2x \geq \frac{5}{x} + \frac{6}{x + 2},$$

$$(x - 2)(2x^2 + 8x + 5) \geq 0.$$

For $b = c = 1$, the original inequality is

$$1 + \frac{4}{a^2 + 1} \geq \frac{7}{a^2 + 2} + \frac{6}{(a + 2)^2},$$

$$a(a^5 + 4a^4 - 2a^3 - 15a + 12) \geq 0,$$

$$a(a-1)^2(a^3 + 6a^2 + 9a + 12) \geq 0.$$

The equality holds for $a = b = c$, and also for $a = 0, b = c$ (or any cyclic permutation).

(c) The proof is also similar to the one of the inequality in (a). For $a = 0$ and

$$x = \frac{b}{c} + \frac{c}{b}, \quad x \geq 2,$$

the original inequality becomes

$$2 \left(\frac{1}{b^2} + \frac{1}{c^2} \right) + \frac{2}{b^2 + c^2} \geq \frac{45}{4(b^2 + c^2) + bc},$$

$$2x + \frac{2}{x} \geq \frac{45}{4x + 1},$$

$$(x-2)(8x^2 + 18x - 1) \geq 0.$$

For $b = c = 1$, the original inequality can be written as

$$1 + \frac{4}{a^2 + 1} \geq \frac{45}{4a^2 + 2a + 9},$$

$$a(2a^3 + a^2 - 8a + 5) \geq 0,$$

$$a(a-1)^2(2a+5) \geq 0.$$

The equality holds for $a = b = c$, and also for $a = 0, b = c$ (or any cyclic permutation).

□

P 5.34. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{1}{a^2 + b^2} + \frac{1}{b^2 + c^2} + \frac{1}{c^2 + a^2} + \frac{3}{a^2 + b^2 + c^2} \geq \frac{4}{ab + bc + ca}.$$

(Vasile C., 2007)

Solution. As shown in the proof of the preceding P 5.33, it suffices to prove the inequality for $a = 0$, and for $0 < a \leq b = c$.

Case 1: $a = 0$. For

$$x = \frac{b}{c} + \frac{c}{b}, \quad x \geq 2,$$

the original inequality becomes

$$\frac{1}{b^2} + \frac{1}{c^2} + \frac{4}{b^2 + c^2} \geq \frac{4}{bc},$$

$$x + \frac{4}{x} \geq 4,$$

$$(x-2)^2 \geq 0.$$

Case 2: $0 < a \leq b = c$. Due to homogeneity, it suffices to prove the original inequality for $0 < a \leq b = c = 1$. Thus, we need to show that

$$\frac{1}{2} + \frac{2}{a^2 + 1} + \frac{3}{a^2 + 2} \geq \frac{4}{2a + 1}.$$

It suffices to show that

$$\frac{2}{a + 1} + \frac{3}{a + 2} \geq \frac{4}{2a + 1} - \frac{1}{2},$$

which is equivalent to

$$\begin{aligned} \frac{5a + 7}{a^2 + 3a + 2} &\geq \frac{7 - 2a}{4a + 2}, \\ a(2a^2 + 19a + 21) &\geq 0, \end{aligned}$$

The equality holds for

$$a = 0, \quad b = c$$

(or any cyclic permutation).

Remark. Actually, the following generalization holds:

- Let a, b, c be nonnegative real numbers, no two of which are zero.

(a) If $-4 \leq k \leq 3$, then

$$\frac{2}{a^2 + b^2} + \frac{2}{b^2 + c^2} + \frac{2}{c^2 + a^2} + \frac{2k}{a^2 + b^2 + c^2} \geq \frac{k + 5}{ab + bc + ca},$$

with equality for

$$a = 0, \quad b = c$$

(or any cyclic permutation).

(b) If $k \geq 3$, then

$$\frac{1}{a^2 + b^2} + \frac{1}{b^2 + c^2} + \frac{1}{c^2 + a^2} + \frac{k}{a^2 + b^2 + c^2} \geq \frac{2\sqrt{k+1}}{ab + bc + ca},$$

with equality for

$$a = 0, \quad \frac{b}{c} + \frac{c}{b} = \sqrt{k+1}$$

(or any cyclic permutation).

□

P 5.35. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$(a) \quad \frac{3}{a^2 + ab + b^2} + \frac{3}{b^2 + bc + c^2} + \frac{3}{c^2 + ca + a^2} \geq \frac{5}{ab + bc + ca} + \frac{4}{a^2 + b^2 + c^2};$$

$$(b) \quad \frac{3}{a^2 + ab + b^2} + \frac{3}{b^2 + bc + c^2} + \frac{3}{c^2 + ca + a^2} \geq \frac{1}{ab + bc + ca} + \frac{24}{(a + b + c)^2};$$

$$(c) \quad \frac{1}{a^2 + ab + b^2} + \frac{1}{b^2 + bc + c^2} + \frac{1}{c^2 + ca + a^2} \geq \frac{21}{2(a^2 + b^2 + c^2) + 5(ab + bc + ca)}.$$

(Vasile C., 2007)

Solution. (a) Due to homogeneity and symmetry, we may consider that

$$a + b + c = 1, \quad 0 \leq a \leq b \leq c.$$

Let

$$p = \frac{1 + a^2 + b^2 + c^2}{2}.$$

Since

$$\frac{1}{2(b^2 + bc + c^2)} = \frac{1}{(a + b + c)^2 + a^2 + b^2 + c^2 - 2a(a + b + c)} = \frac{1}{2(p - a)},$$

the inequality can be written as

$$\frac{3}{p - a} + \frac{3}{p - b} + \frac{3}{p - c} \geq \frac{5}{1 - p} + \frac{4}{2p - 1}.$$

To prove it, we apply Corollary 1 to the function

$$f(u) = \frac{3}{p - u}, \quad 0 \leq u < p.$$

We have $f(p-) = \infty$ and

$$g(x) = f'(x) = \frac{3}{(p - x)^2}, \quad g''(x) = \frac{18}{(p - x)^4}.$$

Since $g''(x) > 0$, g is strictly convex on $[0, p)$. According to Corollary 1 and Note 3, if

$$a + b + c = 1, \quad a^2 + b^2 + c^2 = 2p - 1 = \text{fixed}, \quad 0 \leq a \leq b \leq c,$$

then the sum

$$S_3 = f(a) + f(b) + f(c)$$

is minimum for either $a = 0$ or $0 < a \leq b = c$.

Case 1: $a = 0$. For

$$x = \frac{b}{c} + \frac{c}{b}, \quad x \geq 2,$$

the original inequality becomes

$$3 \left(\frac{1}{b^2} + \frac{1}{c^2} \right) + \frac{3}{b^2 + bc + c^2} \geq \frac{5}{bc} + \frac{4}{b^2 + c^2},$$

which is equivalent to

$$3x + \frac{3}{x + 1} \geq 5 + \frac{4}{x},$$

$$(x - 2)(3x^2 + 4x + 2) \geq 0.$$

Case 2: $0 < a \leq b = c$. Due to homogeneity, it suffices to prove the original inequality for $b = c = 1$. Thus, we need to show that

$$\frac{6}{a^2 + a + 1} + 1 \geq \frac{5}{2a + 1} + \frac{4}{a^2 + 2},$$

which is equivalent to

$$\begin{aligned} a(a^4 - a^3 + 3a^2 - 7a + 4) &\geq 0, \\ a(a-1)^2(a^2 + a + 4) &\geq 0. \end{aligned}$$

The equality holds for $a = b = c$, and also for $a = 0, b = c$ (or any cyclic permutation).

(b) The proof is similar to the one of the inequality in (a). For $a = 0$, the original inequality becomes

$$3\left(\frac{1}{b^2} + \frac{1}{c^2}\right) + \frac{3}{b^2 + bc + c^2} \geq \frac{1}{bc} + \frac{24}{(b+c)^2},$$

which is equivalent to

$$\begin{aligned} 3x + \frac{3}{x+1} &\geq 1 + \frac{24}{x+2}, \quad x = \frac{b}{c} + \frac{c}{b}, \\ (x-2)(3x^2 + 14x + 10) &\geq 0. \end{aligned}$$

For $b = c = 1$, the original inequality becomes

$$\frac{6}{a^2 + a + 1} + 1 \geq \frac{1}{2a+1} + \frac{24}{a^2 + 2},$$

which is equivalent to

$$\begin{aligned} a(a^4 + 5a^3 - 9a^2 - a + 4) &\geq 0, \\ a(a-1)^2(a^2 + 7a + 4) &\geq 0. \end{aligned}$$

The equality holds for $a = b = c$, and also for $a = 0, b = c$ (or any cyclic permutation).

(c) The proof is similar to the one of the inequality in (a). For $a = 0$, the original inequality becomes

$$\frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{b^2 + bc + c^2} \geq \frac{21}{2(b^2 + c^2) + 5bc},$$

which is equivalent to

$$\begin{aligned} x + \frac{1}{x+1} &\geq \frac{21}{2x+5}, \quad x = \frac{b}{c} + \frac{c}{b}, \\ (x-2)(2x^2 + 11x + 8) &\geq 0. \end{aligned}$$

For $b = c = 1$, the original inequality becomes

$$\frac{2}{a^2 + a + 1} + \frac{1}{3} \geq \frac{21}{2a^2 + 10a + 9},$$

which is equivalent to

$$\begin{aligned} a(a^3 + 6a^2 - 15a + 8) &\geq 0, \\ a(a-1)^2(a+8) &\geq 0. \end{aligned}$$

The equality holds for $a = b = c$, and also for $a = 0, b = c$ (or any cyclic permutation).

□

P 5.36. Let f be a real-valued function, continuous on $[0, \infty)$ and differentiable on $(0, \infty)$, so that $f'''(u) \geq 0$ for $u \in (0, \infty)$. If $a, b, c \geq 0$, then

$$f(a^2 + 2bc) + f(b^2 + 2ca) + f(c^2 + 2ab) \leq f(a^2 + b^2 + c^2) + 2f(ab + bc + ca).$$

Solution. Denoting

$$x = a^2 + 2bc, \quad y = b^2 + 2ca, \quad z = c^2 + 2ab,$$

the inequality becomes

$$f(x) + f(y) + f(z) \leq f(a^2 + b^2 + c^2) + 2f(ab + bc + ca).$$

Assume that

$$a + b + c = \text{fixed}, \quad a^2 + b^2 + c^2 = \text{fixed},$$

which involve

$$2(ab + bc + ca) = (a + b + c)^2 - (a^2 + b^2 + c^2) = \text{fixed}.$$

We have

$$x + y + z = (a + b + c)^2 = \text{fixed},$$

$$x^2 + y^2 + z^2 = (a^2 + b^2 + c^2)^2 + 2(ab + bc + ca)^2 = \text{fixed}.$$

According to the EV-Theorem (Corollary 1), since $f'''(u) \geq 0$ for $u \in (0, \infty)$, the sum $f(x) + f(y) + f(z)$ is maximum for $x = y \leq z$, that is

$$a^2 + 2bc = b^2 + 2ca \leq c^2 + 2ab.$$

From $a^2 + 2bc = b^2 + 2ca$, we get $a = b$ or $a + b = 2c$. If $a + b = 2c$, the inequality $b^2 + 2ca \leq c^2 + 2ab$ is equivalent to $(b - c)^2 \leq 0$, which involves $b = c$. Thus it suffices to prove the required inequality for two equal variables, when the inequality is an identity.

The equality holds for $a = b$ or $b = c$ or $c = a$.

Remark 1. The inequality is also true for a real-valued function f , continuous on $(0, \infty)$ and differentiable on $(0, \infty)$, so that $f'''(u) \geq 0$ for $u \in (0, \infty)$ and $\lim_{u \rightarrow 0} f(u) = \pm\infty$.

Remark 2. The following inequalities hold:

$$\begin{aligned} \frac{1}{a^2 + 2bc} + \frac{1}{b^2 + 2ca} + \frac{1}{c^2 + 2ab} &\geq \frac{1}{a^2 + b^2 + c^2} + \frac{2}{ab + bc + ca}, \\ \sqrt{a^2 + 2bc} + \sqrt{b^2 + 2ca} + \sqrt{c^2 + 2ab} &\leq \sqrt{a^2 + b^2 + c^2} + 2\sqrt{ab + bc + ca}, \\ \frac{1}{\sqrt{a^2 + 2bc}} + \frac{1}{\sqrt{b^2 + 2ca}} + \frac{1}{\sqrt{c^2 + 2ab}} &\geq \frac{1}{\sqrt{a^2 + b^2 + c^2}} + \frac{2}{\sqrt{ab + bc + ca}}, \\ (a^2 + 2bc)(b^2 + 2ca)(c^2 + 2ab) &\leq (a^2 + b^2 + c^2)(ab + bc + ca)^2. \end{aligned}$$

□

P 5.37. If a, b, c are the lengths of the side of a triangle, then

$$\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} \leq \frac{85}{36(ab+bc+ca)}.$$

(Vasile C., 2007)

Solution. Use the substitution

$$a = y + z, \quad b = z + x, \quad c = x + y,$$

where x, y, z are nonnegative real numbers. Due to homogeneity and symmetry, we may consider that

$$x + y + z = 2, \quad 0 \leq x \leq y \leq z.$$

We need to show that

$$\frac{1}{(x+2)^2} + \frac{1}{(y+2)^2} + \frac{1}{(z+2)^2} \leq \frac{85}{18(12-x^2-y^2-z^2)},$$

which can be written as

$$f(x) + f(y) + f(z) + \frac{85}{18(12-x^2-y^2-z^2)} \geq 0,$$

where

$$f(u) = \frac{-1}{(u+2)^2}, \quad u \geq 0.$$

We have

$$g(x) = f'(x) = \frac{2}{(x+2)^3}, \quad g''(x) = \frac{24}{(x+2)^5}.$$

Since $g''(x) > 0$ for $x \geq 0$, g is strictly convex on $[0, \infty)$. According to Corollary 1, if

$$x + y + z = 2, \quad x^2 + y^2 + z^2 = \text{fixed}, \quad 0 \leq x \leq y \leq z,$$

then the sum

$$S_3 = f(x) + f(y) + f(z)$$

is minimum for either $x = 0$ or $0 < x \leq y = z$.

Case 1: $x = 0$. This implies $a = b + c$. Since

$$\frac{1}{(a+b)^2} + \frac{1}{(c+a)^2} = \frac{5(b^2+c^2)+8bc}{(2b^2+2c^2+5bc)^2}$$

and

$$ab + bc + ca = a(b+c) + bc = (b+c)^2 + bc = b^2 + c^2 + 3bc,$$

we need to show that

$$\frac{5(b^2+c^2)+8bc}{(2b^2+2c^2+5bc)^2} + \frac{1}{(b+c)^2} \leq \frac{85}{36(b^2+c^2+3bc)}.$$

For $bc = 0$, the inequality is true. For $bc \neq 0$, substituting

$$t = \frac{b}{c} + \frac{c}{b}, \quad t \geq 2,$$

the inequality becomes

$$\begin{aligned} \frac{5t+8}{(2t+5)^2} + \frac{1}{t+2} &\leq \frac{85}{36(t+3)}, \\ \frac{5t+8}{(2t+5)^2} &\leq \frac{49t+62}{36(t+2)(t+3)}. \end{aligned}$$

It suffices to show that

$$\frac{5t+8}{(2t+5)^2} \leq \frac{48t+64}{36(t+2)(t+3)},$$

which is equivalent to

$$\begin{aligned} \frac{5t+8}{(2t+5)^2} &\leq \frac{12t+16}{9(t+2)(t+3)}, \\ 3t^3 + 7t^2 - 10t - 32 &\geq 0, \\ (t-2)(3t^2 + 13t + 16) &\geq 0. \end{aligned}$$

Case 2: $0 < x \leq y = z$. This involves $b = c$. Since the original inequality is homogeneous, we may consider $b = c = 1$ and $0 \leq a \leq b + c = 2$. Thus, we only need to show that

$$\frac{1}{4} + \frac{2}{(a+1)^2} \leq \frac{85}{36(2a+1)},$$

which is equivalent to

$$(a-2)(9a^2 - 2a + 1) \leq 0.$$

The equality holds for a degenerated isosceles triangle with $a = b + c$, $b = c$ (or any cyclic permutation).

□

P 5.38. If a, b, c are the lengths of the side of a triangle so that $a + b + c = 3$, then

$$\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} \leq \frac{3(a^2 + b^2 + c^2)}{4(ab + bc + ca)}.$$

(Vasile C., 2007)

Solution. Write the inequality in the homogeneous form

$$\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} \leq \frac{27(a^2 + b^2 + c^2)}{4(a+b+c)^2(ab + bc + ca)}.$$

As shown in the proof of the preceding P 5.37, it suffices to prove this inequality for $a = b + c$ and for $b = c = 1$.

Case 1: $a = b + c$. Since

$$\frac{1}{(a+b)^2} + \frac{1}{(c+a)^2} = \frac{5(b^2 + c^2) + 8bc}{(2b^2 + 2c^2 + 5bc)^2}$$

and

$$\frac{27(a^2 + b^2 + c^2)}{4(a+b+c)^2(ab+bc+ca)} = \frac{27(b^2 + c^2 + bc)}{8(b+c)^2(b^2 + c^2 + 3bc)},$$

we need to show that

$$\frac{5(b^2 + c^2) + 8bc}{(2b^2 + 2c^2 + 5bc)^2} + \frac{1}{(b+c)^2} \leq \frac{27(b^2 + c^2 + bc)}{8(b+c)^2(b^2 + c^2 + 3bc)}.$$

For $bc = 0$, the inequality is true. For $bc \neq 0$, substituting

$$t = \frac{b}{c} + \frac{c}{b}, \quad t \geq 2,$$

the inequality becomes

$$\begin{aligned} \frac{5t+8}{(2t+5)^2} + \frac{1}{t+2} &\leq \frac{27(t+1)}{8(t+2)(t+3)}, \\ \frac{9t^2+38t+41}{(2t+5)^2} &\leq \frac{27(t+1)}{8(t+3)}. \end{aligned}$$

It suffices to show that

$$\frac{9t^2+45t+27}{(2t+5)^2} \leq \frac{27(t+1)}{8(t+3)},$$

which is equivalent to

$$\begin{aligned} \frac{t^2+5t+3}{(2t+5)^2} &\leq \frac{3(t+1)}{8(t+3)}, \\ 4t^3+t(8t-9)+3 &\geq 0. \end{aligned}$$

Case 2: $b = c = 1$, $a \leq b + c = 2$. The homogeneous inequality becomes

$$\frac{2}{(a+1)^2} + \frac{1}{4} \leq \frac{27(a^2+2)}{4(2a+1)(a+2)^2}.$$

Since

$$4(2a+1)(a+2) \leq 9(a+1)^2,$$

it suffices to show that

$$\frac{2}{(a+1)^2} + \frac{1}{4} \leq \frac{3(a^2+2)}{(a+1)^2(a+2)},$$

which is equivalent to

$$(a-6)(a-1)^2 \leq 0.$$

The equality holds for an equilateral triangle.

□

P 5.39. Let $a, b, c \geq \frac{2}{5}$ so that $a + b + c = 3$. Then,

$$\frac{1}{3 + 2(a^2 + b^2)} + \frac{1}{3 + 2(b^2 + c^2)} + \frac{1}{3 + 2(c^2 + a^2)} \leq \frac{3}{7}.$$

(Vasile C., 2006)

Solution. For $a \leq b \leq c$, we have

$$\frac{2}{5} \leq a \leq b \leq c \leq \frac{11}{5}.$$

Indeed,

$$c = 3 - a - b \leq 3 - \frac{2}{5} - \frac{2}{5} = \frac{11}{5}.$$

Using the substitution

$$m = \frac{3}{2} + a^2 + b^2 + c^2, \quad m \geq \frac{3}{2} + \frac{1}{3}(a + b + c)^2 = \frac{9}{2},$$

we have to show that

$$f(a) + f(b) + f(c) \leq \frac{6}{7}$$

for

$$a + b + c = 3, \quad a^2 + b^2 + c^2 = m - \frac{3}{2}, \quad \frac{2}{5} \leq a \leq b \leq c \leq \frac{11}{5},$$

$$f(u) = \frac{1}{m - u^2}, \quad \frac{2}{5} \leq u \leq \frac{11}{5}.$$

From

$$g(x) = f'(x) = \frac{2x}{(m - x^2)^2}, \quad g''(x) = \frac{24x(m + x^2)}{(m - x^2)^4},$$

it follows that $g''(x) > 0$, hence g is strictly convex. By Corollary 1 and Note 2, if

$$a + b + c = 3, \quad a^2 + b^2 + c^2 = \text{fixed}, \quad \frac{2}{5} \leq a \leq b \leq c \leq \frac{11}{5},$$

then the sum

$$S_3 = f(a) + f(b) + f(c)$$

is maximum for either $c = 11/5$ or $a = b \leq c$. The case $c = 11/5$ leads to $a = b = 2/5$, when the inequality is an equality. In the second case, we need to prove that

$$\frac{1}{3 + 4a^2} + \frac{2}{3 + 2(a^2 + c^2)} \leq \frac{3}{7}$$

for $2a + c = 3$, $\frac{2}{5} \leq a \leq c$. Write the inequality as follows

$$\frac{1}{3 + 4a^2} + \frac{2}{21 - 24a + 10a^2} \leq \frac{3}{7},$$

$$\frac{1}{3+4a^2} \leq \frac{49-72a+30a^2}{7(21-24a+10a^2)},$$

$$a(a-1)^2(5a-2) \geq 0.$$

The equality holds for $a = b = c = 1$, and also for

$$a = b = \frac{2}{5}, \quad c = \frac{11}{5}$$

(or any cyclic permutation).

Remark In the same manner, we can prove the following statement:

• Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$. If $k \geq \frac{n^2 - 1}{n^2 - n - 1}$, then

$$\sum \frac{1}{k + a_1^2 + \dots + a_n^2} \leq \frac{n}{k + n - 1},$$

with equality for $a_1 = a_2 = \dots = a_n = 1$. If $k = \frac{n^2 - 1}{n^2 - n - 1}$, then the equality holds also for

$$a_1 = \dots = a_{n-1} = \frac{1}{n^2 - n - 1}, \quad a_n = n - \frac{n-1}{n^2 - n - 1}$$

(or any cyclic permutation).

□

P 5.40. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$\frac{2}{2+a^2+b^2} + \frac{2}{2+b^2+c^2} + \frac{2}{2+c^2+a^2} \leq \frac{99}{63+a^2+b^2+c^2}.$$

(Vasile C., 2009)

Solution. The proof is similar to the one of P 5.39. Thus, we only need to prove the inequality for $0 \leq a = b \leq c$; that is, to show that $2a + c = 3$ involves

$$\frac{1}{1+a^2} + \frac{4}{2+a^2+c^2} \leq \frac{99}{63+2a^2+c^2}.$$

Write this inequality as follows

$$\frac{1}{a^2+1} + \frac{4}{5a^2-12a+11} \leq \frac{33}{2(a^2-2a+12)},$$

$$49a^4 - 112a^3 + 78a^2 - 16a + 1 \geq 0,$$

$$(a-1)^2(7a-1)^2 \geq 0.$$

The equality holds for $a = b = c = 1$, and also for

$$a = b = \frac{1}{7}, \quad c = \frac{19}{7}$$

(or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization:

- Let a, b, c be nonnegative real numbers so that $a + b + c = 3$. If $\frac{8}{5} \leq k \leq 3$, then

$$\frac{k+2}{k+a^2+b^2} + \frac{k+2}{k+b^2+c^2} + \frac{k+2}{k+c^2+a^2} \leq \frac{9(3k^2+11k+10)}{9(k^2+2k+6) + (5k-8)(a^2+b^2+c^2)},$$

with equality for $a = b = c = 1$, and also for

$$a = b = \frac{3-k}{7}, \quad c = \frac{2k+15}{7}$$

(or any cyclic permutation).

□

P 5.41. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$\frac{1}{3+a^2+b^2} + \frac{1}{3+b^2+c^2} + \frac{1}{3+c^2+a^2} \leq \frac{18}{27+a^2+b^2+c^2}.$$

(Vasile C., 2009)

Solution. The proof is similar to the one of P 5.39. Thus, we only need to prove the inequality for $0 \leq a = b \leq c$. Therefore, we only need to show that $2a + c = 3$ involves

$$\frac{1}{3+2a^2} + \frac{2}{3+a^2+c^2} \leq \frac{18}{27+2a^2+c^2}.$$

Write this inequality as follows

$$\frac{1}{2a^2+3} + \frac{2}{5a^2-12a+12} \leq \frac{3}{a^2-2a+6},$$

$$a^2(a-1)^2 \geq 0.$$

The equality holds for $a = b = c = 1$, and also for

$$a = b = 0, \quad c = 3$$

(or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization:

- Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$. If $k \geq \frac{n}{n-2}$, then

$$\sum \frac{1}{k+a_1^2+\dots+a_n^2} \leq \frac{n^2(n+k)}{n(n^2+kn+k^2) + (kn-n-k)(a_1^2+a_2^2+\dots+a_n^2)},$$

with equality for $a_1 = a_2 = \dots = a_n = 1$, and also for

$$a_1 = \dots = a_{n-1} = 0, \quad a_n = n$$

(or any cyclic permutation).

□

P 5.42. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$\frac{5}{3 + a^2 + b^2} + \frac{5}{3 + b^2 + c^2} + \frac{5}{3 + c^2 + a^2} \geq \frac{27}{6 + a^2 + b^2 + c^2}.$$

(Vasile C., 2014)

Solution. Using the substitution

$$m = 3 + a^2 + b^2 + c^2,$$

we have to show that

$$f(a) + f(b) + f(c) \geq \frac{27}{24 + m}$$

for

$$a + b + c = 3, \quad a^2 + b^2 + c^2 = m - 3, \quad 0 \leq a \leq b \leq c,$$

$$f(u) = \frac{5}{m - u^2}, \quad 0 \leq u \leq \sqrt{m - 3}.$$

From

$$g(x) = f'(x) = \frac{10x}{(m - x^2)^2}, \quad g''(x) = \frac{120x(m + x^2)}{(m - x^2)^4},$$

it follows that $g''(x) \geq 0$ for $0 \leq x \leq \sqrt{m - 3}$, hence g is strictly convex. By Corollary 1, if

$$a + b + c = 3, \quad a^2 + b^2 + c^2 = \text{fixed}, \quad 0 \leq a \leq b \leq c,$$

then the sum

$$S_3 = f(a) + f(b) + f(c)$$

is minimum for either $a = 0$ or $0 < a \leq b = c$. Write the inequality in the homogeneous form

$$\sum \frac{5}{(a + b + c)^2 + 3(a^2 + b^2)} \geq \frac{27}{2(a + b + c)^2 + 3(a^2 + b^2 + c^2)}.$$

Case 1: $a = 0$. The homogeneous inequality becomes

$$\begin{aligned} \frac{5}{(b + c)^2 + 3b^2} + \frac{5}{(b + c)^2 + 3c^2} + \frac{5}{(b + c)^2 + 3(b^2 + c^2)} &\geq \frac{27}{2(b + c)^2 + 3(b^2 + c^2)}, \\ \frac{5[5(b^2 + c^2) + 4bc]}{4(b^2 + c^2)^2 + 10bc(b^2 + c^2) + 13b^2c^2} + \frac{5}{4(b^2 + c^2) + 2bc} &\geq \frac{27}{5(b^2 + c^2) + 4bc}. \end{aligned}$$

For the nontrivial case $bc \neq 0$, substituting

$$\frac{b}{c} + \frac{c}{b} = t, \quad t \geq 2,$$

we may write the inequality as

$$\frac{5(5t + 4)}{4t^2 + 10t + 13} + \frac{5}{4t + 2} \geq \frac{27}{5t + 4},$$

$$\frac{5(5t+4)}{4t^2+10t+13} \geq \frac{83t+34}{2(2t+1)(5t+4)}.$$

Since

$$83t+34 \leq 90t+20,$$

it suffices to show that

$$\frac{5t+4}{4t^2+10t+13} \geq \frac{9t+2}{(2t+1)(5t+4)},$$

which is equivalent to

$$14t^3 + 7t^2 - 65t - 10 \geq 0,$$

$$(t-2)(14t^2+35t+5) \geq 0.$$

Case 2: $0 < a \leq b = c$. We only need to prove the homogeneous inequality for $b = c = 1$; that is,

$$\frac{10}{(a+2)^2+3(a^2+1)} + \frac{5}{(a+2)^2+6} \geq \frac{27}{2(a+2)^2+3(a^2+2)},$$

$$\frac{10}{4a^2+4a+7} + \frac{5}{a^2+4a+10} \geq \frac{27}{5a^2+8a+14},$$

$$a(a^3-3a+2) \geq 0,$$

$$a(a-1)^2(a+2) \geq 0.$$

The equality holds for $a = b = c = 1$, and also for

$$a = 0, \quad b = c = \frac{3}{2}$$

(or any cyclic permutation).

Remark 1. Similarly, we can prove the following generalization:

- Let a, b, c be nonnegative real numbers so that $a + b + c = 3$. If $k \geq 0$, then

$$\frac{1}{k+a^2+b^2} + \frac{1}{k+b^2+c^2} + \frac{1}{k+c^2+a^2} \geq \frac{9(4k+15)}{3(4k^2+15k+9) + (8k+21)(a^2+b^2+c^2)}.$$

with equality for $a = b = c = 1$, and also for

$$a = 0, \quad b = c = \frac{3}{2}$$

(or any cyclic permutation).

For $k = 0$, we get the inequality in P 1.171 from Volume 2:

$$\frac{1}{a^2+b^2} + \frac{1}{b^2+c^2} + \frac{1}{c^2+a^2} \geq \frac{45}{(a+b+c)^2+7(a^2+b^2+c^2)}.$$

Remark 2. More general, the following statement holds:

- Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$. If $k \geq 0$, then

$$\sum \frac{1}{k + a_2^2 + \dots + a_n^2} \geq \frac{p}{q + a_1^2 + a_2^2 + \dots + a_n^2},$$

where

$$p = \frac{n^2(n-1)^2k + n^3(n^2 - n - 1)}{(n-1)^3k + n(n^3 - 2n^2 - n + 1)}, \quad q = \frac{n(n-1)^2k^2 + n^2(n^2 - n - 1)k + n^3}{(n-1)^3k + n(n^3 - 2n^2 - n + 1)},$$

with equality for $a_1 = a_2 = \dots = a_n = 1$, and also for

$$a_1 = 0, \quad a_2 = \dots = a_n = \frac{n}{n-1}$$

(or any cyclic permutation).

For $k = 0$ and $k = n$, we get the inequalities

$$\sum \frac{1}{a_2^2 + \dots + a_n^2} \geq \frac{n^2(n^2 - n - 1)}{n^2 + (n^3 - 2n^2 - n + 1)(a_1^2 + a_2^2 + \dots + a_n^2)},$$

$$\sum \frac{2n-1}{n + a_2^2 + \dots + a_n^2} \geq \frac{n^2(2n-3)}{n(n-1) + (n-2)(a_1^2 + a_2^2 + \dots + a_n^2)}.$$

□

P 5.43. If a, b, c, d are nonnegative real numbers so that $a + b + c + d = 4$, then

$$\sum \frac{3}{3 + 2(a^2 + b^2 + c^2)} \leq \frac{296}{218 + a^2 + b^2 + c^2 + d^2}.$$

(Vasile C., 2009)

Solution. The proof is similar to the one of P 5.39. Thus, we only need to prove the inequality for $0 \leq a = b = c \leq d$, that is to show that $3a + d = 4$ involves

$$\frac{1}{1 + 2a^2} + \frac{9}{3 + 4a^2 + 2d^2} \leq \frac{296}{218 + 3a^2 + d^2}.$$

Write this inequality as follows

$$\frac{1}{1 + 2a^2} + \frac{9}{35 - 48a + 22a^2} \leq \frac{148}{3(39 - 4a + 2a^2)},$$

$$(a-1)^2(14a-1)^2 \geq 0.$$

The equality holds for $a = b = c = d = 1$, and also for

$$a = b = c = \frac{1}{14}, \quad d = \frac{53}{14}$$

(or any cyclic permutation).

□

P 5.44. If a, b, c are nonnegative real numbers so that $ab + bc + ca = 3$, then

$$\frac{4}{2+a^2+b^2} + \frac{4}{2+b^2+c^2} + \frac{4}{2+c^2+a^2} \geq \frac{21}{4+a^2+b^2+c^2}.$$

(Vasile C., 2014)

Solution. The proof is similar to the one of P 5.42. Thus, we only need to prove the inequality for $a = 0$ and for $0 < a \leq b = c$.

Case 1: $a = 0$. We need to show that $bc = 3$ involves

$$\frac{1}{2+b^2} + \frac{1}{2+c^2} + \frac{1}{2+b^2+c^2} \geq \frac{21}{4(4+b^2+c^2)}.$$

Denote

$$x = b^2 + c^2, \quad x \geq 2bc = 6.$$

Since

$$\frac{1}{2+b^2} + \frac{1}{2+c^2} = \frac{4+b^2+c^2}{b^2c^2+2(b^2+c^2)+4} = \frac{x+4}{2x+13},$$

we only need to show that

$$\frac{x+4}{2x+13} + \frac{1}{x+2} \geq \frac{21}{4(x+4)}.$$

Since

$$\frac{x+4}{2x+13} + \frac{1}{x+2} = \frac{x^2+8x+21}{(2x+13)(x+2)} \geq \frac{7(2x+3)}{(2x+13)(x+2)},$$

it suffices to show that

$$\frac{2x+3}{(2x+13)(x+2)} \geq \frac{3}{4(x+4)}.$$

This inequality reduces to

$$(x-6)(2x+5) \geq 0.$$

Case 2: $0 < a \leq b = c$. Let

$$q = ab + bc + ca.$$

We only need to prove the homogeneous inequality

$$\frac{4}{2q+3(a^2+b^2)} + \frac{4}{2q+3(b^2+c^2)} + \frac{4}{2q+3(c^2+a^2)} \geq \frac{21}{4q+3(a^2+b^2+c^2)}$$

for $b = c = 1$. Thus, we need to show that

$$\frac{8}{2(2a+1)+3(a^2+1)} + \frac{4}{2(2a+1)+6} \geq \frac{21}{4(2a+1)+3(a^2+2)},$$

which is equivalent to

$$\begin{aligned} \frac{8}{3a^2+4a+5} + \frac{1}{a+2} &\geq \frac{21}{3a^2+8a+10}, \\ \frac{a^2+4a+7}{(3a^2+4a+5)(a+2)} &\geq \frac{7}{3a^2+8a+10}, \end{aligned}$$

$$a(3a^3 - a^2 - 7a + 5) \geq 0,$$

$$a(a-1)^2(3a+5) \geq 0.$$

The equality holds for $a = b = c = 1$, and also for

$$a = 0, \quad b = c = \sqrt{3}$$

(or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization:

- Let a, b, c be nonnegative real numbers so that $ab + bc + ca = 3$. If $k \geq 0$, then

$$\frac{1}{k + a^2 + b^2} + \frac{1}{k + b^2 + c^2} + \frac{1}{k + c^2 + a^2} \geq \frac{9(k+5)}{3(k^2 + 5k + 2) + 2(k+4)(a^2 + b^2 + c^2)}.$$

with equality for $a = b = c = 1$, and also for

$$a = 0, \quad b = c = \sqrt{3}$$

(or any cyclic permutation).

For $k = 0$, we get the inequality in P 1.171 from Volume 2:

$$\frac{1}{a^2 + b^2} + \frac{1}{b^2 + c^2} + \frac{1}{c^2 + a^2} \geq \frac{45}{2(ab + bc + ca) + 8(a^2 + b^2 + c^2)}.$$

□

P 5.45. If a, b, c are nonnegative real numbers so that $a^2 + b^2 + c^2 = 3$, then

$$\frac{1}{10 - (a+b)^2} + \frac{1}{10 - (b+c)^2} + \frac{1}{10 - (c+a)^2} \leq \frac{1}{2}.$$

(Vasile C., 2006)

Solution. Let

$$s = a + b + c, \quad s \leq 3.$$

We need to show that

$$\frac{1}{10 - (s-a)^2} + \frac{1}{10 - (s-b)^2} + \frac{1}{10 - (s-c)^2} \leq \frac{1}{2}$$

for $a + b + c = s$ and $a^2 + b^2 + c^2 = 3$. Apply Corollary 1 to the function

$$f(u) = \frac{-1}{10 - (s-u)^2}, \quad 0 \leq u \leq s \leq 3.$$

We have

$$g(x) = f'(x) = \frac{2(s-x)}{[10 - (s-x)^2]^2},$$

$$g''(x) = \frac{24(s-x)[10+(s-x)^2]}{[10-(s-x)^2]^4}.$$

Since $g''(x) > 0$ for $x \in [0, s]$, g is strictly convex on $[0, s]$. According to the Corollary 1, if

$$a + b + c = s, \quad a^2 + b^2 + c^2 = 3, \quad 0 \leq a \leq b \leq c,$$

then

$$S_3 = f(a) + f(b) + f(c)$$

is minimum for either $a = 0$ or $0 < a \leq b = c$. Therefore, we only need to prove the homogeneous inequality

$$\sum \frac{1}{10(a^2 + b^2 + c^2) - 3(b+c)^2} \leq \frac{1}{2(a^2 + b^2 + c^2)}$$

for $a = 0$ and for $b = c = 1$.

Case 1: $a = 0$. The homogeneous inequality becomes

$$\frac{1}{7(b^2 + c^2) - 6bc} + \frac{1}{10b^2 + 7c^2} + \frac{1}{7b^2 + 10c^2} \leq \frac{1}{2(b^2 + c^2)}.$$

This is true since

$$\frac{1}{7(b^2 + c^2) - 6bc} \leq \frac{1}{4(b^2 + c^2)}$$

and

$$\begin{aligned} \frac{1}{10b^2 + 7c^2} + \frac{1}{7b^2 + 10c^2} &= \frac{17(b^2 + c^2)}{70(b^2 + c^2) + 149b^2c^2} \\ &\leq \frac{17(b^2 + c^2)}{70(b^2 + c^2) + 140b^2c^2} \\ &= \frac{17}{70(b^2 + c^2)} < \frac{1}{4(b^2 + c^2)}. \end{aligned}$$

Case 2: $b = c = 1$. The homogeneous inequality turns into

$$\begin{aligned} \frac{1}{2(5a^2 + 4)} + \frac{2}{7a^2 - 6a + 17} &\leq \frac{1}{2(a^2 + 2)}, \\ \frac{2}{7a^2 - 6a + 17} &\leq \frac{2a^2 + 1}{(5a^2 + 4)(a^2 + 2)}, \\ 4a^4 - 12a^3 + 13a^2 - 6a + 1 &\geq 0, \\ (a - 1)^2(2a - 1)^2 &\geq 0. \end{aligned}$$

The equality holds for $a = b = c = 1$, and also for

$$2a = b = c = \frac{2}{\sqrt{3}}$$

(or any cyclic permutation).

□

P 5.46. If a, b, c are nonnegative real numbers, no two of which are zero, so that $a^4 + b^4 + c^4 = 3$, then

$$\frac{1}{a^5 + b^5} + \frac{1}{b^5 + c^5} + \frac{1}{c^5 + a^5} \geq \frac{3}{2}.$$

(Vasile C., 2010)

Solution. Using the substitution

$$x = a^4, \quad y = b^4, \quad z = c^4, \quad p = x^{5/4} + y^{5/4} + z^{5/4},$$

we need to show that $x + y + z = 3$ and $x^{5/4} + y^{5/4} + z^{5/4} = p$ involve

$$f(x) + f(y) + f(z) \geq \frac{3}{2},$$

where

$$f(u) = \frac{1}{p - u^{5/4}}, \quad 0 \leq u < p^{4/5}.$$

We will apply the EV-Theorem for $k = 5/4$. We have

$$f'(u) = \frac{5u^{1/4}}{4(p - u^{5/4})^2},$$

$$g(x) = f'\left(x^{\frac{1}{k-1}}\right) = f'(x^4) = \frac{5x}{4(p - x^5)^2},$$

$$g''(x) = \frac{75x^4(2p + 3x^5)}{2(p - x^5)^4}.$$

Since $g''(x) \geq 0$, g is strictly convex. According to the EV-Theorem and Note 3, if

$$x + y + z = 3, \quad x^{5/4} + y^{5/4} + z^{5/4} = p = \text{fixed}, \quad 0 \leq x \leq y \leq z,$$

then

$$S_3 = f(x) + f(y) + f(z)$$

is minimum for either $x = 0$ or $0 < x \leq y = z$. Thus, we only need to prove the homogeneous inequality

$$\frac{1}{a^5 + b^5} + \frac{1}{b^5 + c^5} + \frac{1}{c^5 + a^5} \geq \frac{3}{2} \left(\frac{3}{a^4 + b^4 + c^4} \right)^{5/4}$$

for $a = 0$ and $0 < a \leq b = c = 1$.

Case 1: $a = 0$. The homogeneous inequality becomes

$$\frac{1}{b^5} + \frac{1}{c^5} + \frac{1}{b^5 + c^5} \geq \frac{3}{2} \left(\frac{3}{b^4 + c^4} \right)^{5/4},$$

$$\left(\frac{b}{c}\right)^{5/2} + \left(\frac{c}{b}\right)^{5/2} + \frac{1}{\left(\frac{b}{c}\right)^{5/2} + \left(\frac{c}{b}\right)^{5/2}} \geq \left(\frac{3}{2}\right)^{9/4} \left[\frac{2}{\left(\frac{b}{c}\right)^2 + \left(\frac{c}{b}\right)^2} \right]^{5/4},$$

$$t^{5/2} + t^{-5/2} + \frac{1}{t^{5/2} + t^{-5/2}} \geq \left(\frac{3}{2}\right)^{9/4} \left(\frac{2}{t^2 + t^{-2}}\right)^{5/4},$$

$$2A^{5/2} + \frac{1}{2A^{5/2}} \geq \left(\frac{3}{2}\right)^{9/4} \cdot \frac{1}{B^{5/2}},$$

where

$$A = \left(\frac{t^{5/2} + t^{-5/2}}{2}\right)^{2/5}, \quad B = \left(\frac{t^2 + t^{-2}}{2}\right)^{1/2}, \quad t = \frac{b}{c}.$$

By power mean inequality, we have $A \geq B \geq 1$. Since

$$2A^{5/2} + \frac{1}{2A^{5/2}} - \left(2B^{5/2} + \frac{1}{2B^{5/2}}\right) = (A^{5/2} - B^{5/2}) \left(2 - \frac{1}{2A^{5/2}B^{5/2}}\right) \geq 0,$$

it suffices to show that

$$2B^{5/2} + \frac{1}{2B^{5/2}} \geq \left(\frac{3}{2}\right)^{9/4} \cdot \frac{1}{B^{5/2}},$$

$$4B^5 + 1 \geq \left(\frac{3^9}{2^5}\right)^{1/4},$$

which is true if

$$5 \geq \left(\frac{3^9}{2^5}\right)^{1/4},$$

$$32 \cdot 5^4 \geq 3^9.$$

This inequality follows by multiplying the inequalities

$$5^4 > 23 \cdot 3^3$$

and

$$32 \cdot 23 > 3^6.$$

Case 2: $0 < a \leq 1 = b = c$. The homogeneous inequality becomes

$$\frac{a^5 + 5}{a^5 + 1} \geq 3 \left(\frac{3}{a^4 + 2}\right)^{5/4},$$

which is true if $g(a) \geq 0$, where

$$g(a) = \ln(a^5 + 5) - \ln(a^5 + 1) + \frac{5}{4} \ln(a^4 + 2) - \frac{9 \ln 3}{4},$$

with

$$\begin{aligned} \frac{g'(a)}{5a^3} &= \frac{a}{a^5 + 5} - \frac{a}{a^5 + 1} + \frac{1}{a^4 + 2} = \frac{a^{10} + 2a^5 - 8a + 5}{(a^4 + 5)(a^5 + 1)(a^4 + 2)} \\ &= \frac{(a - 1)(a^9 + a^8 + a^7 + a^6 + a^5 + 3a^4 + 3a^3 + 3a^2 + 3a - 5)}{(a^4 + 5)(a^5 + 1)(a^4 + 2)}. \end{aligned}$$

There exists $d \in (0, 1)$ so that $g'(d) = 0$, $g'(a) > 0$ for $a \in [0, d)$ and $g'(a) < 0$ for $a \in (d, 1)$. Therefore, g is increasing on $[0, d]$ and is decreasing on $[d, 1]$. Since $g(1) = 0$, we only need to show that $g(0) \geq 0$. Indeed,

$$g(0) = \frac{1}{4} \ln \frac{5^4 \cdot 2^5}{3^9} > 0.$$

The equality holds for $a = b = c = 1$.

□

P 5.47. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$, then

$$\sqrt{a_1^2 + 1} + \sqrt{a_2^2 + 1} + \dots + \sqrt{a_n^2 + 1} \geq \sqrt{2 \left(1 - \frac{1}{n}\right) (a_1^2 + a_2^2 + \dots + a_n^2) + 2(n^2 - n + 1)}.$$

(Vasile C., 2014)

Solution. For $n = 2$, we need to show that $a_1 + a_2 = 2$ involves

$$\sqrt{a_1^2 + 1} + \sqrt{a_2^2 + 1} \geq \sqrt{a_1^2 + a_2^2 + 6}.$$

By squaring, the inequality becomes

$$\sqrt{(a_1^2 + 1)(a_2^2 + 1)} \geq 2,$$

which follows immediately from the Cauchy-Schwarz inequality:

$$(a_1^2 + 1)(a_2^2 + 1) = (a_1^2 + 1)(1 + a_2^2) \geq (a_1 + a_2)^2 = 4.$$

Assume further that $n \geq 3$ and $a_1 \leq a_2 \leq \dots \leq a_n$. We will apply Corollary 1 to the function

$$f(u) = -\sqrt{u^2 + 4}, \quad u \geq 0.$$

We have

$$g(x) = f'(x) = \frac{-x}{\sqrt{x^2 + 4}},$$

$$g''(x) = \frac{12x}{(x^2 + 4)^{5/2}}.$$

Since $g''(x) > 0$ for $x > 0$, $g(x)$ is strictly convex for $x \geq 0$. By Corollary 1, if $a_1 \leq a_2 \leq \dots \leq a_n$ and

$$a_1 + a_2 + \dots + a_n = n, \quad a_1^2 + a_2^2 + \dots + a_n^2 = \text{fixed},$$

then the sum

$$S_n = f(a_1) + f(a_2) + \dots + f(a_n)$$

is maximum for $a_1 = a_2 = \dots = a_{n-1}$. Thus, we only need to show that

$$\sqrt{a^2 + 1} + (n-1)\sqrt{b^2 + 1} \geq \sqrt{2 \left(1 - \frac{1}{n}\right) [a^2 + (n-1)b^2] + 2(n^2 - n + 1)}.$$

for

$$a + (n - 1)b = n.$$

By squaring, the inequality becomes

$$2n(n - 1)\sqrt{(a^2 + 1)(b^2 + 1)} \geq (n - 2)a^2 - (n - 2)(n - 1)^2b^2 + n^3,$$

which is equivalent to

$$\sqrt{(b^2 + 1)[(n - 1)^2b^2 - 2n(n - 1)b + n^2 + 1]} \geq n - (n - 2)b.$$

This is true if

$$(b^2 + 1)[(n - 1)^2b^2 - 2n(n - 1)b + n^2 + 1] \geq [n - (n - 2)b]^2,$$

which is equivalent to

$$(n - 1)^2b^4 - 2n(n - 1)b^3 + (n^2 + 2n - 2)b^2 - 2nb + 1 \geq 0,$$

$$(b - 1)^2[(n - 1)b - 1]^2 \geq 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = a_2 = \cdots = a_{n-1} = \frac{1}{n-1}, \quad a_n = n - 1$$

(or any cyclic permutation).

□

P 5.48. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then

$$\sum \sqrt{(3n - 4)a_1^2 + n} \geq \sqrt{(3n - 4)(a_1^2 + a_2^2 + \cdots + a_n^2) + n(4n^2 - 7n + 4)}.$$

(Vasile C., 2009)

Solution. The proof is similar to the one of the preceding P 5.47. Thus, it suffices to prove the inequality for $a_1 = a_2 = \cdots = a_{n-1}$. Write the inequality in the homogeneous form

$$\sum \sqrt{n(3n - 4)a_1^2 + S^2} \geq \sqrt{n(3n - 4)(a_1^2 + a_2^2 + \cdots + a_n^2) + (4n^2 - 7n + 4)S^2},$$

where $S = a_1 + a_2 + \cdots + a_n$. We only need to prove this inequality for $a_1 = a_2 = \cdots = a_{n-1} = 1$, that is

$$\begin{aligned} (n - 1)\sqrt{n(3n - 4) + (n - 1 + a_n)^2} + \sqrt{n(3n - 4)a_n^2 + (n - 1 + a_n)^2} &\geq \\ &\geq \sqrt{n(3n - 4)(n - 1 + a_n^2) + (4n^2 - 7n + 4)(n - 1 + a_n)^2}, \end{aligned}$$

which is equivalent to

$$\begin{aligned} \sqrt{(n - 1)[a_n^2 + 2(n - 1)a_n + 4n^2 - 6n + 1]} + \sqrt{(3n - 1)a_n^2 + 2a_n + n - 1} &\geq \\ &\geq \sqrt{(7n - 4)a_n^2 + 2(4n^2 - 7n + 4)a_n + 4n^3 - 8n^2 + 7n - 4}. \end{aligned}$$

By squaring, the inequality turns into

$$2\sqrt{(n-1)[(3n-1)a_n^2 + 2a_n + n - 1][a_n^2 + 2(n-1)a_n + 4n^2 - 6n + 1]} \geq \\ (3n-2)a_n^2 + 2(n-1)(3n-2)a_n + 2n^2 - n - 2.$$

Squaring again, we get

$$(a_n - 1)^2(a_n - 2n + 3)^2 \geq 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = a_2 = \cdots = a_{n-1} = \frac{a_n}{2n-3} = \frac{n}{3n-4}$$

(or any cyclic permutation).

Remark. For $n = 3$, we get the inequality

$$\sqrt{5a^2 + 3} + \sqrt{5b^2 + 3} + \sqrt{5c^2 + 3} \geq \sqrt{5(a^2 + b^2 + c^2) + 57},$$

where a, b, c are nonnegative real numbers so that $a + b + c = 3$. By squaring, the inequality turns into

$$\sqrt{(5a^2 + 3)(5b^2 + 3)} + \sqrt{(5b^2 + 3)(5c^2 + 3)} + \sqrt{(5c^2 + 3)(5a^2 + 3)} \geq 24,$$

with equality for $a = b = c = 1$, and also for

$$a = b = \frac{c}{3} = \frac{3}{5}$$

(or any cyclic permutation).

□

P 5.49. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$\sqrt{a^2 + 4} + \sqrt{b^2 + 4} + \sqrt{c^2 + 4} \leq \sqrt{\frac{8}{3}(a^2 + b^2 + c^2) + 37}.$$

(Vasile C., 2009)

Solution. Assume that $a \leq b \leq c$, and apply Corollary 1 to the function a

$$f(u) = -\sqrt{u^2 + 4}, \quad u \geq 0.$$

We have

$$g(x) = f'(x) = \frac{-x}{\sqrt{x^2 + 4}}, \\ g''(x) = \frac{12x}{(x^2 + 4)^{5/2}}.$$

Since $g''(x) > 0$ for $x > 0$, $g(x)$ is strictly convex for $x \geq 0$. By Corollary 1, if

$$a + b + c = 3, \quad a^2 + b^2 + c^2 = \text{fixed}, \quad a \leq b \leq c,$$

then the sum

$$S_3 = f(a) + f(b) + f(c)$$

is minimum for either $a = 0$ or $0 < a \leq b = c$. Thus, we only need to prove the desired inequality for these cases.

Case 1: $a = 0$. We need to prove that $b + c = 3$ involves

$$\sqrt{b^2 + 4} + \sqrt{c^2 + 4} \leq \sqrt{\frac{8}{3}(b^2 + c^2) + 37} - 2.$$

Substituting

$$b = \frac{3x}{2}, \quad c = \frac{3y}{2},$$

we need to prove that $x + y = 2$ involves

$$\sqrt{9x^2 + 16} + \sqrt{9y^2 + 16} \leq 2\sqrt{6(x^2 + y^2) + 37} - 4.$$

By squaring, the inequality becomes

$$2\sqrt{(9x^2 + 16)(9y^2 + 16)} \leq 15(x^2 + y^2) + 132 - 16\sqrt{6(x^2 + y^2) + 37}.$$

Denoting

$$p = xy, \quad 0 \leq p \leq 1,$$

we have

$$x^2 + y^2 = 4 - 2p, \quad (9x^2 + 16)(9y^2 + 16) = 81p^2 - 288p + 832,$$

and the inequality becomes

$$\sqrt{81p^2 - 288p + 832} \leq -15p + 96 - 8\sqrt{61 - 12p},$$

$$\sqrt{\frac{81}{4}p^2 - 72p + 208} \leq -\frac{15}{2}p + (48 - 4\sqrt{61 - 12p}),$$

By squaring again (the right hand side is positive), the inequality becomes

$$\frac{81}{4}p^2 - 72p + 208 \leq \frac{225}{4}p^2 - 15p(48 - 4\sqrt{61 - 12p}) + (48 - 4\sqrt{61 - 12p})^2,$$

$$3p^2 - 70p + 256 \geq (32 - 5p)\sqrt{61 - 12p}.$$

Since

$$2\sqrt{61 - 12p} \leq 7 + \frac{61 - 12p}{7} = \frac{2(55 - 6p)}{7},$$

it suffices to show that

$$7(3p^2 - 70p + 256) \geq (32 - 5p)(55 - 6p),$$

which is equivalent to

$$(1 - p)(32 + 9p) \geq 0.$$

Case 2: $b = c$. We need to prove that

$$a + 2b = 3$$

implies

$$\sqrt{a^2 + 4} + 2\sqrt{b^2 + 4} \leq \sqrt{\frac{8}{3}(a^2 + 2b^2) + 37}.$$

By squaring, the inequality becomes

$$12\sqrt{(a^2 + 4)(b^2 + 4)} \leq 5a^2 + 4b^2 + 51,$$

which is equivalent to

$$\sqrt{(4b^2 - 12b + 13)(b^2 + 4)} \leq 2b^2 - 5b + 8.$$

By squaring again, the inequality becomes

$$2b^3 - 7b^2 + 8b - 3 \leq 0,$$

$$(b - 1)^2(2b - 3) \leq 0,$$

$$(b - 1)^2a \geq 0.$$

The equality holds for $a = b = c = 1$, and also for

$$a = 0, \quad b = c = \frac{3}{2}$$

(or any cyclic permutation).

□

P 5.50. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$\sqrt{32a^2 + 3} + \sqrt{32b^2 + 3} + \sqrt{32c^2 + 3} \leq \sqrt{32(a^2 + b^2 + c^2) + 219}.$$

(Vasile C., 2009)

Solution. The proof is similar to the one of P 5.49. Thus, it suffices to prove the homogeneous inequality

$$\sum \sqrt{96a^2 + (a + b + c)^2} \leq \sqrt{96(a^2 + b^2 + c^2) + 73(a + b + c)^2}$$

for $a = 0$ and for $b = c = 1$.

Case 1: $a = 0$. We have to show that

$$b + c + \sqrt{97b^2 + 2bc + c^2} + \sqrt{b^2 + 2bc + 97c^2} \leq \sqrt{169(b^2 + c^2) + 146bc}.$$

Since $2bc \leq b^2 + c^2$, it suffices to prove that

$$b + c + \sqrt{98b^2 + 2c^2} + \sqrt{2b^2 + 98c^2} \leq \sqrt{169(b^2 + c^2) + 146bc}.$$

By squaring, we get

$$\begin{aligned} (b + c) \left(\sqrt{98b^2 + 2c^2} + \sqrt{2b^2 + 98c^2} \right) + 2\sqrt{(49b^2 + c^2)(b^2 + 49c^2)} &\leq \\ &\leq 34(b^2 + c^2) + 72bc. \end{aligned}$$

Since

$$\sqrt{98b^2 + 2c^2} + \sqrt{2b^2 + 98c^2} \leq \sqrt{2(98b^2 + 2c^2 + 2b^2 + 98c^2)} = 10\sqrt{2(b^2 + c^2)}$$

and

$$10(b + c)\sqrt{2(b^2 + c^2)} \leq 20(b + c)^2,$$

it suffices to show that

$$\sqrt{(49b^2 + c^2)(b^2 + 49c^2)} \leq 7(b^2 + c^2) + 36bc.$$

Squaring again, the inequality becomes

$$bc(b - c)^2 \geq 0.$$

Case 2: $b = c = 1$. The homogeneous inequality turns into

$$\sqrt{97a^2 + 4a + 4} + 2\sqrt{a^2 + 4a + 100} \leq \sqrt{169a^2 + 292a + 484}.$$

By squaring, we get

$$\sqrt{(97a^2 + 4a + 4)(a^2 + 4a + 100)} \leq 17a^2 + 68a + 20.$$

Squaring again, the inequality reduces to

$$a(a - 1)^2(a + 12) \geq 0.$$

The equality holds for $a = b = c = 1$, and also for $a = 0$ and $b = c = 3/2$ (or any cyclic permutation).

Remark. By squaring, we deduce the inequality

$$\sqrt{(32a^2 + 3)(32b^2 + 3)} + \sqrt{(32b^2 + 3)(32c^2 + 3)} + \sqrt{(32c^2 + 3)(32a^2 + 3)} \leq 105,$$

with equality for $a = b = c = 1$, and also for

$$a = 0, \quad b = c = \frac{3}{2}$$

(or any cyclic permutation).

□

P 5.51. If $a_1, a_2, \dots, a_n$ are positive real numbers so that $a_1 + a_2 + \dots + a_n = n$, then

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} + \frac{2n\sqrt{n-1}}{a_1^2 + a_2^2 + \dots + a_n^2} \geq n + 2\sqrt{n-1}.$$

(Vasile C., 2009)

Solution. For $n = 2$, the inequality reduces to

$$(a_1 a_2 - 1)^2 \geq 0.$$

Consider further that $n \geq 3$ and $a_1 \leq a_2 \leq \cdots \leq a_n$. By Corollary 5 (case $k = 2$ and $m = -1$), if $a_1 \leq a_2 \leq \cdots \leq a_n$ and

$$a_1 + a_2 + \cdots + a_n = n, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = \text{fixed},$$

then the sum

$$S_n = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$$

is minimum for $a_1 = \cdots = a_{n-1} \leq a_n$. Therefore, we only need to prove that

$$\frac{n-1}{a_1} + \frac{1}{a_n} + \frac{2n\sqrt{n-1}}{(n-1)a_1^2 + a_n^2} \geq n + 2\sqrt{n-1},$$

for $(n-1)a_1 + a_n = n$. The inequality is equivalent to

$$(a_1 - 1)^2 \left(a_1 - \frac{n}{n-1 + \sqrt{n-1}} \right)^2 \geq 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = a_2 = \cdots = a_{n-1} = \frac{a_n}{\sqrt{n-1}}$$

(or any cyclic permutation).

□

P 5.52. If $a, b, c \in [0, 1]$, then

$$(1 + 3a^2)(1 + 3b^2)(1 + 3c^2) \geq (1 + ab + bc + ca)^3.$$

Solution. Since

$$2(ab + bc + ca) = (a + b + c)^2 - (a^2 + b^2 + c^2),$$

we may apply Corollary 1 to the function

$$f(u) = -\ln(1 + 3u^2), \quad u \in [0, 1],$$

to prove the inequality

$$f(a) + f(b) + f(c) + 3\ln(1 + ab + bc + ca) \leq 0.$$

We have

$$g(x) = f'(x) = \frac{-6x}{1 + 3x^2},$$

$$g''(x) = \frac{108x(1 - x^2)}{(1 + 3x^2)^3}.$$

Since $g''(x) > 0$ for $x \in (0, 1)$, g is strictly convex on $[0, 1]$. According to Corollary 1 and Note 2, if

$$a + b + c = \text{fixed}, \quad a^2 + b^2 + c^2 = \text{fixed}, \quad 0 \leq a \leq b \leq c \leq 1,$$

then

$$S_3 = f(a) + f(b) + f(c)$$

is maximum for $a = b \leq c$. or for $c = 1$. Thus, we only need to prove the original inequality for these cases.

Case 1: $a = b \leq c$. We need to show that

$$(1 + 3a^2)^2(1 + 3c^2) \geq (1 + a^2 + 2ac)^3.$$

For $c = 0$, the inequality is an equality. For fixed c , $0 < c \leq 1$, we need to show that $h(a) \geq 0$, where

$$h(a) = 2 \ln(1 + 3a^2) + \ln(1 + 3c^2) - 3 \ln(1 + a^2 + 2ac), \quad a \in [0, c].$$

From

$$h'(a) = \frac{12a}{1 + 3a^2} - \frac{6(a + c)}{1 + a^2 + 2ac} = \frac{6(1 - a^2)(a - c)}{(1 + 3a^2)(1 + a^2 + 2ac)} \leq 0,$$

it follows that h is decreasing on $[0, c]$, hence $h(a) \geq h(c) = 0$.

Case 2: $c = 1$. We need to show that

$$4(1 + 3a^2)(1 + 3b^2) \geq (1 + a)^3(1 + b)^3.$$

This is true because

$$2(1 + 3a^2) \geq (1 + a)^3, \quad 2(1 + 3b^2) \geq (1 + b)^3.$$

The first inequality is equivalent to

$$(1 - a)^3 \geq 0.$$

The proof is completed. The equality holds for $a = b = c$.

Remark. The following statement is true.

- If a, b, c are real numbers so that $ab + bc + ca = 3$, then

$$(1 + 3a^2)(1 + 3b^2)(1 + 3c^2) \geq 64.$$

The inequality is equivalent to

$$(3abc - a - b - c)^2 \geq 0.$$

□

P 5.53. If a, b, c are nonnegative real numbers so that $a + b + c = ab + bc + ca$, then

$$\frac{1}{4 + 5a^2} + \frac{1}{4 + 5b^2} + \frac{1}{4 + 5c^2} \geq \frac{1}{3}.$$

(Vasile C., 2007)

Solution. By expanding, the inequality becomes

$$4(a^2 + b^2 + c^2) + 15 \geq 25a^2b^2c^2 + 5(a^2b^2 + b^2c^2 + c^2a^2).$$

Let $p = a + b + c$. Since

$$a^2 + b^2 + c^2 = p^2 - 2p, \quad a^2b^2 + b^2c^2 + c^2a^2 = p^2 - 2pabc,$$

the inequality becomes

$$(2p - 4)^2 \geq (p - 5abc)^2, \\ (3p - 4 - 5abc)(p + 5abc - 4) \geq 0.$$

We will show that $3p \geq 4 + 5abc$ and $p + 5abc \geq 4$. According to Corollary 4 (case $n = 3$, $k = 2$) or P 3.57 in Volume 1, if

$$a + b + c = \text{fixed}, \quad ab + bc + ca = \text{fixed}, \quad 0 \leq a \leq b \leq c \leq d,$$

then the product abc is maximum for $a = b$, and is minimum for $a = 0$ or $b = c$. Thus, we only need to prove that $3p \geq 4 + 5abc$ for $a = b$, and $p + 5abc \geq 4$ for $a = 0$ and for $b = c$.

For $a = b$, from $a + b + c = ab + bc + ca$ we get

$$c = \frac{a(2 - a)}{2a - 1}, \quad \frac{1}{2} < a \leq 2,$$

hence

$$3p - 4 - 5abc = (3 - 5a^2)c + 6a - 4 = \frac{(a - 1)^2(5a^2 + 4)}{2a - 1} \geq 0.$$

For $a = 0$, from $a + b + c = ab + bc + ca$ we get

$$c = \frac{b}{b - 1}, \quad b > 1,$$

hence

$$p + 5abc - 4 = b + c - 4 = \frac{(b - 2)^2}{b - 1} \geq 0.$$

For $b = c$, from $a + b + c = ab + bc + ca$ we get

$$a = \frac{b(2 - b)}{2b - 1}, \quad \frac{1}{2} < b \leq 2,$$

hence

$$p + 5abc - 4 = a(5b^2 + 1) + 2b - 4 = \frac{(2 - b)(5b^3 - 3b + 2)}{2b - 1} \\ = \frac{(2 - b)[4b^3 + (b - 1)^2(b + 2)]}{2b - 1} \geq 0.$$

The equality holds for $a = b = c = 1$, and also for $a = 0$ and $b = c = 2$ (or any cyclic permutation).

□

P 5.54. If a, b, c, d are positive real numbers so that $a + b + c + d = 4abcd$, then

$$\frac{1}{1+3a} + \frac{1}{1+3b} + \frac{1}{1+3c} + \frac{1}{1+3d} \geq 1.$$

(Vasile C., 2007)

Solution. By expanding, the inequality becomes

$$1 + 3(ab + ac + ad + bc + bd + cd) \geq 19abcd,$$

$$2 + 3(a + b + c + d)^2 \geq 3(a^2 + b^2 + c^2 + d^2) + 38abcd.$$

According to Corollary 5 (case $n = 4$, $k = 0$, $m = 2$), if

$$a + b + c + d = \text{fixed}, \quad abcd = \text{fixed}, \quad 0 < a \leq b \leq c \leq d,$$

then the sum

$$S_4 = a^2 + b^2 + c^2 + d^2$$

is maximum for $a = b = c \leq d$. Thus, we only need to prove that

$$3a + d = 4a^3d, \quad d = \frac{3a}{4a^3 - 1}, \quad a > \frac{1}{\sqrt[3]{4}},$$

involves

$$\begin{aligned} \frac{3}{3a+1} + \frac{1}{3d+1} &\geq 1, \\ \frac{3}{3a+1} + \frac{4a^3-1}{4a^3+9a-1} &\geq 1, \\ 4a^3 - 9a^2 + 6a - 1 &\geq 0, \\ (a-1)^2(4a-1) &\geq 0. \end{aligned}$$

The equality holds for $a = b = c = d = 1$.

Remark. The following generalization holds:

- If $a_1, a_2, \dots, a_n$ ($n \geq 4$) are positive real numbers so that

$$a_1 + a_2 + \dots + a_n = na_1a_2 \cdots a_n,$$

then

$$\frac{1}{1+(n-1)a_1} + \frac{1}{1+(n-1)a_2} + \dots + \frac{1}{1+(n-1)a_n} \geq 1.$$

□

P 5.55. If $a_1, a_2, \dots, a_n$ are positive real numbers so that

$$a_1 + a_2 + \dots + a_n = \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n},$$

then

$$\frac{1}{1+(n-1)a_1} + \frac{1}{1+(n-1)a_2} + \dots + \frac{1}{1+(n-1)a_n} \geq 1.$$

(Vasile C., 1996)

Solution. For $n = 2$, the inequality is an identity. For $n \geq 3$, we consider

$$a_1 \leq a_2 \leq \cdots \leq a_n,$$

and apply Corollary 2 to the function

$$f(u) = \frac{1}{1 + (n-1)u}, \quad u > 0.$$

We have

$$\begin{aligned} f'(u) &= \frac{-(n-1)}{[1 + (n-1)u]^2}, \\ g(x) &= f'\left(\frac{1}{\sqrt{x}}\right) = \frac{-(n-1)x}{[\sqrt{x} + n-1]^2}, \\ g''(x) &= \frac{3(n-1)^2}{2\sqrt{x}(\sqrt{x} + n-1)^4}. \end{aligned}$$

Since $g''(x) > 0$ for $x > 0$, g is strictly convex on $[0, \infty)$. By Corollary 2, if $0 < a_1 \leq a_2 \leq \cdots \leq a_n$ and

$$a_1 + a_2 + \cdots + a_n = \text{fixed}, \quad \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = \text{fixed},$$

then the sum

$$S_n = f(a_1) + f(a_2) + \cdots + f(a_n)$$

is minimum for $a_2 = \cdots = a_n$. Therefore, we only need to show that

$$\frac{1}{1 + (n-1)a} + \frac{n-1}{1 + (n-1)b} \geq 1$$

for

$$a + (n-1)b = \frac{1}{a} + \frac{n-1}{b}, \quad 0 < a \leq b.$$

Write the hypothesis as

$$\frac{1}{a} - a = (n-1) \left(b - \frac{1}{b} \right),$$

which involves $a \leq 1 \leq b$ and

$$\frac{1}{a} - a \geq b - \frac{1}{b}, \quad ab \leq 1.$$

Write the desired inequality as

$$\frac{n-1}{1 + (n-1)b} \geq 1 - \frac{1}{1 + (n-1)a},$$

which is equivalent to

$$\begin{aligned} \frac{n-1}{1 + (n-1)b} &\geq \frac{(n-1)a}{1 + (n-1)a}, \\ 1 - a &\geq (n-1)a(b-1). \end{aligned}$$

For the nontrivial case $b \neq 1$, we have

$$1 - a - (n-1)a(b-1) = 1 - a - \frac{b(1-a^2)}{a(b^2-1)}a(b-1) = \frac{(1-a)(1-ab)}{b+1} \geq 0.$$

If $n \geq 3$, then the equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 5.56. If a, b, c, d, e are nonnegative real numbers so that $a^4 + b^4 + c^4 + d^4 + e^4 = 5$, then

$$7(a^2 + b^2 + c^2 + d^2 + e^2) \geq (a + b + c + d + e)^2 + 10.$$

(Vasile C., 2008)

Solution. According to Corollary 5 (case $n = 5$, $k = 4$, $m = 2$), if

$$a + b + c + d + e = \text{fixed}, \quad a^4 + b^4 + c^4 + d^4 + e^4 = 5, \quad 0 \leq a \leq b \leq c \leq d \leq e,$$

then the sum

$$S_4 = a^2 + b^2 + c^2 + d^2 + e^2$$

is minimum for $a = b = c = d \leq e$. Thus, we only need to prove the homogeneous inequality

$$[7(a^2 + b^2 + c^2 + d^2 + e^2) - (a + b + c + d + e)^2]^2 \geq 20(a^4 + b^4 + c^4 + d^4 + e^4)$$

for $a = b = c = d = 0$ and $a = b = c = d = 1$. The first case is trivial. In the second case, the inequality becomes

$$[7(4 + e^2) - (4 + e)^2]^2 \geq 20(4 + e^4),$$

$$(3e^2 - 4e + 6)^2 \geq 5e^4 + 20,$$

$$e^4 - 6e^3 + 13e^2 - 12e + 4 \geq 0,$$

$$(e - 1)^2(e - 2)^2 \geq 0.$$

The equality holds for $a = b = c = d = e = 1$, and also for

$$a = b = c = d = \frac{e}{2} = \frac{1}{\sqrt{2}}$$

(or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that

$$a_1^4 + a_2^4 + \dots + a_n^4 = n,$$

then

$$(n + \sqrt{n-1})(a_1^2 + a_2^2 + \dots + a_n^2 - n) \geq (a_1 + a_2 + \dots + a_n)^2 - n^2,$$

with equality for $a_1 = a_2 = \dots = a_n = 1$, and also for

$$a_1 = \dots = a_{n-1} = \frac{a_n}{\sqrt{n-1}} = \frac{1}{\sqrt[4]{n-1}}$$

(or any cyclic permutation).

□

P 5.57. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \dots + a_n = n$, then

$$(a_1^2 + a_2^2 + \dots + a_n^2)^2 - n^2 \geq \frac{n(n-1)}{n^2 - n + 1} (a_1^4 + a_2^4 + \dots + a_n^4 - n).$$

(Vasile C., 2008)

Solution. For $n = 2$, the inequality reduces to $(a_1 a_2 - 1)^2 \geq 0$. For $n \geq 3$, we apply Corollary 5 for $k = 2$ and $m = 4$: if $0 \leq a_1 \leq a_2 \leq \dots \leq a_n$ and

$$a_1 + a_2 + \dots + a_n = n, \quad a_1^2 + a_2^2 + \dots + a_n^2 = \text{fixed},$$

then

$$S_n = a_1^4 + a_2^4 + \dots + a_n^4$$

is maximum for $a_1 = \dots = a_{n-1} \leq a_n$. Thus, we only need to prove the homogeneous inequality

$$n^2(n^2 - n + 1)(a_1^2 + a_2^2 + \dots + a_n^2)^2 \geq (n^2 - 2n + 2)(a_1 + a_2 + \dots + a_n)^4 + n^3(n-1)S_n,$$

for $a_1 = \dots = a_{n-1} = 0$ and for $a_1 = \dots = a_{n-1} = 1$. For the nontrivial case $a_1 = \dots = a_{n-1} = 1$, the inequality becomes

$$n^2(n^2 - n + 1)(n - 1 + a_n^2)^2 \geq (n^2 - 2n + 2)(n - 1 + a_n)^4 + n^3(n-1)(n - 1 + a_n^4),$$

$$(a_n - 1)^2[a_n - (n-1)^2]^2 \geq 0.$$

The equality holds for $a_1 = a_2 = \dots = a_n = 1$, and also for

$$a_1 = \dots = a_{n-1} = \frac{1}{n-1}, \quad a_n = n-1$$

(or any cyclic permutation).

□

P 5.58. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that $a_1^2 + a_2^2 + \dots + a_n^2 = n$, then

$$a_1^3 + a_2^3 + \dots + a_n^3 \geq \sqrt{n^2 - n + 1 + \left(1 - \frac{1}{n}\right)(a_1^6 + a_2^6 + \dots + a_n^6)}.$$

(Vasile C., 2008)

Solution. For $n = 2$, the inequality is equivalent to

$$a_1^6 + a_2^6 + 4a_1^3 a_2^3 \geq 6,$$

$$(a_1^2 + a_2^2)^3 - 3a_1^2 a_2^2 (a_1^2 + a_2^2) + 4a_1^3 a_2^3 \geq 6,$$

$$2a_1^3 a_2^3 - 3a_1^2 a_2^2 + 1 \geq 0,$$

$$(a_1 a_2 - 1)^2 (2a_1 a_2 + 1) \geq 0.$$

For $n \geq 3$, we apply Corollary 5 for $k = 3/2$ and $m = 3$: if $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$ and

$$x_1 + x_2 + \cdots + x_n = n, \quad x_1^{3/2} + x_2^{3/2} + \cdots + x_n^{3/2} = \text{fixed},$$

then

$$S_n = x_1^3 + x_2^3 + \cdots + x_n^3$$

is maximum for $x_1 = \cdots = x_{n-1} \leq x_n$. Thus, we only need to prove the homogeneous inequality

$$(a_1^3 + a_2^3 + \cdots + a_n^3)^2 \geq \frac{n^2 - n + 1}{n^3} (a_1^2 + a_2^2 + \cdots + a_n^2)^3 + \left(1 - \frac{1}{n}\right) (a_1^6 + a_2^6 + \cdots + a_n^6)$$

for $a_1 = \cdots = a_{n-1} = 0$ and for $a_1 = \cdots = a_{n-1} = 1$. For the nontrivial case $a_1 = \cdots = a_{n-1} = 1$, the inequality becomes

$$n^3(n-1+a_n^3)^2 \geq (n^2-n+1)(n-1+a_n^2)^3 + n^2(n-1)(n-1+a_n^6),$$

$$(a_n-1)^2(a_n-n+1)^2(a_n^2+2na_n+n-1) \geq 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = \cdots = a_{n-1} = \frac{a_n}{n-1} = \frac{1}{\sqrt{n-1}}$$

(or any cyclic permutation).

□

P 5.59. If a, b, c are positive real numbers so that $abc = 1$, then

$$4 \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) + \frac{50}{a+b+c} \geq 27.$$

(Vasile C., 2012)

Solution. According to Corollary 5 (case $k=0$ and $m = -1$, if

$$a + b + c = \text{fixed}, \quad abc = 1, \quad 0 < a \leq b \leq c,$$

then

$$S_3 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

is minimum for $0 < a = b \leq c$. Thus, we only need to prove that

$$4 \left(\frac{2}{a} + \frac{1}{c} \right) + \frac{50}{2a+c} \geq 27$$

for

$$a^2c = 1, \quad a \leq 1.$$

The inequality is equivalent to

$$8a^6 - 54a^4 - 26a^3 - 27a + 8 \geq 0,$$

$$(2a - 1)^2(2a^4 + 2a^3 - 12a^2 + 5a + 8) \geq 0.$$

It is true for $a \in (0, 1]$ because

$$2a^4 + 2a^3 - 12a^2 + 5a + 8 > -12a^2 + 4a + 8 = 4(1 - a)(2 + 3a) \geq 0.$$

The equality holds for

$$a = b = \frac{1}{2}, \quad c = 4$$

(or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization:

- If $a_1, a_2, \dots, a_n$ are positive real numbers so that $a_1 a_2 \cdots a_n = 1$, then

$$2^n \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right) + \frac{(2^n + n - 1)^2}{a_1 + a_2 + \cdots + a_n} \geq 2n(2^n + 1),$$

with equality for

$$a_1 = \cdots = a_{n-1} = \frac{1}{2}, \quad a_n = 2^{n-1}$$

(or any cyclic permutation).

For

$$a_1 = \cdots = a_{n-1} = a \leq 1, \quad a^{n-1} a_n = 1,$$

the inequality is equivalent to $f(a) \geq 0$, where

$$f(a) = 2^n \left(\frac{n-1}{a} + a^{n-1} \right) + \frac{(2^n + n - 1)^2 a^{n-1}}{(n-1)a^n + 1} - 2n(2^n + 1).$$

We have

$$\begin{aligned} \frac{f'(a)}{n-1} &= \frac{2^n(a^n - 1)}{a^2} - \frac{(2^n + n - 1)^2 a^{n-2}(a^n - 1)}{[(n-1)a^n + 1]^2} \\ &= \frac{(a^n - 1)(2^n a^n - 1)[(n-1)^2 a^n - 2^n]}{a^2[(n-1)a^n + 1]^2}. \end{aligned}$$

Since

$$(n-1)^2 a^n - 2^n \leq (n-1)^2 - 2^n < 0,$$

it follows that $f'(a) < 0$ for $a \in \left(0, \frac{1}{2}\right)$, and $f'(a) > 0$ for $a \in \left(\frac{1}{2}, 1\right)$. Therefore, f is decreasing on $\left(0, \frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2}, 1\right]$, hence

$$f(a) \geq f\left(\frac{1}{2}\right) = 0.$$

□

P 5.60. If a, b, c are positive real numbers so that $abc = 1$, then

$$a^3 + b^3 + c^3 + 15 \geq 6 \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right).$$

(Michael Rozenberg, 2006)

Solution. Replacing a, b, c by their reverses $1/a, 1/b, 1/c$, we need to show that $abc = 1$ involves

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + 15 \geq 6(a + b + c).$$

According to Corollary 5 (case $k=0$ and $m = -3$, if

$$a + b + c = \text{fixed}, \quad abc = 1, \quad 0 < a \leq b \leq c,$$

then

$$S_3 = \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3}$$

is minimum for $0 < a = b \leq c$. Thus, we only need to prove that

$$\frac{2}{a^3} + \frac{1}{c^3} + 15 \geq 6(2a + c)$$

for

$$a^2c = 1, \quad a \leq 1.$$

The inequality is equivalent to

$$\frac{2}{a^3} + a^6 + 15 \geq 6 \left(2a + \frac{1}{a^2} \right),$$

$$a^9 - 12a^4 + 15a^3 - 6a + 2 \geq 0,$$

$$(1 - a)^2(2 - 2a - 6a^2 + 5a^3 + 4a^4 + 3a^5 + 2a^6 + a^7) \geq 0.$$

It suffices to show that

$$2 - 2a - 6a^2 + 5a^3 + 3a^4 \geq 0.$$

Indeed, we have

$$2(2 - 2a - 6a^2 + 5a^3 + 3a^4) = (2 - 3a)^2 \left(1 + 2a + \frac{3}{4}a^2 \right) + a^3 \left(1 - \frac{3}{4}a \right) \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 5.61. Let $a_1, a_2, \dots, a_n$ be positive numbers so that $a_1 a_2 \cdots a_n = 1$. If $k \geq n - 1$, then

$$a_1^k + a_2^k + \cdots + a_n^k + (2k - n)n \geq (2k - n + 1) \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right).$$

(Vasile C., 2008)

Solution. For $n = 2$ and $k = 1$, the inequality is an identity. For $n = 2$ and $k > 1$, we need to show that $f(a) \geq 0$ for $a > 0$, where

$$f(a) = a^k + a^{-k} + 4(k-1) - (2k-1)(a + a^{-1}).$$

We have

$$\begin{aligned} f'(a) &= k(a^{k-1} - a^{-k-1}) - (2k-1)(1 - a^{-2}), \\ f''(a) &= k[(k-1)a^{k-2} + (k+1)a^{-k-2}] - 2(2k-1)a^{-3}. \end{aligned}$$

By the weighted AM-GM inequality, we get

$$(k-1)a^{k-2} + (k+1)a^{-k-2} \geq 2ka^{\frac{(k-1)(k-2)+(k+1)(-k-2)}{2k}} = 2ka^{-3},$$

hence

$$f''(a) \geq 2k^2a^{-3} - 2(2k-1)a^{-3} = 2(k-1)^2a^{-3} > 0,$$

f' is strictly increasing. Since $f'(1) = 0$, it follows that $f'(a) < 0$ for $a < 1$ and $f'(a) > 0$ for $a > 1$, f is decreasing on $(0, 1]$ and increasing on $[1, \infty)$, hence $f(a) \geq f(1) = 0$.

Consider further that $n \geq 3$. Replacing $a_1, a_2, \dots, a_n$ by $1/a_1, 1/a_2, \dots, 1/a_n$, we need to show that $a_1a_2 \cdots a_n = 1$ involves

$$\frac{1}{a_1^k} + \frac{1}{a_2^k} + \cdots + \frac{1}{a_n^k} + (2k-n)n \geq (2k-n+1)(a_1 + a_2 + \cdots + a_n).$$

According to Corollary 5, if $0 < a_1 \leq a_2 \leq \cdots \leq a_n$ and

$$a_1 + a_2 + \cdots + a_n = \text{fixed}, \quad a_1a_2 \cdots a_n = 1,$$

then

$$S_n = \frac{1}{a_1^k} + \frac{1}{a_2^k} + \cdots + \frac{1}{a_n^k}$$

is minimum for $0 < a_1 = \cdots = a_{n-1} \leq a_n$. Thus, we only need to prove the original inequality for $a_1 = \cdots = a_{n-1} \geq 1$; that is, to show that $t \geq 1$ involves $f(t) \geq 0$, where

$$f(t) = (n-1)t^k + \frac{1}{t^{k(n-1)}} + (2k-n)n - (2k-n+1) \left(\frac{n-1}{t} + t^{n-1} \right).$$

We have

$$f'(t) = \frac{(n-1)g(t)}{t^{kn-k+1}}, \quad g(t) = k(t^{kn} - 1) - (2k-n+1)t^{kn-k-1}(t^n - 1),$$

$$g'(t) = t^{kn-k-2}h(t), \quad h(t) = k^2nt^{k+1} - (2k-n+1)[(k+1)(n-1)t^n - kn + k + 1],$$

$$h'(t) = (k+1)nt^{n-1}[k^2t^{k-n+1} - (2k-n+1)(n-1)].$$

If $k = n-1$, then $h(t) = n(n-1)(n-2) > 0$. If $k > n-1$, then

$$k^2t^{k-n+1} - (2k-n+1)(n-1) \geq k^2 - (2k-n+1)(n-1) = (k-n+1)^2 > 0,$$

$h'(t) > 0$ for $t \geq 1$, h is strictly increasing on $[1, \infty)$, hence

$$h(t) \geq h(1) = n[(k-1)^2 + n-2] > 0.$$

From $h > 0$, we get $g' > 0$, g is strictly increasing, $g(t) \geq g(1) = 0$ for $t \geq 1$, $f'(t) > 0$ for $t > 1$, f is strictly increasing, $f(t) \geq f(1) = 0$ for $t \geq 1$.

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $n = 2$ and $k = 1$, then the equality holds for $a_1 a_2 = 1$.

□

P 5.62. Let $a_1, a_2, \dots, a_n$ ($n \geq 3$) be nonnegative numbers so that $a_1 + a_2 + \cdots + a_n = n$, and let k be an integer satisfying $2 \leq k \leq n + 2$. If

$$m = \left(\frac{n}{n-1} \right)^{k-1} - 1,$$

then

$$\frac{a_1^k + a_2^k + \cdots + a_n^k}{n} - 1 \geq m(1 - a_1 a_2 \cdots a_n).$$

(Vasile C., 2005)

Solution. According to Corollary 4, if $0 \leq a_1 \leq a_2 \leq \cdots \leq a_n$ and

$$a_1 + a_2 + \cdots + a_n = n, \quad a_1^k + a_2^k + \cdots + a_n^k = \text{fixed},$$

then the product

$$P = a_1 a_2 \cdots a_n$$

is minimum for either $a_1 = 0$ or $0 < a_1 \leq a_2 = \cdots = a_n$.

Case 1: $a_1 = 0$. We need to show that

$$a_2^k + \cdots + a_n^k \geq \frac{n^k}{(n-1)^{k-1}}$$

for $a_2 + \cdots + a_n = n$. This follows by Jensen's inequality

$$a_2^k + \cdots + a_n^k \geq (n-1) \left(\frac{a_2 + \cdots + a_n}{n-1} \right)^k.$$

Case 2: $0 < a_1 \leq a_2 = \cdots = a_n$. Denoting $a_1 = x$ and $a_2 = y$ ($x \leq y$), we only need to show that

$$f(x) \geq 0,$$

where

$$f(x) = x^k + (n-1)y^k + nmxy^{n-1} - n(m+1), \quad y = \frac{n-x}{n-1}, \quad 0 < x \leq 1 \leq y.$$

It is easy to check that

$$f(0) = f(1) = 0.$$

Since

$$y' = \frac{-1}{n-1},$$

we have

$$\begin{aligned} f'(x) &= k(x^{k-1} - y^{k-1}) + nmy^{n-2}(y - x) \\ &= (y - x)[nmy^{n-2} - k(y^{k-2} + y^{k-3}x + \cdots + x^{k-2})] \\ &= (y - x)y^{n-2}[nm - kg(x)], \end{aligned}$$

where

$$g(x) = \frac{1}{y^{n-k}} + \frac{x}{y^{n-k+1}} + \cdots + \frac{x^{k-2}}{y^{n-2}}.$$

We see that $f'(x)$ has the same sign as

$$h(x) = nm - kg(x).$$

Since the function

$$y(x) = \frac{n - x}{n - 1}$$

is strictly decreasing, g is strictly increasing for $2 \leq k \leq n$. Also, g is strictly increasing for $k = n + 1$, when

$$\begin{aligned} g(x) &= y + x + \frac{x^2}{y} + \cdots + \frac{x^{n-1}}{y^{n-2}} \\ &= \frac{(n-2)x + n}{n-1} + \frac{x^2}{y} + \cdots + \frac{x^{n-1}}{y^{n-2}}, \end{aligned}$$

and for $k = n + 2$, when

$$\begin{aligned} g(x) &= y^2 + yx + x^2 + \frac{x^3}{y} + \cdots + \frac{x^n}{y^{n-2}} \\ &= \frac{(n^2 - 3n + 3)x^2 + n(n-3)x + n^2}{(n-1)^2} + \frac{x^3}{y} + \cdots + \frac{x^n}{y^{n-2}}. \end{aligned}$$

Therefore, the function $h(x)$ is strictly decreasing for $x \in [0, 1]$. Since $f(0) = f(1) = 0$, there exists $x_1 \in (0, 1)$ so that $f(x)$ is increasing on $[0, x_1]$ and decreasing on $[x_1, 1]$. As a consequence, $f(x) \geq 0$ for $x \in [0, 1]$.

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = 0, \quad a_2 = \cdots = a_n = \frac{n}{n-1}$$

(or any cyclic permutation).

Remark 1. For $k = 2$, we obtain the following inequality

$$(n-1)(a_1^2 + a_2^2 + \cdots + a_n^2) + na_1a_2 \cdots a_n \geq n^2,$$

which holds for any nonnegative real numbers $a_1, a_2, \dots, a_n$ satisfying $a_1 + a_2 + \cdots + a_n = n$. Since $a_1a_2 \cdots a_n \leq 1$ (by the AM-GM inequality), this inequality implies

$$(n-1)(a_1^2 + a_2^2 + \cdots + a_n^2) + n(a_1a_2 \cdots a_n)^{2/n} \geq n^2,$$

which is equivalent to Shleifer's homogeneous inequality:

$$(n-1)(a_1^2 + a_2^2 + \cdots + a_n^2) + n(a_1 a_2 \cdots a_n)^{2/n} \geq (a_1 + a_2 + \cdots + a_n)^2.$$

For $n = 4$, Shleifer's inequality becomes Turkevich's inequality:

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 + 2\sqrt{a_1 a_2 a_3 a_4} \geq a_1 a_2 + a_1 a_3 + a_1 a_4 + a_2 a_3 + a_2 a_4 + a_3 a_4.$$

Remark 2. For the particular case $k = n$, the inequality has been posted in 2004 on Art of Problem Solving website by *Gabriel Dospinescu* and *Calin Popa*.

□

P 5.63. If a, b, c are positive real numbers so that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 3$, then

$$4(a^2 + b^2 + c^2) + 9 \geq 21abc.$$

(Vasile C., 2006)

Solution. Replacing a, b, c by their reverses $1/a, 1/b, 1/c$, we need to show that $a + b + c = 3$ involves

$$4\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) + 9 \geq \frac{21}{abc}.$$

According to Corollary 5 (case $k=0$ and $m = -2$), if

$$a + b + c = 3, \quad abc = \text{fixed}, \quad 0 < a \leq b \leq c,$$

then

$$S_3 = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$$

is minimum for $0 < a = b \leq c$. Thus, we only need to prove that

$$4\left(\frac{2}{a^2} + \frac{1}{c^2}\right) + 9 \geq \frac{21}{a^2 c}$$

for $2a + b = 3$. The inequality is equivalent to

$$(9a^2 + 8)c^2 - 21c + 4a^2 \geq 0,$$

$$4a^4 - 12a^3 + 13a^2 - 6a + 1 \geq 0,$$

$$(a-1)^2(2a-1)^2 \geq 0.$$

The equality holds for $a = b = c = 1$, and also for

$$a = b = 2, \quad c = \frac{1}{2}$$

(or any cyclic permutation).

□

P 5.64. If $a_1, a_2, \dots, a_n$ are positive real numbers so that $\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} = n$, then

$$a_1 + a_2 + \dots + a_n - n \leq e_{n-1}(a_1 a_2 \cdots a_n - 1),$$

where

$$e_{n-1} = \left(1 + \frac{1}{n-1}\right)^{n-1}.$$

(Gabriel Dospinescu and Calin Popa, 2004)

Solution. For $n = 2$, the inequality is an identity. For $n \geq 3$, replacing $a_1, a_2, \dots, a_n$ by $1/a_1, 1/a_2, \dots, 1/a_n$, we need to show that $a_1 + a_2 + \dots + a_n = n$ involves

$$a_1 a_2 \cdots a_n \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} - n + e_{n-1} \right) \leq e_{n-1}.$$

According to Corollary 5 (case $k = 0$ and $m = -1$), if $0 < a_1 \leq a_2 \leq \dots \leq a_n$ and

$$a_1 + a_2 + \dots + a_n = n, \quad a_1 a_2 \cdots a_n = \text{fixed},$$

then

$$S_n = \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}$$

is maximum for $0 < a_1 \leq a_2 = \dots = a_n$. Using the notation $a_1 = x$ and $a_2 = y$, we only need to show that $f(x) \leq 0$ for

$$x + (n-1)y = n, \quad 0 < x \leq 1,$$

where

$$\begin{aligned} f(x) &= xy^{n-1} \left(\frac{1}{x} + \frac{n-1}{y} - n + e_{n-1} \right) - e_{n-1} \\ &= y^{n-1} + (n-1)xy^{n-2} - (n - e_{n-1})xy^{n-1} - e_{n-1}. \end{aligned}$$

Since

$$y' = \frac{-1}{n-1},$$

we get

$$\frac{f'(x)}{y^{n-3}} = (y-x)h(x),$$

where

$$h(x) = n - 2 - (n - e_{n-1})y = n - 2 - (n - e_{n-1})\frac{n-x}{n-1}$$

is a linear increasing function. Since

$$h(0) = \frac{n}{n-1} \left(e_{n-1} - 3 + \frac{2}{n} \right) < 0$$

and

$$h(1) = e_{n-1} - 2 > 0,$$

there exists $x_1 \in (0, 1)$ so that $h(x_1) = 0$, $h(x) < 0$ for $x \in [0, x_1)$, and $h(x) > 0$ for $x \in (x_1, 1]$. Consequently, f is strictly decreasing on $[0, x_1]$ and strictly increasing on $[x_1, 1]$. From

$$f(0) = f(1) = 0,$$

it follows that $f(x) \leq 0$ for $x \in [0, 1]$.

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $n = 2$, then the equality holds for $a_1 + a_2 = 2a_1a_2$. □

P 5.65. If $a_1, a_2, \dots, a_n$ are positive real numbers, then

$$\frac{a_1^n + a_2^n + \cdots + a_n^n}{a_1 a_2 \cdots a_n} + n(n-1) \geq (a_1 + a_2 + \cdots + a_n) \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right).$$

(Vasile C., *Cruz Mathematicorum*, 8, 2006)

Solution. For $n = 2$, the inequality is an identity. For $n \geq 3$, according to Corollary 5 (case $k = 0$ and $m \in \{-1, n\}$), if $0 < a_1 \leq a_2 \leq \cdots \leq a_n$ and

$$a_1 + a_2 + \cdots + a_n = \text{fixed}, \quad a_1 a_2 \cdots a_n = \text{fixed},$$

then the sum $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$ is maximum and the sum $a_1^n + a_2^n + \cdots + a_n^n$ is minimum for

$$0 < a_1 \leq a_2 = \cdots = a_n.$$

Consequently, we only need to prove the desired homogeneous inequality for $a_2 = \cdots = a_n = 1$, when it becomes

$$a_1^n + (n-2)a_1 \geq (n-1)a_1^2.$$

Indeed, by the AM-GM inequality, we have

$$a_1^n + (n-2)a_1 \geq (n-1) \sqrt[n-1]{a_1^n \cdot a_1^{n-2}} = (n-1)a_1^2.$$

For $n \geq 3$, the equality holds when $a_1 = a_2 = \cdots = a_n$. □

P 5.66. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers, then

$$(n-1)(a_1^n + a_2^n + \cdots + a_n^n) + na_1 a_2 \cdots a_n \geq (a_1 + a_2 + \cdots + a_n)(a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1}).$$

(Janos Suranyi, *MSC-Hungary*)

Solution. For $n = 2$, the inequality is an identity. For $n \geq 3$, according to Corollary 5 (case $k = n$ and $m = n - 1$), if $0 \leq a_1 \leq a_2 \leq \cdots \leq a_n$ and

$$a_1 + a_2 + \cdots + a_n = \text{fixed}, \quad a_1^n + a_2^n + \cdots + a_n^n = \text{fixed},$$

then the sum $a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1}$ is maximum and the product $a_1 a_2 \cdots a_n$ is minimum for either $a_1 = 0$ or $0 < a_1 \leq a_2 = \cdots = a_n$. Consequently, we only need to consider these cases.

Case 1: $a_1 = 0$. The inequality reduces to

$$(n-1)(a_2^n + \cdots + a_n^n) \geq (a_2 + \cdots + a_n)(a_2^{n-1} + \cdots + a_n^{n-1}),$$

which follows immediately from Chebyshev's inequality.

Case 2: $0 < a_1 \leq a_2 = \cdots = a_n$. Due to homogeneity, we may set $a_2 = \cdots = a_n = 1$, when the inequality becomes

$$(n-2)a_1^n + a_1 \geq (n-1)a_1^{n-1}.$$

Indeed, by the AM-GM inequality, we have

$$(n-2)a_1^n + a_1 \geq (n-1) \sqrt[n-1]{a_1^{n(n-2)} \cdot a_1} = (n-1)a_1^{n-1}.$$

For $n \geq 3$, the equality holds when $a_1 = a_2 = \cdots = a_n$, and also when

$$a_1 = 0, \quad a_2 = \cdots = a_n$$

(or any cyclic permutation).

□

P 5.67. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers, then

$$(n-1)(a_1^{n+1} + a_2^{n+1} + \cdots + a_n^{n+1}) \geq (a_1 + a_2 + \cdots + a_n)(a_1^n + a_2^n + \cdots + a_n^n - a_1 a_2 \cdots a_n).$$

(Vasile C., 2006)

Solution. For $n = 2$, the inequality is an identity. For $n \geq 3$, according to Corollary 5 (case $k = n + 1$ and $m = n$), if $0 \leq a_1 \leq a_2 \leq \cdots \leq a_n$ and

$$a_1 + a_2 + \cdots + a_n = \text{fixed}, \quad a_1^{n+1} + a_2^{n+1} + \cdots + a_n^{n+1} = \text{fixed},$$

then the sum $a_1^n + a_2^n + \cdots + a_n^n$ is maximum and the product $a_1 a_2 \cdots a_n$ is minimum for either $a_1 = 0$ or $0 < a_1 \leq a_2 = \cdots = a_n$. Consequently, we only need to consider these cases.

Case 1: $a_1 = 0$. The inequality reduces to

$$(n-1)(a_2^{n+1} + \cdots + a_n^{n+1}) \geq (a_2 + \cdots + a_n)(a_2^n + \cdots + a_n^n),$$

which follows immediately from Chebyshev's inequality.

Case 2: $0 < a_1 \leq a_2 = \cdots = a_n$. Due to homogeneity, we may set $a_2 = \cdots = a_n = 1$, when the inequality becomes

$$(n-2)a_1^{n+1} + a_1^2 \geq (n-1)a_1^n.$$

Indeed, by the AM-GM inequality, we have

$$(n-2)a_1^{n+1} + a_1^2 \geq (n-1) \sqrt[n-1]{a_1^{(n+1)(n-2)} \cdot a_1^2} = (n-1)a_1^n.$$

For $n \geq 3$, the equality holds when $a_1 = a_2 = \cdots = a_n$, and also when

$$a_1 = 0, \quad a_2 = \cdots = a_n$$

(or any cyclic permutation).

□

P 5.68. If $a_1, a_2, \dots, a_n$ are positive real numbers, then

$$(a_1 + a_2 + \cdots + a_n - n) \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n \right) + a_1 a_2 \cdots a_n + \frac{1}{a_1 a_2 \cdots a_n} \geq 2.$$

(Vasile C., 2006)

Solution. For $n = 2$, the inequality reduces to

$$(1 - a_1)^2(1 - a_2)^2 \geq 0.$$

Consider further that $n \geq 3$. Since the inequality remains unchanged by replacing each a_i with $1/a_i$, we may consider $a_1 a_2 \cdots a_n \geq 1$. By the AM-GM inequality, we get

$$a_1 + a_2 + \cdots + a_n \geq n \sqrt[n]{a_1 a_2 \cdots a_n} \geq n.$$

According to Corollary 5 (case $k = 0$ and $m = -1$), if $0 < a_1 \leq a_2 \leq \cdots \leq a_n$ and

$$a_1 + a_2 + \cdots + a_n = \text{fixed}, \quad a_1 a_2 \cdots a_n = \text{fixed},$$

then the sum

$$S_n = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$$

is minimum for $0 < a_1 = a_2 = \cdots = a_{n-1} \leq a_n$. Consequently, we only need to consider

$$a_1 = a_2 = \cdots = a_{n-1} = x, \quad a_n = y, \quad x \leq y.$$

The inequality becomes

$$\begin{aligned} & [(n-1)x + y - n] \left(\frac{n-1}{x} + \frac{1}{y} - n \right) + x^{n-1}y + \frac{1}{x^{n-1}y} \geq 2, \\ & \left(x^{n-1} + \frac{n-1}{x} - n \right) y + \left[\frac{1}{x^{n-1}} + (n-1)x - n \right] \frac{1}{y} \geq \frac{n(n-1)(x-1)^2}{x}. \end{aligned}$$

Since

$$\begin{aligned} x^{n-1} + \frac{n-1}{x} - n &= \frac{x-1}{x} [(x^{n-1} - 1) + (x^{n-2} - 1) + \cdots + (x - 1)] \\ &= \frac{(x-1)^2}{x} [x^{n-2} + 2x^{n-3} + \cdots + (n-1)], \end{aligned}$$

and

$$\frac{1}{x^{n-1}} + (n-1)x - n = \frac{(x-1)^2}{x} \left[\frac{1}{x^{n-2}} + \frac{2}{x^{n-3}} + \cdots + (n-1) \right],$$

it is enough to prove the inequality

$$\left[x^{n-2} + 2x^{n-3} + \cdots + (n-1) \right] y + \left[\frac{1}{x^{n-2}} + \frac{2}{x^{n-3}} + \cdots + (n-1) \right] \frac{1}{y} \geq n(n-1),$$

which is equivalent to

$$\begin{aligned} \left(x^{n-2}y + \frac{1}{x^{n-2}y} - 2 \right) + 2 \left(x^{n-3}y + \frac{1}{x^{n-3}y} - 2 \right) + \cdots + (n-1) \left(y + \frac{1}{y} - 2 \right) &\geq 0, \\ \frac{(x^{n-2}y - 1)^2}{x^{n-2}y} + \frac{2(x^{n-3}y - 1)^2}{x^{n-3}y} + \cdots + \frac{(n-1)(y - 1)^2}{y} &\geq 0. \end{aligned}$$

The equality holds if $n-1$ of the numbers a_i are equal to 1.

□

P 5.69. If $a_1, a_2, \dots, a_n$ are positive real numbers so that $a_1 a_2 \cdots a_n = 1$, then

$$\left| \frac{1}{\sqrt{a_1 + a_2 + \cdots + a_n - n}} - \frac{1}{\sqrt{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n}} \right| < 1.$$

(Vasile C., 2006)

Solution. Let

$$A = a_1 + a_2 + \cdots + a_n - n, \quad B = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n.$$

By the AM-GM inequality, it follows that $A > 0$ and $B > 0$. According to the preceding P 5.68, the following inequality holds

$$(a_1 + \cdots + a_{n+1} - n - 1) \left(\frac{1}{a_1} + \cdots + \frac{1}{a_{n+1}} - n - 1 \right) + a_1 \cdots a_{n+1} + \frac{1}{a_1 \cdots a_{n+1}} \geq 2,$$

which is equivalent to

$$(A - 1 + a_{n+1}) \left(B - 1 + \frac{1}{a_{n+1}} \right) + a_{n+1} + \frac{1}{a_{n+1}} \geq 2,$$

$$\frac{A}{a_{n+1}} + B a_{n+1} + AB - A - B \geq 0.$$

Choosing

$$a_{n+1} = \sqrt{\frac{A}{B}},$$

we get

$$\begin{aligned} 2\sqrt{AB} + AB - A - B &\geq 0, \\ AB &\geq \left(\sqrt{A} - \sqrt{B}\right)^2, \\ 1 &\geq \left|\frac{1}{\sqrt{A}} - \frac{1}{\sqrt{B}}\right|. \end{aligned}$$

□

P 5.70. If $a_1, a_2, \dots, a_n$ are positive real numbers so that $a_1 a_2 \cdots a_n = 1$, then

$$a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1} + \frac{n^2(n-2)}{a_1 + a_2 + \cdots + a_n} \geq (n-1) \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right).$$

Solution. For $n = 2$, the inequality is an identity. Consider further that $n \geq 3$. According to Corollary 5 (case $k = 0$), if $0 < a_1 \leq a_2 \leq \cdots \leq a_n$ and

$$a_1 + a_2 + \cdots + a_n = \text{fixed}, \quad a_1 a_2 \cdots a_n = 1,$$

then the sum $a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1}$ is minimum and the sum $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$ is maximum for $0 < a_1 \leq a_2 \leq \cdots = a_n$. Thus, we only need to prove the homogeneous inequality

$$a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1} + \frac{n^2(n-2)a_1 a_2 \cdots a_n}{a_1 + a_2 + \cdots + a_n} \geq (n-1)a_1 a_2 \cdots a_n \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right)$$

for $a_2 = \cdots = a_n = 1$; that is, to show that $f(x) \geq 0$ for $x \in [0, 1]$, where

$$f(x) = x^{n-2} + \frac{n^2(n-2)}{x+n-1} - (n-1)^2,$$

$$\frac{f'(x)}{n-2} = x^{n-3} - \frac{n^2}{(x+n-1)^2}.$$

Since f' is increasing, we have $f'(x) \leq f'(1) = 0$ for $x \in [0, 1]$, f is decreasing on $[0, 1]$, hence $f(x) \geq f(1) = 0$.

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $n = 2$, then the equality holds for $a_1 a_2 = 1$.

□

P 5.71. If a, b, c are nonnegative real numbers, then

$$(a+b+c-3)^2 \geq \frac{abc-1}{abc+1}(a^2+b^2+c^2-3).$$

(Vasile C., 2006)

Solution. For $a = 0$, the inequality reduces to

$$b^2 + c^2 + bc + 3 \geq 3(b + c),$$

which is equivalent to

$$(b - c)^2 + 3(b + c - 2)^2 \geq 0.$$

For $abc > 0$, according to Corollary 5 (case $k = 0$ and $m = 2$), if

$$a + b + c = \text{fixed}, \quad abc = \text{fixed},$$

then

$$S_3 = a^2 + b^2 + c^2$$

is minimum and maximum when two of a, b, c are equal. Thus, we only need to prove the desired inequality for $a = b$; that is,

$$(2a + c - 3)^2 \geq \frac{a^2c - 1}{a^2c + 1}(2a^2 + c^2 - 3),$$

which is equivalent to

$$(a - 1)^2[ca^2 + 2c(c - 2)a + c^2 - 3c + 3] \geq 0.$$

For $c \geq 2$, the inequality is clearly true. It is also true for $c \leq 2$, because

$$ca^2 + 2c(c - 2)a + c^2 - 3c + 3 = c(a + c - 2)^2 + (1 - c)^2(3 - c) \geq 0.$$

The equality holds if two of a, b, c are equal to 1.

□

P 5.72. If $a_1, a_2, \dots, a_n$ are positive real numbers so that $a_1 + a_2 + \dots + a_n = n$, then

$$(a_1 a_2 \cdots a_n)^{\frac{1}{\sqrt{n-1}}}(a_1^2 + a_2^2 + \dots + a_n^2) \leq n.$$

(Vasile C., 2006)

Solution. For $n = 2$, the inequality is equivalent to

$$(a_1 a_2 - 1)^2 \geq 0.$$

For $n \geq 3$, according to Corollary 5 (case $k = 0$, $m = 2$), if $0 < a_1 \leq a_2 \leq \dots \leq a_n$ and

$$a_1 + a_2 + \dots + a_n = n, \quad a_1 a_2 \cdots a_n = \text{fixed},$$

then the sum

$$S_n = a_1^2 + a_2^2 + \dots + a_n^2$$

is maximum for $a_1 = a_2 = \dots = a_{n-1}$. Therefore, we only need to prove the homogeneous inequality

$$(a_1 a_2 \cdots a_n)^{\frac{1}{\sqrt{n-1}}} \cdot \frac{a_1^2 + a_2^2 + \dots + a_n^2}{n} \leq \left(\frac{a_1 + a_2 + \dots + a_n}{n} \right)^{2 + \frac{n}{\sqrt{n-1}}}$$

for $a_1 = a_2 = \cdots = a_{n-1} = 1$. The inequality is equivalent to $f(x) \geq 0$ for $x \geq 1$, where

$$f(x) = \left(2 + \frac{n}{\sqrt{n-1}}\right) \ln \frac{x+n-1}{n} - \frac{\ln x}{\sqrt{n-1}} - \ln \frac{x^2+n-1}{n}.$$

Let

$$p = \frac{1}{\sqrt{n-1}}.$$

Since

$$\begin{aligned} f'(x) &= \frac{2+np}{x+n-1} - \frac{p}{x} - \frac{2x}{x^2+n-1} \\ &= \frac{(n-1)(x-1)}{x+n-1} \left(\frac{p}{x} - \frac{2}{x^2+n-1} \right) \\ &= \frac{p(n-1)(x-1)(x-\sqrt{n-1})^2}{x(x+n-1)(x^2+n-1)} \geq 0, \end{aligned}$$

$f(x)$ is increasing for $x \geq 1$, hence

$$f(x) \geq f(1) = 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Remark. For $n = 5$, from the homogeneous inequality above, we get the following nice inequalities:

- If a, b, c, d, e are positive real numbers so that

$$a^2 + b^2 + c^2 + d^2 + e^2 = 5,$$

then

$$(a) \quad abcde(a^4 + b^4 + c^4 + d^4 + e^4) \leq 5;$$

$$(b) \quad a + b + c + d + e \geq 5\sqrt[9]{abcde}.$$

□

P 5.73. If $a_1, a_2, \dots, a_n$ are positive real numbers such that $a_1 + a_2 + \cdots + a_n = n - 1$, then

$$\sqrt[n]{\frac{n-1}{a_1 a_2 \cdots a_n}} \geq 4 \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n(n-1)}}.$$

(Vasile Cîrtoaje and KaiRain, 2020)

Solution. For $n = 2$, we need to show that $a_1 + a_2 = 1$ involves

$$\frac{1}{a_1 a_2} \geq 8(a_1^2 + a_2^2),$$

which is equivalent to

$$(4a_1 a_2 - 1)^2 \geq 0.$$

For $n \geq 3$, write the inequality in the homogeneous form

$$\left(\frac{a_1 + a_2 + \cdots + a_n}{n-1} \right)^2 \sqrt[n]{\frac{n-1}{a_1 a_2 \cdots a_n}} \geq 4 \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n(n-1)}}.$$

According to Corollary 4, for $a_1 + a_2 + \cdots + a_n = \text{fixed}$ and $a_1^2 + a_2^2 + \cdots + a_n^2 = \text{fixed}$, the product $a_1 a_2 \cdots a_n$ is maximum for $a_1 = a_2 = \cdots = a_{n-1} \leq a_n$. Due to homogeneity, we may set $a_1 = a_2 = \cdots = a_{n-1} = 1$, when the inequality becomes

$$\frac{A(x+n-1)^2}{\sqrt[n]{x}} \geq \sqrt{x^2 + n-1},$$

where

$$A = \frac{\sqrt{n}}{4(n-1)^{(3n-2)/(2n)}}, \quad x \geq 1.$$

The inequality is true if $f(x) \geq 0$, where

$$f(x) = \ln A + 2 \ln(x+n-1) - \frac{1}{n} \ln x - \frac{1}{2} \ln(x^2 + n-1).$$

From

$$\begin{aligned} f'(x) &= \frac{2}{x+n-1} - \frac{1}{nx} - \frac{x}{x^2+n-1} \\ &= \frac{(n-1)[x^3 - (n+1)x^2 + (2n-1)x - n+1]}{nx(x+n-1)(x^2+n-1)} \\ &= \frac{(n-1)(x-1)^2(x-n+1)}{nx(x+n-1)(x^2+n-1)}, \end{aligned}$$

it follows that f is decreasing on $[1, n-1]$ and increasing on $[n-1, \infty)$, therefore

$$f(x) \geq f(n-1) = 0.$$

The equality occurs for $a_1 = a_2 = \cdots = a_{n-1} = \frac{1}{2}$ and $a_n = \frac{n-1}{2}$ (or any cyclic permutation). $\square$

P 5.74. If $a_1, a_2, \dots, a_n$ are positive real numbers so that $a_1^3 + a_2^3 + \cdots + a_n^3 = n$, then

$$a_1 + a_2 + \cdots + a_n \geq n \sqrt[n+1]{a_1 a_2 \cdots a_n}.$$

(Vasile C., 2007)

Solution. For $n = 2$, we need to show that $a_1^3 + a_2^3 = 2$ involves $(a_1 + a_2)^3 \geq 8a_1 a_2$. Let

$$x = a_1 + a_2.$$

From

$$2 = a_1^3 + a_2^3 = x^3 - 3a_1 a_2 x,$$

we get

$$a_1 a_2 = \frac{x^3 - 2}{3x}.$$

Thus,

$$(a_1 + a_2)^3 - 8a_1 a_2 = x^3 - \frac{8(x^3 - 2)}{3x} = \frac{(x - 2)^2(3x^2 + 4x + 4)}{3x} \geq 0.$$

For $n \geq 3$, according to Corollary 4, if $0 < a_1 \leq a_2 \leq \dots \leq a_n$ and

$$a_1 + a_2 + \dots + a_n = \text{fixed}, \quad a_1^3 + a_2^3 + \dots + a_n^3 = n,$$

then the product

$$P = a_1 a_2 \dots a_n$$

is maximum for $a_1 = a_2 = \dots = a_{n-1}$. Therefore, we only need to prove the homogeneous inequality

$$\left(\frac{a_1 + a_2 + \dots + a_n}{n} \right)^{n+1} \geq a_1 a_2 \dots a_n \sqrt[3]{\frac{a_1^3 + a_2^3 + \dots + a_n^3}{n}}$$

for $a_1 = a_2 = \dots = a_{n-1} = 1$. The inequality is equivalent to $f(x) \geq 0$ for $x \geq 1$, where

$$f(x) = (n+1) \ln \frac{x+n-1}{n} - \ln x - \frac{1}{3} \ln \frac{x^3+n-1}{n}.$$

Since

$$\begin{aligned} f'(x) &= \frac{n+1}{x+n-1} - \frac{1}{x} - \frac{x^2}{x^3+n-1} \\ &= \frac{(n-1)(x-1)(x^3-x^2-x+n-1)}{x(x+n-1)(x^3+n-1)} \\ &\geq \frac{(n-1)(x-1)(x^3-x^2-x+1)}{x(x+n-1)(x^3+n-1)} \\ &= \frac{(n-1)(x-1)^3(x+1)}{x(x+n-1)(x^3+n-1)}, \end{aligned}$$

$f(x)$ is increasing for $x \geq 1$, hence

$$f(x) \geq f(1) = 0.$$

The equality holds for $a_1 = a_2 = \dots = a_n = 1$.

□

P 5.75. Let a, b, c be nonnegative real numbers so that $ab + bc + ca = 3$. If

$$k \geq 2 - \frac{\ln 4}{\ln 3} \approx 0.738,$$

then

$$a^k + b^k + c^k \geq 3.$$

(Vasile C., 2004)

Solution. Let

$$r = 2 - \frac{\ln 4}{\ln 3}.$$

By the power mean inequality, we have

$$\frac{a^k + b^k + c^k}{3} \geq \left(\frac{a^r + b^r + c^r}{3} \right)^{k/r}.$$

Thus, it suffices to show that

$$a^r + b^r + c^r \geq 3.$$

Since

$$2(ab + bc + ca) = (a + b + c)^2 - (a^2 + b^2 + c^2),$$

according to Corollary 5 (case $k = 2$, $m = r$), if $a \leq b \leq c$ and

$$a + b + c = \text{fixed}, \quad a^2 + b^2 + c^2 = \text{fixed},$$

then

$$S_3 = a^r + b^r + c^r$$

is minimum for either $a = 0$ or $0 < a \leq b = c$.

Case 1: $a = 0$. We need to show that $bc = 3$ implies $b^r + c^r \geq 3$. Indeed, by the AM-GM inequality, we have

$$b^r + c^r \geq 2\sqrt{(bc)^r} = 2 \cdot 3^{r/2} = 3.$$

Case 2: $0 < a \leq b = c$. We only need to show that the homogeneous inequality

$$a^r + b^r + c^r \geq 3 \left(\frac{ab + bc + ca}{3} \right)^{r/2}$$

holds for $b = c = 1$; that is, to show that $a \in (0, 1]$ involves

$$a^r + 2 \geq 3 \left(\frac{2a + 1}{3} \right)^{r/2},$$

which is equivalent to $f(a) \geq 0$, where

$$f(a) = \ln \frac{a^r + 2}{3} - \frac{r}{2} \ln \frac{2a + 1}{3}.$$

The derivative

$$f'(a) = \frac{ra^{r-1}}{a^r + 2} - \frac{r}{2a + 1} = \frac{rg(a)}{a^{1-r}(a^r + 2)(2a + 1)},$$

where

$$g(a) = a - 2a^{1-r} + 1.$$

From

$$g'(a) = 1 - \frac{2(1-r)}{a^r},$$

it follows that $g'(a) < 0$ for $a \in (0, a_1)$, and $g'(a) > 0$ for $a \in (a_1, 1]$, where

$$a_1 = (2 - 2r)^{1/r} \approx 0.416.$$

Then, g is strictly decreasing on $[0, a_1]$ and strictly increasing on $[a_1, 1]$. Since $g(0) = 1$ and $g(1) = 0$, there exists $a_2 \in (0, 1)$ so that $g(a_2) = 0$, $g(a) > 0$ for $a \in [0, a_2)$, and $g(a) < 0$ for $a \in (a_2, 1]$. Consequently, f is increasing on $[0, a_2]$ and decreasing on $[a_2, 1]$. Since $f(0) = f(1) = 0$, we have $f(a) \geq 0$ for $0 < a \leq 1$.

The equality holds for $a = b = c = 1$. If $k = 2 - \frac{\ln 4}{\ln 3}$, then the equality holds also for

$$a = 0, \quad b = c = \sqrt{3}$$

(or any cyclic permutation).

Remark. For $k = 3/4$, we get the following nice results (see P 3.33 in Volume 1):

- Let a, b, c be positive real numbers.

(a) If $a^4b^4 + b^4c^4 + c^4a^4 = 3$, then

$$a^3 + b^3 + c^3 \geq 3.$$

(b) If $a^3 + b^3 + c^3 = 3$, then

$$a^4b^4 + b^4c^4 + c^4a^4 \leq 3.$$

□

P 5.76. Let a, b, c be nonnegative real numbers so that $a + b + c = 3$. If

$$k \geq \frac{\ln 9 - \ln 8}{\ln 3 - \ln 2} \approx 0.29,$$

then

$$a^k + b^k + c^k \geq ab + bc + ca.$$

(Vasile C., 2005)

Solution. For $k \geq 1$, by Jensen's inequality, we get

$$a^k + b^k + c^k \geq 3 \left(\frac{a + b + c}{3} \right)^k = 3 = \frac{1}{3}(a + b + c)^2 \geq ab + bc + ca.$$

Let

$$r = \frac{\ln 9 - \ln 8}{\ln 3 - \ln 2}.$$

Assume further that

$$r \leq k < 1,$$

and write the inequality as

$$2(a^k + b^k + c^k) + a^2 + b^2 + c^2 \geq 9.$$

By Corollary 5, if $a \leq b \leq c$ and

$$a + b + c = 3, \quad a^2 + b^2 + c^2 = \text{fixed},$$

then the sum

$$S_3 = a^k + b^k + c^k$$

is minimum for either $a = 0$ or $0 < a \leq b = c$. Thus, we only need to prove the desired inequality for these cases.

Case 1: $a = 0$. We need to show that $b + c = 3$ involves $b^k + c^k \geq bc$. Indeed, by the AM-GM inequality, we have

$$\begin{aligned} b^k + c^k - bc &\geq 2(bc)^{k/2} - bc = (bc)^{k/2} [2 - (bc)^{1-k/2}] \\ &\geq (bc)^{k/2} \left[2 - \left(\frac{b+c}{2} \right)^{2-k} \right] = (bc)^{k/2} \left[2 - \left(\frac{3}{2} \right)^{2-k} \right] \\ &\geq (bc)^{k/2} \left[2 - \left(\frac{3}{2} \right)^{2-r} \right] = 0. \end{aligned}$$

Case 2: $0 < a \leq b = c$. We only need to show that the homogeneous inequality

$$(a^k + b^k + c^k) \left(\frac{a+b+c}{3} \right)^{2-k} \geq ab + bc + ca$$

holds for $b = c = 1$; that is, to show that $a \in (0, 1]$ involves

$$(a^k + 2) \left(\frac{a+2}{3} \right)^{2-k} \geq 2a + 1,$$

which is equivalent to $f(a) \geq 0$, where

$$f(a) = \ln(a^k + 2) + (2-k) \ln \frac{a+2}{3} - \ln(2a+1).$$

We have

$$f'(a) = \frac{ka^{k-1}}{a^k + 2} + \frac{2-k}{a+2} - \frac{2}{2a+1} = \frac{2g(a)}{a^{1-k}(a^k + 2)(2a+1)},$$

where

$$g(a) = a^2 + (2k-1)a + k + 2(1-k)a^{2-k} - (k+2)a^{1-k},$$

with

$$g'(a) = 2a + 2k - 1 + 2(1-k)(2-k)a^{1-k} - (k+2)(1-k)a^{-k},$$

$$g''(a) = 2 + 2(1-k)^2(2-k)a^{-k} + k(k+2)(1-k)a^{-k-1}.$$

Since $g'' > 0$, g' is strictly increasing. From $g'(0_+) = -\infty$ and $g'(1) = 3(1-k) + 3k^2 > 0$, it follows that there exists $a_1 \in (0, 1)$ so that $g'(a_1) = 0$, $g'(a) < 0$ for $a \in (0, a_1)$ and $g'(a) > 0$ for $a \in (a_1, 1]$. Therefore, g is strictly decreasing on $[0, a_1]$ and strictly increasing on $[a_1, 1]$. Since $g(0) = k > 0$ and $g(1) = 0$, there exists $a_2 \in (0, a_1)$ so that $g(a_2) = 0$, $g(a) > 0$ for $a \in [0, a_2]$

and $g(a) < 0$ for $a \in (a_2, 1]$. Consequently, f is increasing on $[0, a_2]$ and decreasing on $[a_2, 1]$. Since

$$f(0) = \ln 2 + (3 - k) \ln \frac{2}{3} \geq \ln 2 + (3 - r) \ln \frac{2}{3} = 0$$

and $f(1) = 0$, we get $f(a) \geq 0$ for $0 \leq a \leq 1$.

The equality holds for $a = b = c = 1$. If $k = \frac{\ln 9 - \ln 8}{\ln 3 - \ln 2}$, then the equality holds also for

$$a = 0, \quad b = c = \frac{3}{2}$$

(or any cyclic permutation).

□

P 5.77. If $a_1, a_2, \dots, a_n$ ($n \geq 4$) are nonnegative numbers so that $a_1 + a_2 + \dots + a_n = n$, then

$$\frac{1}{n+1-a_2a_3 \cdots a_n} + \frac{1}{n+1-a_3a_4 \cdots a_1} + \dots + \frac{1}{n+1-a_1a_2 \cdots a_{n-1}} \leq 1.$$

(Vasile C., 2004)

Solution. Let $a_1 \leq a_2 \leq \dots \leq a_n$ and

$$e_{n-1} = \left(1 + \frac{1}{n-1}\right)^{n-1}.$$

By the AM-GM inequality, we have

$$a_2a_3 \cdots a_n \leq \left(\frac{a_2 + a_3 + \dots + a_n}{n-1}\right)^{n-1} \leq \left(\frac{a_1 + a_2 + \dots + a_n}{n-1}\right)^{n-1} = e_{n-1},$$

hence

$$n+1-a_2a_3 \cdots a_n \geq n+1-e_{n-1} = (n-2) + (3-e_{n-1}) > 0.$$

Consider the cases $a_1 = 0$ and $a_1 > 0$.

Case 1: $a_1 = 0$. We need to show that $a_2 + a_3 + \dots + a_n = n$ involves

$$\frac{1}{n+1-a_2a_3 \cdots a_n} + \frac{n-1}{n+1} \leq 1,$$

which is equivalent to

$$a_2a_3 \cdots a_n \leq \frac{n+1}{2}.$$

Since

$$a_2a_3 \cdots a_n \leq \left(\frac{a_2 + a_3 + \dots + a_n}{n-1}\right)^{n-1} = e_{n-1},$$

it suffices to show that

$$e_{n-1} \leq \frac{n+1}{2}.$$

For $n = 4$, we have

$$\frac{n+1}{2} - e_{n-1} = \frac{7}{54} > 0.$$

For $n \geq 5$, we get

$$\frac{n+1}{2} \geq 3 > e_{n-1}.$$

Case 2: $0 < a_1 \leq a_2 \leq \cdots \leq a_n$. Denote

$$a_1 a_2 \cdots a_n = (n+1)r, \quad r > 0.$$

From $a_2 a_3 \cdots a_n \leq e_{n-1}$, we get

$$a_1 \geq a, \quad a = \frac{(n+1)r}{e_{n-1}} > r.$$

Write the inequality as follows

$$\frac{a_1}{a_1 - r} + \frac{a_2}{a_2 - r} + \cdots + \frac{a_n}{a_n - r} \leq n+1,$$

$$\frac{1}{a_1 - r} + \frac{1}{a_2 - r} + \cdots + \frac{1}{a_n - r} \leq \frac{1}{r},$$

$$f(a_1) + f(a_2) + \cdots + f(a_n) + \frac{1}{r} \geq 0,$$

where

$$f(u) = \frac{-1}{u - r}, \quad u \geq a.$$

We will apply Corollary 3 to the function f . We have

$$f'(u) = \frac{1}{(u - r)^2},$$

$$g(x) = f'\left(\frac{1}{x}\right) = \frac{x^2}{(1 - rx)^2}, \quad g''(x) = \frac{4rx + 2}{(1 - rx)^4} > 0.$$

According to Corollary 3, if $a \leq a_1 \leq a_2 \leq \cdots \leq a_n$ and

$$a_1 + a_2 + \cdots + a_n = n, \quad a_1 a_2 \cdots a_n = (n+1)r = \text{fixed},$$

then the sum $S_3 = f(a_1) + f(a_2) + \cdots + f(a_n)$ is minimum for $a \leq a_1 \leq a_2 = \cdots = a_n$. Thus, we only need to prove the homogeneous inequality

$$\frac{1}{n+1 - \frac{a_2 a_3 \cdots a_n}{s^{n-1}}} + \frac{1}{n+1 - \frac{a_3 a_4 \cdots a_1}{s^{n-1}}} + \cdots + \frac{1}{n+1 - \frac{a_1 a_2 \cdots a_{n-1}}{s^{n-1}}} \leq 1$$

for $0 < a_1 \leq a_2 = a_3 = \cdots = a_n = 1$, where

$$s = \frac{a_1 + a_2 + \cdots + a_n}{n};$$

that is,

$$\frac{s^{n-1}}{(n+1)s^{n-1}-1} + \frac{(n-1)s^{n-1}}{(n+1)s^{n-1}-a_1} \leq 1, \quad s = \frac{a_1+n-1}{n},$$

which is equivalent to

$$f(s) \geq 0, \quad s_1 < s \leq 1,$$

where

$$s_1 = \frac{n-1}{n}$$

and

$$f(s) = (n+1)s^{2n-2} - n^2s^n + (n+1)(n-2)s^{n-1} + ns - n + 1.$$

We have

$$\begin{aligned} f'(s) &= 2(n^2-1)s^{2n-3} - n^3s^{n-1} + (n^2-1)(n-2)s^{n-2} + n, \\ f''(s) &= (n-1)s^{n-3}g(s), \end{aligned}$$

where

$$\begin{aligned} g(s) &= 2(2n-3)(n+1)s^{n-1} - n^3s + (n-2)^2(n+1), \\ g'(s) &= 2(2n-3)(n^2-1)s^{n-2} - n^3. \end{aligned}$$

Since

$$\begin{aligned} g'(s) &\geq g'(s_1) = \frac{2n(2n-3)(n+1)}{e_{n-1}} - n^3 \\ &> \frac{2n(2n-3)(n+1)}{3} - n^3 = \frac{n(n^2-2n-6)}{3} > 0, \end{aligned}$$

g is increasing. There are two cases to consider: $g(s_1) \geq 0$ and $g(s_1) < 0$.

Subcase A: $g(s_1) \geq 0$. Then, $g(s) \geq 0$, $f''(s) \geq 0$, f' is increasing. Since $f'(1) = 0$, it follows that $f'(s) \leq 0$ for $s \in [s_1, 1]$, f is decreasing, hence $f(s) \geq f(1) = 0$.

Subcase B: $g(s_1) < 0$. Then, since $g(1) = n^2 - 2n + 4 > 0$, there exists $s_2 \in (s_1, 1)$ so that $g(s_2) = 0$, $g(s) < 0$ for $s \in [s_1, s_2)$ and $g(s) > 0$ for $s \in (s_2, 1]$, f' is decreasing on $[s_1, s_2]$ and increasing on $[s_2, 1]$. We see that $f'(1) = 0$. If $f'(s_1) \leq 0$, then $f'(s) \leq 0$ for $s \in [s_1, 1]$, f is decreasing, hence $f(s) \geq f(1) = 0$. If $f'(s_1) > 0$, then there exists $s_3 \in (s_1, s_2)$ so that $f'(s_3) = 0$, $f'(s) > 0$ for $s \in [s_1, s_3)$ and $g(s) < 0$ for $s \in (s_3, 1]$, hence f is increasing on $[s_1, s_3]$ and decreasing on $[s_3, 1]$. Since $f(1) = 0$, it suffices to show that $f(s_1) \geq 0$. This is true since $s = s_1$ involves $a_1 = 0$, and we have shown that the desired inequality holds for $a_1 = 0$.

The equality occurs for $a_1 = a_2 = \dots = a_n = 1$.

□

P 5.78. If a, b, c are nonnegative real numbers so that

$$a + b + c \geq 2, \quad ab + bc + ca \geq 1,$$

then

$$\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} \geq 2.$$

(Vasile C., 2005)

Solution. According to Corollary 5 (case $k = 2$ and $m = 1/3$), if $0 \leq a \leq b \leq c$ and

$$a + b + c = \text{fixed}, \quad ab + bc + ca = \text{fixed},$$

then the sum $S_3 = \sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}$ is minimum for either $a = 0$ or $0 < a \leq b = c$.

Case 1: $a = 0$. The hypothesis $ab + bc + ca \geq 1$ implies $bc \geq 1$; consequently,

$$\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} = \sqrt[3]{b} + \sqrt[3]{c} \geq 2\sqrt[6]{bc} \geq 2.$$

Case 2: $0 < a \leq b = c$. If $c \geq 1$, then

$$\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} \geq 2\sqrt[3]{c} \geq 2.$$

If $c < 1$, then

$$\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} \geq a + b + c \geq 2.$$

The equality holds for

$$a = 0, \quad b = c = 1$$

(or any cyclic permutation).

□

P 5.79. If a, b, c, d are positive real numbers so that $abcd = 1$, then

$$(a + b + c + d)^4 \geq 36\sqrt{3} (a^2 + b^2 + c^2 + d^2).$$

(Vasile C., 2008)

Solution. According to Corollary 5 (case $k = 0$ and $m = 2$), if $a \leq b \leq c \leq d$ and

$$a + b + c + d = \text{fixed}, \quad abcd = 1,$$

then the sum

$$S_4 = a^2 + b^2 + c^2 + d^2$$

is maximum for $a = b = c \leq d$. Thus, we only need to show that

$$(3a + d)^4 \geq 36\sqrt{3} (3a^2 + d^2)$$

for $a^3d = 1$. Write this inequality as $f(a) \geq 0$, where

$$f(a) = 4 \ln \left(3a + \frac{1}{a^3} \right) - \ln \left(3a^2 + \frac{1}{a^6} \right) - \ln 36\sqrt{3}, \quad 0 < a \leq 1.$$

Since

$$f'(a) = \frac{12(a^4 - 1)}{a(3a^4 + 1)} - \frac{6(a^8 - 1)}{a(3a^8 + 1)} = \frac{6(a^4 - 1)^2(3a^4 - 1)}{a(3a^4 + 1)(3a^8 + 1)},$$

f is decreasing on $[0, 1/\sqrt[4]{3}]$ and increasing on $[1/\sqrt[4]{3}, 1]$; therefore,

$$f(a) \geq f\left(\frac{1}{\sqrt[4]{3}}\right) = 0.$$

The equality holds for

$$a = b = c = \frac{1}{\sqrt[4]{3}}, \quad d = \sqrt[4]{27}$$

(or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization:

- If $a_1, a_2, \dots, a_n$ are positive real numbers so that $a_1 a_2 \cdots a_n = 1$, then

$$(a_1 + a_2 + \cdots + a_n)^4 \geq \frac{16}{n} \sqrt[n]{(n-1)^{3n-2}} (a_1^2 + a_2^2 + \cdots + a_n^2),$$

with equality for

$$a_1 = a_2 = \cdots = a_{n-1} = \frac{1}{\sqrt[n]{n-1}}, \quad a_n = \sqrt[n]{(n-1)^{n-1}}$$

(or any cyclic permutation).

□

P 5.80. If a, b, c are nonnegative real numbers so that $ab + bc + ca = 1$, then

$$\sqrt{33a^2 + 16} + \sqrt{33b^2 + 16} + \sqrt{33c^2 + 16} \leq 9(a + b + c).$$

(Vasile C., 2006)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) + 297(a + b + c) \geq 0,$$

where

$$f(u) = -\frac{1}{33} \sqrt{33u^2 + 16}, \quad u \geq 0.$$

We have

$$g(x) = f'(x) = \frac{-x}{\sqrt{33x^2 + 16}},$$

$$g''(x) = \frac{33 \cdot 48x}{(33x^2 + 16)^{5/2}}.$$

Since $g''(x) > 0$ for $x > 0$, g is strictly convex on $[0, \infty)$. According to Corollary 1, if $0 \leq a \leq b \leq c$ and

$$a + b + c = \text{fixed}, \quad a^2 + b^2 + c^2 = \text{fixed},$$

then the sum

$$S_n = f(a) + f(b) + f(c)$$

is minimum for either $a = 0$ or $0 < a \leq b = c$.

Case 1: $a = 0$. We need to show that $bc = 1$ involves

$$\sqrt{33b^2 + 16} + \sqrt{33c^2 + 16} \leq 9(b + c) - 4.$$

We see that

$$9(b+c) - 4 \geq 18\sqrt{bc} - 4 = 14 > 0.$$

By squaring, the inequality becomes

$$\sqrt{528t^2 + 289} \leq 24t^2 - 36t + 25,$$

where

$$t = b + c \geq 2.$$

Since

$$24t^2 - 36t + 25 \geq 6t^2 + 25,$$

it suffices to show that

$$528t^2 + 289 \leq (6t^2 + 25)^2,$$

which is equivalent to

$$(t^2 - 4)(3t^2 - 7) \geq 0.$$

Case 2: $0 < a \leq b = c$. Write the inequality in the homogeneous form

$$\sum \sqrt{33a^2 + 16(ab + bc + ca)} \leq 9(a + b + c).$$

Without loss of generality, assume that $b = c = 1$, when the inequality becomes

$$\sqrt{33a^2 + 32a + 16} + 2\sqrt{32a + 49} \leq 9a + 18.$$

By squaring twice, the inequality becomes as follows:

$$\sqrt{(33a^2 + 32a + 16)(32a + 49)} \leq 12a^2 + 41a + 28,$$

$$72a(2a^3 - a^2 - 4a + 3) \geq 0,$$

$$72a(a-1)^2(2a+3) \geq 0.$$

The equality holds for $a = b = c = \frac{1}{\sqrt{3}}$, and also for

$$a = 0, \quad b = c = 1$$

(or any cyclic permutation).

□

P 5.81. *If a, b, c are positive real numbers so that $a + b + c = 3$, then*

$$a^2b^2 + b^2c^2 + c^2a^2 \leq \frac{3}{\sqrt[3]{abc}}.$$

(Vasile C., 2006)

Solution. Write the inequality in the homogeneous form

$$\left(\frac{a+b+c}{3}\right)^{15} \geq abc \left(\frac{a^2b^2+b^2c^2+c^2a^2}{3}\right)^3.$$

Since

$$\begin{aligned} a^2b^2 + b^2c^2 + c^2a^2 &= (ab + bc + ca)^2 - 2abc(a + b + c) \\ &= \frac{1}{4}(9 - a^2 - b^2 - c^2) - 6abc, \end{aligned}$$

we will apply Corollary 5 (case $k = 0$ and $m = 2$):

- If $0 \leq a \leq b \leq c$ and

$$a + b + c = 3, \quad abc = \text{fixed},$$

then the sum

$$S_3 = a^2 + b^2 + c^2$$

is minimum for $0 < a \leq b = c$.

Therefore, we only need to prove the homogeneous inequality for $0 < a \leq 1$ and $b = c = 1$. Taking logarithms, we have to show that $f(a) \geq 0$, where

$$f(a) = 15 \ln \frac{a+2}{3} - \ln a - 3 \ln \frac{2a^2+1}{3}.$$

Since the derivative

$$f'(a) = \frac{15}{a+2} - \frac{1}{a} - \frac{12a}{2a^2+1} = \frac{2(a-1)(2a-1)(4a-1)}{a(a+2)(2a^2+1)}$$

is negative for $a \in \left(0, \frac{1}{4}\right) \cup \left(\frac{1}{2}, 1\right)$ and positive for $a \in \left(\frac{1}{4}, \frac{1}{2}\right)$, f is decreasing on $\left(0, \frac{1}{4}\right] \cup \left[\frac{1}{2}, 1\right]$ and increasing on $\left[\frac{1}{4}, \frac{1}{2}\right]$. Therefore, it suffices to show that $f\left(\frac{1}{4}\right) \geq 0$ and $f(1) \geq 0$. Indeed, we have $f(1) = 0$ and

$$f\left(\frac{1}{4}\right) = \ln \frac{3^{12}}{2^{19}} > 0.$$

The equality holds for $a = b = c = 1$.

□

P 5.82. If $a_1, a_2, \dots, a_n$ ($n \leq 81$) are nonnegative real numbers so that

$$a_1^2 + a_2^2 + \dots + a_n^2 = a_1^5 + a_2^5 + \dots + a_n^5,$$

then

$$a_1^6 + a_2^6 + \dots + a_n^6 \leq n.$$

(Vasile C., 2006)

Solution. Setting $a_n = 1$, we obtain the statement for $n - 1$ numbers a_i . Consequently, it suffices to prove the inequality for $n = 81$. We need to show that the following homogeneous inequality holds:

$$81(a_1^5 + a_2^5 + \cdots + a_{81}^5)^2 \geq (a_1^6 + a_2^6 + \cdots + a_{81}^6)(a_1^2 + a_2^2 + \cdots + a_{81}^2)^2.$$

According to Corollary 5 (case $k = 3$ and $m = 5/2$), if $0 \leq a_1 \leq a_2 \leq \cdots \leq a_{81}$ and

$$a_1^2 + a_2^2 + \cdots + a_{81}^2 = \text{fixed}, \quad a_1^6 + a_2^6 + \cdots + a_{81}^6 = \text{fixed},$$

then the sum $a_1^5 + a_2^5 + \cdots + a_{81}^5$ is minimum for $a_1 = a_2 = \cdots = a_{80} \leq a_{81}$. Therefore, we only need to prove the homogeneous inequality for $a_1 = a_2 = \cdots = a_{80} = 0$ and for $a_1 = a_2 = \cdots = a_{80} = 1$. The first case is trivial. In the second case, denoting a_{81} by x , the homogeneous inequality becomes as follows:

$$\begin{aligned} 81(80 + x^5)^2 &\geq (80 + x^6)(80 + x^2)^2, \\ x^{10} - 2x^8 - 80x^6 + 162x^5 - x^4 - 160x^2 + 80 &\geq 0, \\ (x - 1)^2(x - 2)^2(x^6 + 6x^5 + 21x^4 + 60x^3 + 75x^2 + 60x + 20) &\geq 0. \end{aligned}$$

Thus, the proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $n = 81$, then the equality holds also for

$$a_1 = a_2 = \cdots = a_{80} = \frac{a_{81}}{2} = \sqrt[3]{\frac{3}{4}}$$

(or any cyclic permutation).

□

P 5.83. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$1 + \sqrt{1 + a^3 + b^3 + c^3} \geq \sqrt{3(a^2 + b^2 + c^2)}.$$

(Vasile C., 2006)

Solution. Write the inequality as

$$\sqrt{1 + a^3 + b^3 + c^3} \geq \sqrt{3(a^2 + b^2 + c^2)} - 1.$$

By squaring, we may rewrite the inequality in the homogeneous form

$$a^3 + b^3 + c^3 + 2\left(\frac{a + b + c}{3}\right)^2 \sqrt{3(a^2 + b^2 + c^2)} \geq (a + b + c)(a^2 + b^2 + c^2).$$

According to Corollary 5 (case $k = 2$ and $m = 3$), if $0 \leq a \leq b \leq c$ and

$$a + b + c = \text{fixed}, \quad a^2 + b^2 + c^2 = \text{fixed},$$

then the sum

$$S_3 = a^3 + b^3 + c^3$$

is minimum for either $a = 0$ or $0 < a \leq b = c$. Thus, we only need to prove the homogeneous inequality for $a = 0$ and for $b = c = 1$.

Case 1: $a = 0$. We need to show that

$$b^3 + c^3 + 2 \left(\frac{b+c}{3} \right)^2 \sqrt{3(b^2 + c^2)} \geq (b+c)(b^2 + c^2).$$

Simplifying by $b+c$, it remains to show that

$$(b+c)\sqrt{b^2 + c^2} \geq \frac{3\sqrt{3}}{2}bc.$$

Indeed,

$$(b+c)\sqrt{b^2 + c^2} \geq (2\sqrt{bc})\sqrt{2bc} \geq \frac{3\sqrt{3}}{2}bc.$$

Case 2: $b = c = 1$. We need to prove that

$$(a+2)^2\sqrt{3(a^2 + 2)} \geq 9(a^2 + a + 1).$$

By squaring, the inequality becomes

$$a^6 + 8a^5 - a^4 - 6a^3 - 17a^2 + 10a + 5 \geq 0,$$

$$(a-1)^2(a^4 + 10a^3 + 18a^2 + 20a + 5) \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 5.84. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$\sqrt{a+b} + \sqrt{b+c} + \sqrt{c+a} \leq \sqrt{16 + \frac{2}{3}(ab + bc + ca)}.$$

(Lorian Saceanu, 2017)

Solution. Write the inequality in the form

$$f(a) + f(b) + f(c) + \sqrt{16 + \frac{2}{3}(ab + bc + ca)} \geq 0,$$

where

$$f(u) = -\sqrt{3-u}, \quad 0 \leq u \leq 3.$$

We have

$$g(x) = f'(x) = \frac{1}{2\sqrt{3-x}},$$

$$g''(x) = \frac{3}{8}(3-x)^{-5/2}.$$

Since $g''(x) > 0$ for $x \in [0, 3]$, g is strictly convex on $[0, 3]$. According to Corollary 1, if $0 \leq a \leq b \leq c$ and

$$a + b + c = 3, \quad ab + bc + ca = \text{fixed},$$

then the sum $S_3 = f(a) + f(b) + f(c)$ is minimum for either $a = 0$ or $0 < a \leq b = c$. Therefore, we only need to prove the homogeneous inequality

$$\sqrt{a+b} + \sqrt{b+c} + \sqrt{c+a} \leq \sqrt{\frac{16}{3}(a+b+c) + \frac{2(ab+bc+ca)}{a+b+c}}$$

for $a = 0$ and $b = c = 1$.

Case 1: $a = 0$. We need to show that

$$\sqrt{b} + \sqrt{c} + \sqrt{b+c} \leq \sqrt{\frac{16}{3}(b+c) + \frac{2bc}{b+c}}.$$

Consider the nontrivial case $b, c > 0$, use the substitution

$$x = \sqrt{\frac{b}{c}} + \sqrt{\frac{c}{b}}, \quad x \geq 2,$$

and write the inequality as

$$\sqrt{b+c+2\sqrt{bc}} + \sqrt{b+c} \leq \sqrt{\frac{16}{3}(b+c) + \frac{2bc}{b+c}},$$

$$\sqrt{x+2} + \sqrt{x} \leq \sqrt{\frac{16}{3}x + \frac{2}{x}}.$$

By squaring twice, the inequality becomes as follows:

$$\sqrt{x(x+2)} \leq \frac{5}{3}x - 1 + \frac{1}{x},$$

$$16x^4 - 48x^3 + 39x^2 - 18x + 9 \geq 0,$$

$$(x-2)[16x^2(x-1) + 7x - 4] + 1 \geq 0.$$

Case 2: $b = c = 1$. We need to prove that

$$2\sqrt{a+1} + \sqrt{2} \leq \sqrt{\frac{16}{3}(a+2) + \frac{2(2a+1)}{a+2}}$$

By squaring twice, the inequality becomes as follows:

$$6(a+2)\sqrt{2(a+1)} \leq 2a^2 + 17a + 17,$$

$$4a^4 - 4a^3 - 3a^2 + 2a + 1 \geq 0,$$

$$(a-1)^2(2a+1)^2 \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 5.85. If $a, b, c \in [0, 4]$ and $ab + bc + ca = 4$, then

$$\sqrt{a+b} + \sqrt{b+c} + \sqrt{c+a} \leq 3 + \sqrt{5}.$$

(Vasile Cîrtoaje, 2019)

First Solution. Denote $s = a + b + c$, consider s fixed and write the inequality as

$$f(a) + f(b) + f(c) \geq -3 - \sqrt{5},$$

where

$$f(x) = -\sqrt{s-x}. \quad 0 \leq x < s.$$

From

$$g(x) = f'(x) = \frac{1}{2}(s-x)^{-1/2}, \quad g''(x) = \frac{3}{8}(s-x)^{-5/2} > 0,$$

it follows that g is strictly convex. Thus, by Corollary 1 and Note 2, the sum $f(a) + f(b) + f(c)$ is minimum for either $a \leq b = c$ or $a = 0$.

Case 1: $a \leq b = c$. We need to show that $2ac + c^2 = 4$ yields

$$2\sqrt{a+c} + \sqrt{2c} \leq 3 + \sqrt{5},$$

that is

$$\sqrt{\frac{2(c^2+1)}{c}} + \sqrt{2c} \leq 3 + \sqrt{5}.$$

From $2ac + c^2 = 4$, it follows that

$$\frac{2}{\sqrt{3}} \leq c \leq 2.$$

Since $\sqrt{2c} \leq 2$, it is enough to show that

$$\sqrt{\frac{2(c^2+1)}{c}} \leq 1 + \sqrt{5},$$

that is

$$c^2 - (3 + \sqrt{5})c + 4 \leq 0.$$

Indeed,

$$c^2 - (3 + \sqrt{5})c + 4 \leq c^2 - 5c + 4 = (c-1)(c-4) < 0.$$

Case 2: $a = 0$. We need to show that $bc = 4$ yields

$$\sqrt{b} + \sqrt{c} + \sqrt{b+c} \leq 3 + \sqrt{5}.$$

From $(4-b)(4-c) \geq 0$, we get $b+c \leq 5$. Thus,

$$\begin{aligned} \sqrt{b} + \sqrt{c} + \sqrt{b+c} &\leq \sqrt{b+c+2\sqrt{bc}} + \sqrt{b+c} \\ &\leq \sqrt{5+2\sqrt{4}} + \sqrt{5} = 3 + \sqrt{5}. \end{aligned}$$

The equality occurs for $a = 0$, $b = 1$ and $c = 4$ (or any permutation).

Second Solution (by Kiyoras-2001) Assume that $a \geq b \geq c$, denote

$$S = ab + bc + ca$$

and show that

$$f(a, b, c) \leq f\left(a, \frac{S}{a}, 0\right) \leq 3 + \sqrt{5},$$

where

$$f(a, b, c) = \sqrt{a+b} + \sqrt{b+c} + \sqrt{c+a}.$$

The left homogeneous inequality is true because

$$\begin{aligned} f\left(a, \frac{S}{a}, 0\right) - f(a, b, c) &= \\ &= \sqrt{a + \frac{S}{a}} - \sqrt{a+b} + \sqrt{\frac{S}{a}} - \sqrt{b+c} + \sqrt{a} - \sqrt{c+a} \\ &= \frac{\frac{c}{a}(a+b)}{\sqrt{\frac{(a+b)(a+c)}{a}} + \sqrt{a+b}} + \frac{\frac{bc}{a}}{\sqrt{\frac{S}{a}} + \sqrt{b+c}} - \frac{c}{\sqrt{a} + \sqrt{c+a}} \\ &\geq \frac{c}{a} \left(\frac{\sqrt{a(a+b)}}{\sqrt{a+c} + \sqrt{a}} - \frac{a}{\sqrt{a} + \sqrt{c+a}} \right) \geq 0. \end{aligned}$$

Also, the right inequality is true for $S = 4$ and $a, b, c \in [0, 4]$ since $a > 1$ and

$$\begin{aligned} f\left(a, \frac{4}{a}, 0\right) - 3 - \sqrt{5} &= \\ &= \sqrt{a + \frac{4}{a}} - \sqrt{5} + \frac{2}{\sqrt{a}} + \sqrt{a} - 3 \\ &= \frac{(a-1)\left(1 - \frac{4}{a}\right)}{\sqrt{a + \frac{4}{a}} + \sqrt{5}} + (\sqrt{a} - 1) \left(1 - \frac{2}{\sqrt{a}}\right) \leq 0. \end{aligned}$$

□

P 5.86. If a, b, c are positive real numbers so that $abc = 1$, then

$$(a) \quad \frac{a+b+c}{3} \geq \sqrt[3]{\frac{2+a^2+b^2+c^2}{5}};$$

$$(b) \quad a^3 + b^3 + c^3 \geq \sqrt{3(a^4 + b^4 + c^4)}.$$

(Vasile C., 2006)

Solution. (a) According to Corollary 5 (case $k = 0$ and $m = 2$), if $a \leq b \leq c$ and

$$a + b + c = \text{fixed}, \quad abc = 1,$$

the sum $S_3 = a^2 + b^2 + c^2$ is maximum for $0 < a = b \leq c$. Thus, we only need to show that $a^2c = 1$ involves

$$\frac{2a + c}{3} \geq \sqrt[3]{\frac{2 + 2a^2 + c^2}{5}},$$

which is equivalent to

$$\begin{aligned} 5 \left(2a + \frac{1}{a^2} \right)^3 &\geq 27 \left(2 + 2a^2 + \frac{1}{a^4} \right), \\ 40a^9 - 54a^8 + 6a^6 + 30a^3 - 27a^2 + 5 &\geq 0, \\ (a - 1)^2(40a^7 + 26a^6 + 12a^5 + 4a^4 - 4a^3 - 12a^2 + 10a + 5) &\geq 0. \end{aligned}$$

The inequality is true since

$$\begin{aligned} 12a^5 + 4a^4 - 4a^3 - 12a^2 + 10a + 5 &> 2a^5 + 4a^4 - 4a^3 - 12a^2 + 10a \\ &= 2a(a - 1)^2(a^2 + 4a + 5) \geq 0. \end{aligned}$$

The equality holds for $a = b = c = 1$.

(b) According to Corollary 5 (case $k = 0$ and $m = 4/3$), if $a \leq b \leq c$ and

$$a^3 + b^3 + c^3 = \text{fixed}, \quad a^3b^3c^3 = 1,$$

the sum $S_3 = a^4 + b^4 + c^4$ is maximum for $0 < a = b \leq c$. Thus, we only need to show that

$$2a^3 + c^3 \geq \sqrt{3(2a^4 + c^4)}$$

for $a^2c = 1$, $a \leq 1$. The inequality is equivalent to

$$\left(2a^3 + \frac{1}{a^6} \right)^2 \geq 3 \left(2a^4 + \frac{1}{a^8} \right).$$

Substituting $a = 1/t$, $t \geq 1$, the inequality becomes

$$\left(\frac{2}{t^3} + t^6 \right)^2 \geq 3 \left(\frac{2}{t^4} + t^8 \right),$$

which is equivalent to $f(t) \geq 0$, where

$$f(t) = t^{18} - 3t^{14} + 4t^9 - 6t^2 + 4.$$

We have

$$\begin{aligned} f'(t) &= 6tg(t), & g(t) &= 3t^{16} - 7t^{12} + 6t^7 - 2, \\ g'(t) &= 6t^6h(t), & h(t) &= 8t^9 - 14t^5 + 7, \\ h'(t) &= 2t^4(36t^2 - 35). \end{aligned}$$

Since $h'(t) > 0$ for $t \geq 1$, h is increasing, $h(t) \geq h(1) = 1$ for $t \geq 1$, g is increasing, $g(t) \geq g(1) = 0$ for $t \geq 1$, f is increasing, hence $f(t) \geq f(1) = 0$ for $t \geq 1$.

The equality holds for $a = b = c = 1$.

□

P 5.87. If a, b, c, d are nonnegative real numbers so that $a + b + c + d = 4$, then

$$(a^2 + b^2 + c^2 + d^2 - 4)(a^2 + b^2 + c^2 + d^2 + 18) \leq 10(a^3 + b^3 + c^3 + d^3 - 4).$$

(Vasile Cîrtoaje, 2010)

Solution. Apply Corollary 2 for $n = 4$, $k = 2$, $m = 3$:

- If a, b, c, d are real numbers so that $0 \leq a \leq b \leq c \leq d$ and

$$a + b + c + d = 4, \quad a^2 + b^2 + c^2 + d^2 = \text{fixed},$$

then

$$S_4 = a^3 + b^3 + c^3 + d^3$$

is minimum for either $0 < a \leq b = c = d$ or $a = 0$.

Case 1: $0 < a \leq b = c = d$. We need to show that $a + 3d = 4$ involves

$$(a^2 + 3d^2 - 4)(a^2 + 3d^2 + 18) \leq 10(a^3 + 3d^3 - 4).$$

This inequality is equivalent to

$$(1 - d)^2(1 + d)(4 - 3d) \geq 0,$$

$$(1 - d)^2(1 + d)a \geq 0.$$

Case 2: $a = 0$. Let

$$s = b^2 + c^2 + d^2.$$

We need to show that $b + c + d = 4$ involves

$$(s - 4)(s + 18) \leq 10(b^3 + c^3 + d^3 - 4).$$

By the Cauchy-Schwarz inequality, we have

$$s \geq \frac{1}{3}(b + c + d)^2 = \frac{16}{3}$$

and

$$(b + c + d)(b^3 + c^3 + d^3) \geq (b^2 + c^2 + d^2)^2, \quad b^3 + c^3 + d^3 \geq \frac{s^2}{4}.$$

Thus, it suffices to show that

$$(s - 4)(s + 18) \leq 10 \left(\frac{s^2}{4} - 4 \right),$$

which is equivalent to the obvious inequality

$$(s - 4)(3s - 16) \geq 0.$$

The equality holds for $a = b = c = d = 1$, and also for

$$a = 0, \quad b = c = d = \frac{4}{3}$$

(or any cyclic permutation).

□

P 5.88. If a, b, c, d are nonnegative real numbers such that

$$a + b + c + d = 4,$$

then

$$(a^4 + b^4 + c^4 + d^4)^2 \geq (a^2 + b^2 + c^2 + d^2)(a^5 + b^5 + c^5 + d^5).$$

(Vasile C., 2020)

Solution. Consider the inequality

$$(a_1^4 + a_2^4 + \cdots + a_n^4)^2 \geq (a_1^2 + a_2^2 + \cdots + a_n^2)(a_1^5 + a_2^5 + \cdots + a_n^5),$$

where $a_1, a_2, \dots, a_n$ are nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$. Write this inequality in the homogeneous form

$$n(a_1^4 + a_2^4 + \cdots + a_n^4)^2 \geq (a_1 + a_2 + \cdots + a_n)(a_1^2 + a_2^2 + \cdots + a_n^2)(a_1^5 + a_2^5 + \cdots + a_n^5).$$

Replacing $a_1, a_2, \dots, a_n$ with $x_1^{1/4}, x_2^{1/4}, \dots, x_n^{1/4}$, the inequality becomes

$$\begin{aligned} n(x_1 + x_2 + \cdots + x_n)^2 &\geq \\ &\geq \left(x_1^{1/4} + x_2^{1/4} + \cdots + x_n^{1/4}\right) \left(x_1^{1/2} + x_2^{1/2} + \cdots + x_n^{1/2}\right) \left(x_1^{5/4} + x_2^{5/4} + \cdots + x_n^{5/4}\right). \end{aligned}$$

By Corollary 5 (case $k = 5/4$), if

$$x_1 + x_2 + \cdots + x_n = \text{fixed}, \quad x_1^{5/4} + x_2^{5/4} + \cdots + x_n^{5/4} = \text{fixed},$$

then the sums $x_1^{1/4} + x_2^{1/4} + \cdots + x_n^{1/4}$ and $x_1^{1/2} + x_2^{1/2} + \cdots + x_n^{1/2}$ are maximum for

$$0 \leq x_1 = x_2 = \cdots = x_{n-1} \leq x_n.$$

Since the case $a_1 = a_2 = \cdots = a_{n-1} = 0$ is trivial, it suffices to consider the case $a_1 = a_2 = \cdots = a_{n-1} = 1$, when the required inequality becomes $f(a) \geq 0$, where

$$f(a) = (a^4 + n - 1)^2 - (a + n - 1)(a^2 + n - 1)(a^5 + n - 1), \quad a \geq 1.$$

We have

$$\begin{aligned} \frac{f(a)}{n-1} &= a^8 - a^7 - a^6 - (n-1)a^5 + 2na^4 - a^3 - (n-1)a^2 - (n-1)a + n-1 \\ &= a^3A - (n-1)B, \end{aligned}$$

where

$$A = a^5 - a^4 - a^3 + 2a - 1, \quad B = a^5 - 2a^4 + a^2 + a - 1.$$

Since

$$A = (a-1)^2(a^3 + a^2 - 1), \quad B = (a-1)^2(a^3 - a - 1),$$

we have

$$f(a) = (n-1)(a-1)^2g(a),$$

where

$$g(a) = a^6 + a^5 - na^3 + (n-1)a + n - 1.$$

The inequality is true if $g(a) \geq 0$. For $n = 4$, we have

$$g(a) = a^6 + a^5 - 4a^3 + 3a + 3 > 2a^5 - 4a^3 + 2a = 2a(a^2 - 1)^2 \geq 0.$$

The equality occurs for $a = b = c = d = 1$.

Remark 1. Since $g(a) \geq 0$ for $n \leq 16$, the homogeneous inequality is true for all $n \leq 16$.

Remark 2. Since

$$\begin{aligned} (a_1 + a_2 + \cdots + a_n)(a_1^5 + a_2^5 + \cdots + a_n^5) &\leq |(a_1 + a_2 + \cdots + a_n)(a_1^5 + a_2^5 + \cdots + a_n^5)| \\ &\leq (|a_1| + |a_2| + \cdots + |a_n|)(|a_1|^5 + |a_2|^5 + \cdots + |a_n|^5), \end{aligned}$$

the homogeneous inequality is true for $n \leq 16$ and real $a_1, a_2, \dots, a_n$. □

P 5.89. If a, b, c, d are nonnegative real numbers such that

$$a + b + c + d = 4,$$

then

$$13(a^2 + b^2 + c^2 + d^2)^2 \geq 12(a^4 + b^4 + c^4 + d^4) + 160.$$

(Vasile Cîrtoaje, 2020)

Solution. Write the inequality in the homogeneous form

$$104(a^2 + b^2 + c^2 + d^2)^2 \geq 96(a^4 + b^4 + c^4 + d^4) + 5(a + b + c + d)^4.$$

According to Corollary 5, for $a + b + c + d = \text{fixed}$ and $a^2 + b^2 + c^2 + d^2 = \text{fixed}$, the sum

$$S = a^4 + b^4 + c^4 + d^4$$

is maximum when $a \geq b = c = d$. Therefore, it suffices to consider this case. Due to homogeneity, for the nontrivial case $b = c = d \neq 0$, we may consider that $b = c = d = 1$. Thus we only need to prove that

$$104(a^2 + 3)^2 \geq 96(a^4 + 3) + 5(a + 3)^4,$$

which is equivalent to

$$(a - 1)^2(a - 9)^2 \geq 0.$$

The equality occurs for $a = b = c = d = 1$, and also for $a = 3$ and $b = c = d = \frac{1}{3}$ (or any cyclic permutation). □

P 5.90. If $a_1, a_2, \dots, a_8$ are nonnegative real numbers, then

$$19(a_1^2 + a_2^2 + \dots + a_8^2)^2 \geq 12(a_1 + a_2 + \dots + a_8)(a_1^3 + a_2^3 + \dots + a_8^3).$$

(Vasile C., 2007)

Solution. By Corollary 5 (case $n = 8, k = 2, m = 3$), if $0 \leq a_1 \leq a_2 \leq \dots \leq a_8$ and

$$a_1 + a_2 + \dots + a_8 = \text{fixed}, \quad a_1^2 + a_2^2 + \dots + a_8^2 = \text{fixed},$$

then the sum

$$S_8 = a_1^3 + a_2^3 + \dots + a_8^3$$

is maximum for $a_1 = a_2 = \dots = a_7 \leq a_8$. Due to homogeneity, we only need to consider the cases $a_1 = a_2 = \dots = a_7 = 0$ and $a_1 = a_2 = \dots = a_7 = 1$. For the second case (nontrivial), we need to show that

$$19(7 + a_8^2)^2 \geq 12(7 + a_8)(7 + a_8^3),$$

which is equivalent to

$$\begin{aligned} a_8^4 - 12a_8^3 + 38a_8^2 - 12a_8 + 49 &\geq 0, \\ (a_8^2 - 6a_8 + 1)^2 + 48 &\geq 0. \end{aligned}$$

The equality holds for $a_1 = a_2 = \dots = a_8 = 0$.

□

P 5.91. If a, b, c are nonnegative real numbers so that

$$5(a^2 + b^2 + c^2) = 17(ab + bc + ca),$$

then

$$3\sqrt{\frac{3}{5}} \leq \sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} \leq \frac{1+\sqrt{7}}{\sqrt{2}}.$$

(Vasile C., 2006)

Solution. Due to homogeneity, we may assume that $a + b + c = 9$. From the hypothesis $5(a^2 + b^2 + c^2) = 17(ab + bc + ca)$, which is equivalent to

$$27(a^2 + b^2 + c^2) = 17(a + b + c)^2,$$

we get

$$a^2 + b^2 + c^2 = 51.$$

Also, from $2(b^2 + c^2) \geq (b + c)^2$ and

$$b + c = 9 - a, \quad b^2 + c^2 = 51 - a^2,$$

we get $a \leq 7$. Write the desired inequality in the form

$$3\sqrt{\frac{3}{5}} \leq f(a) + f(b) + f(c) \leq \frac{1+\sqrt{7}}{\sqrt{2}}.$$

where

$$f(u) = \sqrt{\frac{u}{9-u}}, \quad 0 \leq u \leq 7.$$

We have

$$g(x) = f'(x) = \frac{9}{2x^{1/2}(9-x)^{3/2}},$$

$$g''(x) = \frac{27(8x^2 - 36x + 81)}{8x^{5/2}(9-x)^{7/2}}.$$

Since $g''(x) > 0$ for $x \in (0, 7]$, g is strictly convex on $(0, 7]$. According to Corollary 1, if $0 \leq a \leq b \leq c$ and

$$a + b + c = 9, \quad a^2 + b^2 + c^2 = 51,$$

then the sum $S_3 = f(a) + f(b) + f(c)$ is maximum for $a = b \leq c$, and is minimum for either $a = 0$ or $0 < a \leq b = c$.

(a) To prove the right inequality, it suffices to consider the case $a = b \leq c$. From

$$a + b + c = 9, \quad a^2 + b^2 + c^2 = 51,$$

we get $a = b = 1$ and $c = 7$, therefore

$$\sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} = \frac{1+\sqrt{7}}{\sqrt{2}}.$$

The original right inequality is an equality for $a = b = c/7$ (or any cyclic permutation).

(b) To prove the left inequality, it suffices to consider the cases $a = 0$ and $0 < a \leq b = c$. For $a = 0$, from

$$a + b + c = 9, \quad a^2 + b^2 + c^2 = 51,$$

we get

$$\frac{b}{c} + \frac{c}{b} = \frac{17}{5},$$

therefore

$$\sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} = \sqrt{\frac{b}{c}} + \sqrt{\frac{c}{b}} = \sqrt{\frac{b}{c} + \frac{c}{b} + 2} = 3\sqrt{\frac{3}{5}}.$$

The case $0 < a \leq b = c$ is not possible, because from

$$a + b + c = 9, \quad a^2 + b^2 + c^2 = 51,$$

we get $a = 7$ and $b = c = 1$, which don't satisfy the condition $a \leq b$. The original left inequality is an equality for

$$a = 0, \quad \frac{b}{c} + \frac{c}{b} = \frac{17}{5}$$

(or any cyclic permutation).

□

P 5.92. If a, b, c are nonnegative real numbers so that

$$8(a^2 + b^2 + c^2) = 9(ab + bc + ca),$$

then

$$\frac{19}{12} \leq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \leq \frac{141}{88}.$$

(Vasile C., 2006)

Solution. The proof is similar to the one of the preceding P 5.91. Assume that $a + b + c = 15$, which involves $a^2 + b^2 + c^2 = 81$ and $a \in [3, 7]$, then write the inequality in the form

$$\frac{19}{12} \leq f(a) + f(b) + f(c) \leq \frac{141}{88},$$

where

$$f(u) = \frac{u}{15-u}, \quad 3 \leq u \leq 7.$$

We have

$$g(x) = f'(x) = \frac{1}{5}(15-x)^2, \quad g''(x) = \frac{90}{(15-x)^4}.$$

Since g is strictly convex on $[3, 7]$, according to Corollary 1, if $0 \leq a \leq b \leq c$ and

$$a + b + c = 15, \quad a^2 + b^2 + c^2 = 81,$$

then the sum $S_3 = f(a) + f(b) + f(c)$ is maximum for $a = b \leq c$, and is minimum for either $a = 0$ or $0 < a \leq b = c$.

(a) To prove the right inequality, it suffices to consider the case $a = b \leq c$, which involves

$$a = b = 4, \quad c = 7,$$

and

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = \frac{141}{88}.$$

The original right inequality is an equality for $a = b = 4c/7$ (or any cyclic permutation).

(b) To prove the left inequality, it suffices to consider the cases $a = 0$ and $0 < a \leq b = c$. The first case is not possible, while the second case involves

$$a = 3, \quad b = c = 6,$$

and

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = \frac{19}{12}.$$

The original left inequality is an equality for $2a = b = c$ (or any cyclic permutation).

□

P 5.93. If $a, b, c \in (0, 2]$ such that $a + b + c = 3$, then

$$\sqrt{\frac{2(b+c)}{a}} - 1 + \sqrt{\frac{2(c+a)}{b}} - 1 + \sqrt{\frac{2(a+b)}{c}} - 1 \geq \frac{9}{\sqrt{ab+bc+ca}}.$$

(Vasile C., 2020)

Solution. Write the inequality in the form

$$f(a) + f(b) + f(c) \leq \frac{-3\sqrt{3}}{\sqrt{ab+bc+ca}},$$

where

$$f(u) = -\sqrt{\frac{2}{u}} - 1, \quad 0 < u \leq 2.$$

We have $f(0+) = -\infty$ and

$$\begin{aligned} g(x) = f'(x) &= x^{-3/2}(2-x)^{-1/2}, & g'(x) &= (2x-3)x^{-5/2}(2-x)^{-3/2}, \\ g''(x) &= (7x^2 - 20x + 15)x^{-7/2}(2-x)^{-5/2} > 0. \end{aligned}$$

Since g is strictly convex on $(0, 2)$, according to Corollary 1, Note 1 and Note 2, if $a \geq b \geq c > 0$ and

$$a + b + c = 3, \quad ab + bc + ca = \text{fixed},$$

then the sum $S_3 = f(a) + f(b) + f(c)$ is maximum for $a = 2$ or $a \geq b = c$. Thus, it suffices to prove the desired inequality for these cases.

Case 1: $a = 2$. We need to prove the homogeneous inequality

$$\sqrt{\frac{2(b+c)}{a}} - 1 + \sqrt{\frac{2(c+a)}{b}} - 1 + \sqrt{\frac{2(a+b)}{c}} - 1 \geq \frac{3(a+b+c)}{\sqrt{ab+bc+ca}}$$

for

$$a = 2(b+c).$$

The inequality is equivalent to

$$\sqrt{\frac{2b}{c} + 1} + \sqrt{\frac{2c}{b} + 1} \geq \frac{3\sqrt{3}(b+c)}{\sqrt{2(b+c)^2 + bc}}.$$

Let

$$x = \frac{(b+c)^2}{4bc}, \quad x \geq 1.$$

Since

$$\sqrt{\frac{2c}{b} + 1} + \sqrt{\frac{2b}{c} + 1} \geq 2\sqrt[4]{\left(\frac{2b}{c} + 1\right)\left(\frac{2c}{b} + 1\right)} = 2\sqrt[4]{8x+1},$$

the inequality becomes

$$\sqrt[4]{8x+1} \geq \frac{3\sqrt{3x}}{\sqrt{8x+1}},$$

$$(8x + 1)^3 \geq 729x^2.$$

Since

$$8x + 1 \geq 3(2x + 1),$$

it suffices to show that

$$(2x + 1)^3 \geq 27x^2.$$

This is true because

$$2x + 1 = x + x + 1 \geq 3\sqrt[3]{x^2}.$$

Case 2: $a \geq b = c$. We need to show that $a + 2c = 3$ implies

$$\sqrt{\frac{4c}{a} - 1} + 2\sqrt{\frac{2(a+c)}{c} - 1} \geq \frac{9}{\sqrt{2ac + c^2}},$$

that is

$$\begin{aligned} \sqrt{\frac{2-a}{a}} + 2\sqrt{\frac{1+a}{3-a}} &\geq \frac{6}{\sqrt{(1+a)(3-a)}}, \\ \sqrt{\frac{2-a}{a}} &\geq \frac{2(2-a)}{\sqrt{(1+a)(3-a)}}. \end{aligned}$$

It is true if

$$\frac{1}{\sqrt{a}} \geq \frac{2\sqrt{2-a}}{\sqrt{(1+a)(3-a)}},$$

which, by squaring, reduces to

$$(a-1)^2 \geq 0.$$

The equality occurs for $a = b = c = 1$, and also for $a = b = \frac{1}{2}$ and $c = 2$ (or any cyclic permutation).

□

P 5.94. Let a, b, c and x, y, z be nonnegative real numbers such that

$$x^3 + y^3 + z^3 = a^3 + b^3 + c^3.$$

Then,

$$\frac{(a+b+c)(x+y+z)}{ab+bc+ca+xy+yz+zx} \geq \sqrt[3]{3}.$$

(Vasile Cîrtoaje, 2019)

Solution. Assume that

$$x + y + z \geq a + b + c$$

and denote

$$t = \frac{x+y+z}{3}, \quad t \geq \frac{a+b+c}{3}.$$

Since

$$\frac{a+b+c}{3} \leq \frac{x+y+z}{3} \leq \sqrt[3]{\frac{x^3+y^3+z^3}{3}} = \sqrt[3]{\frac{a^3+b^3+c^3}{3}},$$

we have

$$t_1 \leq t \leq t_2,$$

where

$$t_1 = \frac{a+b+c}{3}, \quad t_2 = \sqrt[3]{\frac{a^3+b^3+c^3}{3}}.$$

It is enough to prove the inequality

$$\frac{1}{\sqrt[3]{3}} (a+b+c)(x+y+z) \geq ab+bc+ca + \frac{1}{3}(x+y+z)^2.$$

For fixed a, b, c , we may write the required inequality as $f(t) \leq 0$, where

$$f(t) = 3t^2 - \sqrt[3]{9} (a+b+c)t + ab+bc+ca$$

is a quadratic convex function. Thus, it is enough to show that $f(t_1) \leq 0$ and $f(t_2) \leq 0$. We have

$$\begin{aligned} 3f(t_1) &= 3(ab+bc+ca) - (\sqrt[3]{9} - 1)(a+b+c)^2 \\ &\leq 3(2 - \sqrt[3]{9})(ab+bc+ca) \leq 0. \end{aligned}$$

To prove the inequality $f(t_2) \leq 0$, we write it as

$$3t_2^2 - \sqrt[3]{9} (a+b+c)t_2 + ab+bc+ca \leq 0.$$

According to Corollary 5, for $a+b+c = \text{fixed}$ and $a^n+b^n+c^n = \text{fixed}$, the sum $a^2+b^2+c^2$ is minimum (hence the sum $ab+bc+ca$ is maximum) for $a \geq b = c$. Thus, due to homogeneity, it is enough to prove the inequality for $a = 1$ and $b = c \leq 1$. So, we need to prove that $g(u) \leq 0$, where

$$g(u) = u^2 - (2c+1)u + \frac{c^2+2c}{\sqrt[3]{3}},$$

with

$$u = \sqrt[3]{2c^3+1}, \quad c \in [0, 1].$$

Consider two cases: $c \in [0, 4/5]$ and $c \in [4/5, 1]$.

Case 1: $c \in [0, 4/5]$. Since $\sqrt[3]{3} > 4/3$, we have

$$g(u) \leq u^2 - (2c+1)u + \frac{3(c^2+2c)}{4} = \frac{(2u-3c)(2u-c-2)}{4}.$$

Thus, we need to show that

$$\frac{3c}{2} \leq u \leq \frac{c+2}{2}.$$

The left inequality is equivalent to

$$c \leq \sqrt{\frac{8}{11}}.$$

This is true since

$$c \leq \frac{4}{5} < \sqrt{\frac{8}{11}}.$$

The right inequality is equivalent to

$$c(2c + 6 - 5c^2) \geq 0.$$

Case 2: $c \in [4/5, 1]$. Since $\sqrt[3]{3} > 7/5$, we have $g(u) < h(u)$, where

$$h(u) = u^2 - (2c + 1)u + \frac{5(c^2 + 2c)}{7}.$$

It suffices to prove that $h(u) \leq 0$. From

$$h'(u) = 2u - 2c - 1$$

and

$$(2u)^3 - (2c + 1)^3 = 7 + 8c^3 - 12c^2 - 6c \leq 7 - 4c^2 - 6c \leq 7 - \frac{64}{25} - \frac{24}{5} = \frac{-9}{25} < 0,$$

it follows that $h'(u) < 0$, hence $h(u)$ is a decreasing function. Since

$$u > 1 + \frac{c^3}{3},$$

it follows that

$$h(u) < h\left(1 + \frac{c^3}{3}\right) = c\left(\frac{5c}{7} + \frac{c^2}{3} + \frac{c^5}{9} - \frac{4}{7} - \frac{2c^3}{3}\right).$$

Since

$$\frac{5c}{7} + \frac{c^2}{3} + \frac{c^5}{9} \leq \frac{5c}{7} + \frac{c}{3} + \frac{c^3}{9} = \frac{22c}{21} + \frac{c^3}{9},$$

it suffices to show that

$$\frac{22c}{21} + \frac{c^3}{9} - \frac{4}{7} - \frac{2c^3}{3} \leq 0,$$

that is

$$\frac{22c}{21} - \frac{4}{7} - \frac{5c^3}{9} \leq 0.$$

Indeed, we have

$$\frac{4}{7} + \frac{5c^3}{9} = \frac{2}{7} + \frac{2}{7} + \frac{5c^3}{9} \geq 3\sqrt[3]{\frac{20c^3}{49 \cdot 9}} > \frac{22c}{21}.$$

Thus, the proof is completed. If $a \geq b \geq c$ and $x \geq y \geq z$, then the equality occurs for $a = b = c = \frac{x}{\sqrt[n]{3}}$ and $y = z = 0$, and for $x = y = z = \frac{a}{\sqrt[n]{3}}$ and $b = c = 0$.

□

P 5.95. If a, b, c, d are positive numbers such that

$$a + b + c + d = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d},$$

then

$$ab + ac + ad + bc + bd + cd + 3abcd \geq 9.$$

(Vasile Cîrtoaje, 2019)

Solution. Write the inequality as

$$(a + b + c + d)^2 + 6abcd \geq 18 + a^2 + b^2 + c^2 + d^2$$

and apply Corollary 4 for $k = -1$, and Corollary 5 for $k = -1$ and $m = 2$:

- If a, b, c, d are positive numbers such that

$$a + b + c + d = \text{fixed}, \quad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = \text{fixed}, \quad a \leq b \leq c \leq d,$$

then the product $abcd$ is minimum and the sum $a^2 + b^2 + c^2 + d^2$ is maximum for $a = b = c = d$. Thus, it suffices to consider this case. We need to show that

$$3a + d = \frac{3}{a} + \frac{1}{d}$$

involve

$$a^2 + ad + a^3d \geq 3.$$

From the hypothesis, we get

$$d = \frac{3(1 - a^2) + \sqrt{9a^4 - 14a^2 + 9}}{2a}.$$

So, the required inequality becomes as follows:

$$\begin{aligned} a^2 + (a^2 + 1)ad &\geq 3, \\ (a^2 + 1)\sqrt{9a^4 - 14a^2 + 9} &\geq 3a^4 - 2a^2 + 3, \\ (a^2 + 1)^2(9a^4 - 14a^2 + 9) &\geq (3a^4 - 2a^2 + 3)^2, \\ 16a^2(a^2 - 1)^2 &\geq 0. \end{aligned}$$

The equality occurs for $a = b = c = d = 1$.

□

P 5.96. If a_1, a_2, a_3, a_4, a_5 are nonnegative real numbers, then

$$\frac{(a_1^3 + a_2^3 + a_3^3 + a_4^3 + a_5^3)^2}{a_1^4 + a_2^4 + a_3^4 + a_4^4 + a_5^4} \geq \frac{1}{2} \sum_{i < j} a_i a_j.$$

(Vasile Cîrtoaje, 2019)

Solution. Write the inequality in the form

$$\frac{4(a_1^3 + a_2^3 + a_3^3 + a_4^3 + a_5^3)^2}{a_1^4 + a_2^4 + a_3^4 + a_4^4 + a_5^4} + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 \geq (a_1 + a_2 + a_3 + a_4 + a_5)^2.$$

According to Corollary 5, for $a_1 + a_2 + a_3 + a_4 + a_5 = \text{fixed}$ and $a_1^3 + a_2^3 + a_3^3 + a_4^3 + a_5^3 = \text{fixed}$, the sum $a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2$ is minimum and the sum $a_1^4 + a_2^4 + a_3^4 + a_4^4 + a_5^4$ is maximum for $a_1 = a_2 = a_3 = a_4 \leq a_5$. Thus, it is enough to show that

$$\frac{4(4x^3 + y^3)^2}{4x^4 + y^4} + 4x^2 + y^2 \geq (4x + y)^2,$$

which can be written as

$$4x^6 - 8x^5y + 8x^3y^3 - 3x^2y^4 - 2xy^5 + y^6 \geq 0,$$

$$(x - y)^2(2x^2 - y^2)^2 \geq 0.$$

The proof is completed. The equality occurs for $a_1 = a_2 = a_3 = a_4 = a_5$, and also for $a_1 = a_2 = a_3 = a_4 = \frac{a_5}{\sqrt{2}}$ (or any cyclic permutation). □

P 5.97. If $a_1, a_2, \dots, a_n \geq 0$ such that

$$a_1 + a_2 + \dots + a_n = n,$$

then

$$\sqrt{a_1} + \sqrt{a_2} + \dots + \sqrt{a_n} \leq \sqrt{2n - 1 + 2 \left(1 - \frac{1}{n}\right) \sum_{i < j} a_i a_j}.$$

(Vasile C., 2018)

Solution. Since

$$2 \sum_{i < j} a_i a_j = (a_1 + a_2 + \dots + a_n)^2 - a_1^2 - a_2^2 - \dots - a_n^2 = n^2 - a_1^2 - a_2^2 - \dots - a_n^2,$$

we can write the inequality as

$$(\sqrt{a_1} + \sqrt{a_2} + \dots + \sqrt{a_n})^2 \leq n^2 + n - 1 - \left(1 - \frac{1}{n}\right) (a_1^2 + a_2^2 + \dots + a_n^2).$$

Now, we can apply Corollary 5 for $k = 2$ and $m = 1/2$:

- If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that

$$a_1 + a_2 + \dots + a_n = n, \quad a_1^2 + a_2^2 + \dots + a_n^2 = \text{fixed}, \quad a_1 \leq a_2 \leq \dots \leq a_n,$$

then the sum

$$\sqrt{a_1} + \sqrt{a_2} + \dots + \sqrt{a_n}$$

is maximum for $0 \leq a_1 = \cdots = a_{n-1} \leq a_n$.

Thus, it suffices to show that

$$[(n-1)x + y]^2 \leq n^2 + n - 1 - \left(1 - \frac{1}{n}\right) [(n-1)x^4 + y^4].$$

for

$$(n-1)x^2 + y^2 = n, \quad 0 \leq x \leq y$$

Write this inequality in the homogeneous form

$$[(n-1)x + y]^2 \leq \frac{(n^2 + n - 1) \frac{[(n-1)x^2 + y^2]^2}{n} - (n-1)[(n-1)x^4 + y^4]}{(n-1)x^2 + y^2},$$

which is equivalent to

$$(n-1)^2 x^4 - 2n(n-1)x^3 y + (n^2 + 2n - 2)x^2 y^2 - 2nxy^3 + y^4 \geq 0,$$

$$(x-y)^2 [(n-1)x - y]^2 \geq 0.$$

The inequality is an equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = \cdots = a_{n-1} = \frac{1}{n-1}$ and $a_n = n-1$ (or any cyclic permutation).

□

P 5.98. If $a_1, a_2, \dots, a_n \geq 0$ such that

$$a_1 + a_2 + \cdots + a_n = \sum_{i < j} a_i a_j > 0,$$

then

$$\frac{(n-1)(n-2)}{2} (a_1 + a_2 + \cdots + a_n) + \sum_{i < j} \sqrt{a_i a_j} \geq n(n-1).$$

(Vasile C., 2020)

Solution. For $n = 2$, we need to show that $a_1 + a_2 = a_1 a_2$ involves $a_1 a_2 \geq 4$. Indeed, this follows from

$$a_1 a_2 = a_1 + a_2 \geq 2\sqrt{a_1 a_2},$$

Since

$$2 \sum_{i < j} a_i a_j = (a_1 + a_2 + \cdots + a_n)^2 - a_1^2 - a_2^2 - \cdots - a_n^2$$

and

$$2 \sum_{i < j} \sqrt{a_i a_j} = (\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n})^2 - a_1 - a_2 - \cdots - a_n,$$

we can apply Corollary 5 for $k = 2$ and $m = 1/2$:

- If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that

$$a_1 + a_2 + \cdots + a_n = \text{fixed}, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = \text{fixed}, \quad a_1 \leq a_2 \leq \cdots \leq a_n,$$

then the sum

$$\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n}$$

is minimum for either $0 < a_1 \leq a_2 = \cdots = a_n$ or $a_1 = 0$.

Thus, it suffices to consider the case $a_1 = x^2$, $a_2 = \cdots = a_n = y^2$, $0 < x \leq y$, and the case $a_1 = 0$. In addition, we will use the induction method.

Case 1: $a_1 = x^2$, $a_2 = \cdots = a_n = y^2$. We need to show that

$$x^2 + (n-1)y^2 = (n-1)x^2y^2 + \frac{(n-1)(n-2)}{2}y^4$$

implies

$$\frac{(n-2)}{2}[x^2 + (n-1)y^2] + xy + \frac{(n-2)}{2}y^2 \geq n,$$

which can be written in the homogeneous form

$$(n-2)x^2 + 2xy + n(n-2)y^2 \geq n \frac{2(n-1)x^2y^2 + (n-1)(n-2)y^4}{x^2 + (n-1)y^2}.$$

For $y = 1$, the inequality becomes

$$(x^2 + n-1)[(n-2)x^2 + 2x + n(n-2)] \geq 2n(n-1)x^2 + n(n-1)(n-2),$$

$$(n-2)x^4 + 2x^3 - (3n-2)x^2 + 2(n-1)x \geq 0,$$

$$x(x-1)^2[(n-2)x + 2(n-1)] \geq 0.$$

Case 2: $a_1 = 0$. We need to show that

$$a_2 + a_3 + \cdots + a_n = \sum_{2 \leq i < j} a_i a_j > 0 \quad (1)$$

involves

$$\frac{(n-1)(n-2)}{2}(a_2 + a_3 + \cdots + a_n) + \sum_{2 \leq i < j} \sqrt{a_i a_j} \geq n(n-1). \quad (2)$$

From

$$\begin{aligned} (a_2 + a_3 + \cdots + a_n)^2 &\leq (n-1)(a_2^2 + a_3^2 + \cdots + a_n^2) \\ &= (n-1)(a_2 + a_3 + \cdots + a_n)^2 - 2(n-1) \sum_{2 \leq i < j} a_i a_j, \end{aligned}$$

we get

$$(n-2)(a_2 + a_3 + \cdots + a_n)^2 \geq 2(n-1) \sum_{2 \leq i < j} a_i a_j = 2(n-1)(a_2 + a_3 + \cdots + a_n),$$

hence

$$a_2 + a_3 + \cdots + a_n \geq \frac{2(n-1)}{n-2}. \quad (3)$$

On the other hand, by the induction hypothesis, (1) involves

$$\frac{(n-2)(n-3)}{2}(a_2 + a_3 + \cdots + a_n) + \sum_{2 \leq i < j} \sqrt{a_i a_j} \geq (n-1)(n-2).$$

According to this inequality, (2) is true if

$$\begin{aligned} \frac{(n-1)(n-2)}{2}(a_2 + a_3 + \cdots + a_n) + (n-1)(n-2) - \frac{(n-2)(n-3)}{2}(a_2 + a_3 + \cdots + a_n) \\ \geq n(n-1), \end{aligned}$$

which is equivalent to (3).

The inequality is an equality for $a_1 = a_2 = \cdots = a_n = \frac{2}{n-1}$, and also for $a_1 = 0$ and $a_2 = a_3 = \cdots = a_n = \frac{2}{n-2}$ (or any cyclic permutation). □

P 5.99. *Let*

$$F(a_1, a_2, \dots, a_n) = n(a_1^2 + a_2^2 + \cdots + a_n^2) - (a_1 + a_2 + \cdots + a_n)^2,$$

where $a_1, a_2, \dots, a_n$ are positive real numbers such that $a_1 = \min\{a_1, a_2, \dots, a_n\}$ and

$$a_1^2(a_2^2 + a_3^2 + \cdots + a_n^2) \geq n-1.$$

Then,

$$F(a_1, a_2, \dots, a_n) \geq F\left(\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_n}\right).$$

(Vasile C., 2020)

Solution. Assume that $a_1 \leq a_2 \leq \cdots \leq a_n$. For $n = 2$, we need to show that $a_1 a_2 \geq 1$ involves

$$(a_1^2 a_2^2 - 1)(a_1 - a_2)^2 \geq 0,$$

which is clearly true. For $n \geq 3$, since $F(a_1, a_2, \dots, a_n) \geq 0$ and $F\left(\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_n}\right) \geq 0$, it suffices to prove the homogeneous inequality

$$F(a_1, a_2, \dots, a_n) \geq \frac{a_1^2(a_2^2 + \cdots + a_n^2)}{n-1} \cdot F\left(\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_n}\right)$$

for $a_1 \leq a_2 \leq \cdots \leq a_n$ and without the inequality constraint $a_1^2(a_2^2 + a_3^2 + \cdots + a_n^2) \geq n-1$. Due to homogeneity, we may set $a_1 = 1$, hence $1 \leq a_2 \leq \cdots \leq a_n$. Thus, we need to show the inequality

$$F(1, a_2, \dots, a_n) \geq \frac{a_2^2 + \cdots + a_n^2}{n-1} \cdot F\left(1, \frac{1}{a_2}, \dots, \frac{1}{a_n}\right),$$

which is equivalent to

$$n(1 + S_2) - (1 + a_2 + \cdots + a_n)^2 - \frac{S_2}{n-1} \cdot \left[n(1 + \Sigma_2) - \left(1 + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)^2 \right] \geq 0,$$

where $S_2 = a_2^2 + \cdots + a_n^2$ and $\Sigma_2 = \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2}$. We will show first that the left hand side of the inequality increases when S_2 increases. This is true if

$$n - \frac{1}{n-1} \left[n(1 + \Sigma_2) - \left(1 + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right)^2 \right] \geq 0.$$

By Lemma below, it suffices to show that

$$n - \frac{1}{n-1} \cdot \frac{n^2}{4} \geq 0,$$

which is clearly true. So, the left hand side of the inequality increases when S_2 increases and when Σ_2 decreases. According to Corollary 5 (case $k = -1$), we have:

- If $a_2, a_3, \dots, a_n$ are positive real numbers so that

$$a_2 + a_3 + \cdots + a_n = \text{fixed}, \quad \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_n} = \text{fixed}, \quad a_2 \leq a_3 \leq \cdots \leq a_n,$$

then the sum S_2 is minimum and the sum Σ_2 is maximum for $a_2 \leq a_3 = \cdots = a_n$.

Thus, it suffices to consider the case $1 \leq a_2 \leq a_3 = \cdots = a_n := x$. We need to show that

$$n[1 + a_2^2 + (n-2)x^2] - [1 + a_2 + (n-2)x]^2 \geq A \left[n \left(1 + \frac{1}{a_2^2} + \frac{n-2}{x^2} \right) - \left(1 + \frac{1}{a_2} + \frac{n-2}{x} \right)^2 \right],$$

where $A = \frac{a_2^2 + (n-2)x^2}{n-1}$. Write this inequality as follows

$$(a_2-1)^2 + (n-2)(x-1)^2 + (n-2)(a_2-x)^2 \geq A \left[\frac{(a_2-1)^2}{a_2^2} + \frac{(n-2)(x-1)^2}{x^2} + \frac{(n-2)(a_2-x)^2}{a_2^2 x^2} \right],$$

$$(a_2-1)^2 \left(1 - \frac{A}{a_2^2} \right) + (n-2)(x-1)^2 \left(1 - \frac{A}{x^2} \right) + (n-2)(a_2-x)^2 \left(1 - \frac{A}{a_2^2 x^2} \right) \geq 0.$$

Since $a_2^2 x^2 \geq A$, it suffices to show that

$$(n-2)(x-1)^2 \left(1 - \frac{A}{x^2} \right) \geq (a_2-1)^2 \left(\frac{A}{a_2^2} - 1 \right),$$

i.e.

$$\left(1 - \frac{1}{x^2} \right)^2 (x^2 - a_2^2) \geq \left(1 - \frac{1}{a_2^2} \right)^2 (x^2 - a_2^2),$$

which is true.

The equality occurs for $a_1 = a_2 = \cdots = a_n \geq 1$ and for $\frac{1}{a_1} = a_2 = \cdots = a_n \geq 1$.

Lemma. If $x_1, x_2, \dots, x_n \in [0, 1]$, then

$$n(1 + x_2^2 + \cdots + x_n^2) - (1 + x_2 + \cdots + x_n)^2 \leq \frac{n^2}{4}.$$

Proof. Since the function $f(x) = x^2$ is convex on $[0, 1]$, the left hand side of the inequality is maximum when $x_1, x_2, \dots, x_n \in \{0, 1\}$ (see P 2.104 from Volume 1). Assuming that m numbers x_i are 1 and the others are 0, it suffices to show that

$$n(1+m) - (1+m)^2 \leq \frac{n^2}{4}.$$

Indeed,

$$\frac{n^2}{4} - n(1+m) + (1+m)^2 = \left(1+m - \frac{n}{2}\right)^2 \geq 0.$$

Remark. Since $a_1(a_2 + a_3 + \dots + a_n) \geq n-1$ yields $a_1^2(a_2^2 + a_3^2 + \dots + a_n^2) \geq n-1$, the inequality is also true for

$$a_1(a_2 + a_3 + \dots + a_n) \geq n-1.$$

In addition, it is true in the particular case

$$a_1, a_2, \dots, a_n \geq 1.$$

□

P 5.100. Let

$$F(a_1, a_2, \dots, a_n) = a_1 + a_2 + \dots + a_n - n\sqrt[n]{a_1 a_2 \dots a_n},$$

where $a_1, a_2, \dots, a_n$ are positive real numbers such that $a_1 = \min\{a_1, a_2, \dots, a_n\}$ and

$$a_1(a_2 + a_3 + \dots + a_n) \geq n-1.$$

Then,

$$F(a_1, a_2, \dots, a_n) \geq F\left(\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_n}\right).$$

(Vasile C., 2020)

Solution. Assume that $a_1 \leq a_2 \leq \dots \leq a_n$. For $n = 2$, we need to show that $a_1 a_2 \geq 1$ involves

$$(a_1 a_2 - 1)(\sqrt{a_1} - \sqrt{a_2})^2 \geq 0,$$

which is true. For $n \geq 3$, the inequality has the form

$$a_1 + a_2 + \dots + a_n - n\sqrt[n]{a_1 a_2 \dots a_n} \geq \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} - \frac{n}{\sqrt[n]{a_1 a_2 \dots a_n}}.$$

According to Corollary 5 (case $k = 0$ and $m = -1$), we have:

- If $a_2, a_3, \dots, a_n$ are positive real numbers so that

$$a_2 + a_3 + \dots + a_n = \text{fixed}, \quad a_2 a_3 \dots a_n = \text{fixed}, \quad a_2 \leq a_3 \leq \dots \leq a_n,$$

then the sum $\frac{1}{a_2} + \frac{1}{a_3} + \dots + \frac{1}{a_n}$ is maximum for $a_2 \leq a_3 = \dots = a_n$.

Thus, we only need to show that

$$x + y + (n-2)z - n\sqrt[n]{xyz^{n-2}} \geq \frac{1}{x} + \frac{1}{y} + \frac{n-2}{z} - \frac{n}{\sqrt[n]{xyz^{n-2}}}$$

for $0 < x \leq y \leq z$ and $x[y + (n-2)z] \geq n-1$. Since both sides of the inequality are nonnegative, it suffices to prove the homogeneous inequality

$$\left[x + y + (n-2)z - n\sqrt[n]{xyz^{n-2}} \right] \geq \frac{x[y + (n-2)z]}{n-1} \left[\frac{1}{x} + \frac{1}{y} + \frac{n-2}{z} - \frac{n}{\sqrt[n]{xyz^{n-2}}} \right],$$

that is

$$\begin{aligned} & (n-1) \left[x + y + (n-2)z - n\sqrt[n]{xyz^{n-2}} \right] \geq \\ & \geq y + (n-2)z + \frac{[y + (n-2)z][(n-2)y + z]}{yz}x - n[y + (n-2)z] \sqrt[n]{\frac{x^{n-1}}{yz^{n-2}}}. \end{aligned}$$

For fixed y and z , write this inequality as $f(x) \geq 0$, $x \in (0, y]$. We will show that

$$f(x) \geq f(y) \geq 0.$$

To prove that $f(x) \geq f(y)$, we show that $f'(x) \leq 0$, which is equivalent to

$$\begin{aligned} & n-1 - (n-1)\sqrt[n]{\frac{yz^{n-2}}{x^{n-1}}} - \frac{[y + (n-2)z][(n-2)y + z]}{yz} + (n-1)\frac{y + (n-2)z}{\sqrt[n]{xyz^{n-2}}} \leq 0, \\ & (n-2) \left(\frac{y}{z} + \frac{z}{y} + n-3 \right) + (n-1)\sqrt[n]{\frac{yz^{n-2}}{x^{n-1}}} \geq (n-1)\frac{y + (n-2)z}{\sqrt[n]{xyz^{n-2}}}. \end{aligned}$$

By the AM-GM inequality, we have

$$\begin{aligned} & (n-2) \cdot \left(\frac{y}{z} + \frac{z}{y} + n-3 \right) + (n-1)\sqrt[n]{\frac{yz^{n-2}}{x^{n-1}}} \geq \\ & \geq (n-1)^{\frac{n-1}{n}} \sqrt[n-1]{\left(\frac{y}{z} + \frac{z}{y} + n-3 \right)^{n-2}} \cdot (n-1)\sqrt[n]{\frac{yz^{n-2}}{x^{n-1}}}. \end{aligned}$$

Thus, it suffices to show that

$$\sqrt[n-1]{\left(\frac{y}{z} + \frac{z}{y} + n-3 \right)^{n-2}} \cdot (n-1)\sqrt[n]{\frac{yz^{n-2}}{x^{n-1}}} \geq \frac{y + (n-2)z}{\sqrt[n]{xyz^{n-2}}},$$

which is equivalent to

$$(n-1) \left(\frac{y}{z} + \frac{z}{y} + n-3 \right)^{n-2} yz^{n-2} \geq [y + (n-2)z]^{n-1}.$$

Due to homogeneity, we may set $z = 1$, when the inequality becomes

$$(n-1)Ay \geq y + n-2,$$

where

$$A = \left(\frac{y + 1/y + n - 3}{y + n - 2} \right)^{n-2}, \quad 0 < y \leq 1.$$

By Bernoulli's inequality, we have

$$A = \left(1 + \frac{1/y - 1}{y + n - 2} \right)^{n-2} \geq 1 + \frac{(n-2)(1/y - 1)}{y + n - 2} = \frac{y^2 + n - 2}{y(y + n - 2)},$$

hence

$$\begin{aligned} (n-1)Ay - (y + n - 2) &\geq \frac{(n-1)(y^2 + n - 2)}{y + n - 2} - (y + n - 2) \\ &= \frac{(n-2)(y-1)^2}{y + n - 2} \geq 0. \end{aligned}$$

The inequality $f(y) \geq 0$ has the form

$$2y + (n-2)z - n\sqrt[n]{y^2 z^{n-2}} \geq \frac{y[y + (n-2)z]}{n-1} \left[\frac{2}{y} + \frac{n-2}{z} - \frac{n}{\sqrt[n]{y^2 z^{n-2}}} \right].$$

Due to homogeneity, we may set $z = 1$ (hence $0 < y \leq 1$), when the inequality becomes

$$2y + n - 2 - n\sqrt[n]{y^2} \geq \frac{y(y + n - 2)}{n-1} \left(\frac{2}{y} + n - 2 - \frac{n}{\sqrt[n]{y^2}} \right).$$

Denoting

$$t = \sqrt[n]{y}, \quad 0 < t \leq 1,$$

we need to show that $g(t) \geq 0$, where

$$\begin{aligned} g(t) &= (n-1)(2t^n - nt^2 + n - 2) - (t^n + n - 2)[(n-2)t^n - nt^{n-2} + 2] \\ &= -(n-2)t^{2n} + nt^{2n-2} - (n-2)(n-4)t^n + n(n-2)t^{n-2} - n(n-1)t^2 + (n-2)(n-3). \end{aligned}$$

For $n = 3$, we have

$$g(t) = t(1-t)^3(3+3t+t^2) \geq 0,$$

and for $n = 4$, we have

$$g(t) = 2(1-t^2)^3(1+t^2) \geq 0.$$

For $n \geq 5$, we have

$$\begin{aligned} g'(t) &= ntg_1(t), \\ g_1(t) &= -2(n-2)t^{2n-2} + 2(n-1)t^{2n-4} - (n-2)(n-4)t^{n-2} + (n-2)^2t^{n-4} - 2(n-1), \\ g'_1(t) &= (n-2)t^{n-5}(1-t^2)[4(n-1)t^n + n-2] \geq 0, \end{aligned}$$

hence $g_1(t)$ is increasing, $g_1(t) \leq g_1(1) = 0$, $g'(t) \leq 0$, $g(t)$ is decreasing, $g(t) \geq g(1) = 0$. Thus, the proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n \geq 1$.

Remark 1. Since $a_1^{n-1}a_2a_3 \cdots a_n \geq 1$ yields $a_1(a_2 + a_3 + \cdots + a_n) \geq n-1$, the inequality

$$F(a_1, a_2, \dots, a_n) \geq F\left(\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_n}\right)$$

is also valid if $a_1, a_2, \dots, a_n$ are positive real numbers such that

$$a_1 \leq a_2 \leq \dots \leq a_n, \quad a_1^{n-1} a_2 a_3 \dots a_n \geq 1.$$

Also, it is valid in the particular case

$$a_1, a_2, \dots, a_n \geq 1.$$

Remark 2. Since $a_1 a_2 \dots a_n \geq 1$, from P 5.100 it follows that

$$a_1 + a_2 + \dots + a_n \geq \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}$$

for

$$a_1(a_2 + a_3 + \dots + a_n) \geq n - 1.$$

□

P 5.101. Let

$$F(a_1, a_2, \dots, a_n) = \sqrt{\frac{a_1^2 + a_2^2 + \dots + a_n^2}{n}} - \frac{a_1 + a_2 + \dots + a_n}{n},$$

where $a_1, a_2, \dots, a_n$ are positive real numbers such that $a_1 = \min\{a_1, a_2, \dots, a_n\}$ and

$$a_1^{n-1}(a_2 + a_3 + \dots + a_n) \geq n - 1.$$

Then,

$$F(a_1, a_2, \dots, a_n) \geq F\left(\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_n}\right).$$

(Vasile C., 2020)

Solution. Assume that $a_1 \leq a_2 \leq \dots \leq a_n$. For $n = 2$, we need to show that $a_1 a_2 \geq 1$ involves

$$(a_1 a_2 - 1)(\sqrt{2(a_1^2 + a_2^2)} - a_1 - a_2) \geq 0,$$

which is true. For $n \geq 3$, write the inequality in the form

$$\begin{aligned} & \sqrt{n(a_1^2 + a_2^2 + \dots + a_n^2)} - (a_1 + a_2 + \dots + a_n) \\ & \geq \sqrt{n\left(\frac{1}{a_1^2} + \frac{1}{a_2^2} + \dots + \frac{1}{a_n^2}\right)} - \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \geq 0. \end{aligned}$$

According to Corollary 5 (case $k = -1$), we have:

- If $a_2, a_3, \dots, a_n$ are positive real numbers so that

$$a_2 + a_3 + \dots + a_n = \text{fixed}, \quad \frac{1}{a_2} + \frac{1}{a_3} + \dots + \frac{1}{a_n} = \text{fixed}, \quad a_2 \leq a_3 \leq \dots \leq a_n,$$

then the sum $a_2^2 + a_3^2 + \cdots + a_n^2$ is minimum and the sum $\frac{1}{a_2^2} + \frac{1}{a_3^2} + \cdots + \frac{1}{a_n^2}$ is maximum for $a_2 \leq a_3 = \cdots = a_n$.

Thus, it suffices to consider the case $a_2 \leq a_3 = \cdots = a_n$. We need to show that if x, y, z are positive real numbers such that $x \leq y \leq z$ and

$$x^{n-1}[y + (n-2)z] \geq n-1,$$

then $E(x, y, z) \geq 0$, where

$$E(x, y, z) = \sqrt{x^2 + y^2 + (n-2)z^2} - \frac{x + y + (n-2)z}{\sqrt{n}} \\ - \sqrt{\frac{1}{x^2} + \frac{1}{y^2} + \frac{n-2}{z^2}} + \frac{1}{\sqrt{n}} \left(\frac{1}{x} + \frac{1}{y} + \frac{n-2}{z} \right).$$

We will show that

$$E(x, y, z) \geq E(x, w, w) \geq 0,$$

where

$$w = \frac{y + (n-2)z}{n-1}, \quad x \leq y \leq w \leq z.$$

Write the inequality $E(x, y, z) \geq E(x, w, w)$ as follows:

$$\frac{y^2 + (n-2)z^2 - (n-1)w^2}{\sqrt{x^2 + y^2 + (n-2)z^2} + \sqrt{x^2 + (n-1)w^2}} + \frac{1}{\sqrt{n}} \left(\frac{1}{y} + \frac{n-2}{z} - \frac{n-1}{w} \right) \\ \geq \frac{\frac{1}{y^2} + \frac{n-2}{z^2} + \frac{n-1}{w^2}}{\sqrt{\frac{1}{x^2} + \frac{1}{y^2} + \frac{n-2}{z^2}} + \sqrt{\frac{1}{x^2} + \frac{n-1}{w^2}}}, \\ \frac{(n-2)(y-z)^2}{n-1} \cdot \frac{1}{\sqrt{x^2 + y^2 + (n-2)z^2} + \sqrt{x^2 + (n-1)w^2}} + \frac{(n-2)(y-z)^2}{\sqrt{nyz}[y + (n-2)z]} \\ \geq \frac{(n-2)(y-z)^2[y^2 + 2(n-1)yz + (n-2)z^2]}{y^2z^2[y + (n-2)z]^2} \cdot \frac{1}{\sqrt{\frac{1}{x^2} + \frac{1}{y^2} + \frac{n-2}{z^2}} + \sqrt{\frac{1}{x^2} + \frac{n-1}{w^2}}},$$

which is true if

$$\frac{1}{n-1} \cdot \frac{1}{\sqrt{x^2 + y^2 + (n-2)z^2} + \sqrt{x^2 + (n-1)w^2}} + \frac{1}{\sqrt{nyz}[y + (n-2)z]} \\ \geq \frac{y^2 + 2(n-1)yz + (n-2)z^2}{y^2z^2[y + (n-2)z]^2} \cdot \frac{1}{\sqrt{\frac{1}{x^2} + \frac{1}{y^2} + \frac{n-2}{z^2}} + \sqrt{\frac{1}{x^2} + \frac{n-1}{w^2}}}.$$

Since $x \leq y$, it is enough to show that

$$\frac{1}{n-1} \cdot \frac{1}{\sqrt{2y^2 + (n-2)z^2} + \sqrt{y^2 + (n-1)w^2}} + \frac{1}{\sqrt{nyz}[y + (n-2)z]}$$

$$\geq \frac{y^2 + 2(n-1)yz + (n-2)z^2}{y^2 z^2 [y + (n-2)z]^2} \cdot \frac{1}{\sqrt{\frac{2}{y^2} + \frac{n-2}{z^2}} + \sqrt{\frac{1}{y^2} + \frac{n-1}{w^2}}}.$$

In addition, since $w \leq z$, it suffices to show that

$$\begin{aligned} & \frac{1}{n-1} \cdot \frac{1}{\sqrt{2y^2 + (n-2)z^2} + \sqrt{y^2 + (n-1)z^2}} + \frac{1}{\sqrt{nyz[y + (n-2)z]}} \\ & \geq \frac{y^2 + 2(n-1)yz + (n-2)z^2}{y^2 z^2 [y + (n-2)z]^2} \cdot \frac{1}{\sqrt{\frac{2}{y^2} + \frac{n-2}{z^2}} + \sqrt{\frac{1}{y^2} + \frac{n-1}{z^2}}}. \end{aligned}$$

Since

$$y^2 + 2(n-1)yz + (n-2)z^2 = [y^2 + (n-2)z^2] + 2(n-1)yz,$$

we rewrite the inequality as

$$A + B \geq C + D,$$

where

$$\begin{aligned} A &= \frac{1}{n-1} \cdot \frac{1}{\sqrt{2y^2 + (n-2)z^2} + \sqrt{y^2 + (n-1)z^2}}, \\ B &= \frac{1}{\sqrt{nyz[y + (n-2)z]}}, \\ C &= \frac{y^2 + (n-2)z^2}{y^2 z^2 [y + (n-2)z]^2} \cdot \frac{1}{\sqrt{\frac{2}{y^2} + \frac{n-2}{z^2}} + \sqrt{\frac{1}{y^2} + \frac{n-1}{z^2}}}, \\ D &= \frac{2(n-1)yz}{y^2 z^2 [y + (n-2)z]^2} \cdot \frac{1}{\sqrt{\frac{2}{y^2} + \frac{n-2}{z^2}} + \sqrt{\frac{1}{y^2} + \frac{n-1}{z^2}}}. \end{aligned}$$

We will show that

$$A \geq C, \quad B \geq D.$$

Since the inequality $B \geq D$ is homogeneous, we may consider $y = 1$ and $z \geq 1$, when it becomes

$$[(n-2)z + 1] \left[\sqrt{2z^2 + n-2} + \sqrt{z^2 + n-1} \right] \geq 2\sqrt{n}(n-1)z.$$

Since

$$\sqrt{2z^2 + n-2} + \sqrt{z^2 + n-1} \geq \frac{2z + n-2}{\sqrt{n}} + \frac{z + n-1}{\sqrt{n}} = \frac{3z + 2n-3}{\sqrt{n}},$$

it is sufficient to show that

$$[(n-2)z + 1](3z + 2n-3) \geq 2n(n-1),$$

which is equivalent to

$$(z-1)[3(n-2)z + 2n^2 - 4n + 3] \geq 0.$$

To show that $A \geq C$, we see that $x^{n-1}[y + (n-2)z] \geq n-1$ yields

$$y^{n-1}[y + (n-2)z] \geq n-1.$$

Thus, it suffices to prove the homogeneous inequality

$$A \geq C_0 C, \quad C_0 = \left[\frac{y^{n-1}[y + (n-2)z]}{n-1} \right]^{2/n},$$

that is

$$\begin{aligned} & \frac{1}{\sqrt{2y^2 + (n-2)z^2} + \sqrt{y^2 + (n-1)z^2}} \geq \\ & \geq \frac{(n-1)[y^2 + (n-2)z^2]}{y^2 z^2 [y + (n-2)z]} \cdot \frac{C_0}{\sqrt{\frac{2}{y^2} + \frac{n-2}{z^2}} + \sqrt{\frac{1}{y^2} + \frac{n-1}{z^2}}}, \end{aligned}$$

Due to homogeneity, we may set $y = 1$, hence $z \geq 1$. The inequality becomes

$$\begin{aligned} & \sqrt{2z^2 + n-2} + \sqrt{z^2 + n-1} \geq \\ & \geq \frac{(n-1)[1 + (n-2)z^2]C_1}{z[1 + (n-2)z]^2} \left[\sqrt{2 + (n-2)z^2} + \sqrt{1 + (n-1)z^2} \right], \end{aligned}$$

where

$$C_1 = \left[\frac{1 + (n-2)z}{n-1} \right]^{2/n}.$$

By Bernoulli's inequality, we have

$$C_1 = \left[1 + \frac{(n-2)(z-1)}{n-1} \right]^{2/n} \leq 1 + \frac{2(n-2)(z-1)}{n(n-1)} = \frac{2(n-2)z + n^2 - 3n + 4}{n(n-1)}.$$

Thus, it suffices to show that

$$\begin{aligned} & \sqrt{2z^2 + n-2} + \sqrt{z^2 + n-1} \geq \\ & \geq \frac{[1 + (n-2)z^2][2(n-2)z + n^2 - 3n + 4]}{nz[1 + (n-2)z]^2} \left[\sqrt{2 + (n-2)z^2} + \sqrt{1 + (n-1)z^2} \right]. \end{aligned}$$

We will show that

$$\sqrt{2z^2 + n-2} \geq \frac{[1 + (n-2)z^2][2(n-2)z + n^2 - 3n + 4]}{nz[1 + (n-2)z]^2} \sqrt{(n-1)z^2 + 1}$$

and

$$\sqrt{z^2 + n-1} \geq \frac{[1 + (n-2)z^2][2(n-2)z + n^2 - 3n + 4]}{nz[1 + (n-2)z]^2} \sqrt{(n-2)z^2 + 2}.$$

Since

$$\frac{2z^2 + n-2}{(n-1)z^2 + 1} - \frac{z^2 + n-1}{(n-2)z^2 + 2} = \frac{(n-3)(z^2 - 1)^2}{[n-1]z^2 + 1][(n-2)z^2 + 2]} \geq 0,$$

it suffices to prove the second inequality. After squaring and making many calculations, this inequality can be written as $(z-1)P(z) \geq 0$, where $P(z) \geq 0$ for $z \geq 1$.

To complete the proof, we need to show that $E(x, w, w) \geq 0$ for $x^{n-1}w \geq 1$. Write the required inequality as follows:

$$\sqrt{n[x^2 + (n-1)w^2]} - [x + (n-1)w] \geq \sqrt{n \left[\frac{1}{x^2} + \frac{n-1}{w^2} \right]} - \left(\frac{1}{x} + \frac{n-1}{w} \right),$$

$$\frac{(n-1)(x-w)^2}{\sqrt{x^2 + (n-1)w^2} + \frac{x+(n-1)w}{\sqrt{n}}} \geq \frac{1}{xw} \cdot \frac{(n-1)(x-w)^2}{\sqrt{(n-1)x^2 + w^2} + \frac{(n-1)x+w}{\sqrt{n}}}.$$

This is true if

$$\sqrt{(n-1)x^2 + w^2} + \frac{(n-1)x + w}{\sqrt{n}} \geq \frac{1}{xw} \cdot \left[\sqrt{x^2 + (n-1)w^2} + \frac{x + (n-1)w}{\sqrt{n}} \right].$$

Since $x^{n-1}w \geq 1$, it suffices to prove the homogeneous inequality

$$\sqrt{(n-1)x^2 + w^2} + \frac{(n-1)x + w}{\sqrt{n}} \geq \frac{(x^{n-1}w)^{2/n}}{xw} \cdot \left[\sqrt{x^2 + (n-1)w^2} + \frac{x + (n-1)w}{\sqrt{n}} \right].$$

Due to homogeneity, we may set $w = 1$, which yields $x \leq 1$. The inequality becomes

$$\sqrt{(n-1)x^2 + 1} + \frac{(n-1)x + 1}{\sqrt{n}} \geq x^{\frac{n-2}{n}} \left[\sqrt{x^2 + n-1} + \frac{x + n-1}{\sqrt{n}} \right].$$

We can get this by summing the inequalities

$$\sqrt{(n-1)x^2 + 1} \geq x^{\frac{n-2}{n}} \cdot \sqrt{x^2 + n-1}$$

and

$$\frac{(n-1)x + 1}{\sqrt{n}} \geq x^{\frac{n-2}{n}} \cdot \frac{x + n-1}{\sqrt{n}}.$$

Replacing x with x^2 in the second inequality gives the first inequality. Thus, it suffices to prove the second inequality, which can be rewritten as $f(x) \geq 0$, where

$$f(x) = \ln[(n-1)x + 1] - \ln(x + n-1) - \frac{n-2}{n} \ln x.$$

From

$$f'(x) = \frac{n-1}{(n-1)x + 1} - \frac{1}{x + n-1} - \frac{n-2}{nx} = \frac{-(n-1)(n-2)(x-1)^2}{nx[(n-1)x + 1]_x + n-1} \leq 0,$$

it follows that f is decreasing, hence $f(x) \geq f(1) = 0$.

The proof is completed. The equality holds for $a_1 = a_2 = \dots = a_n \geq 1$.

Remark. The inequality

$$F(a_1, a_2, \dots, a_n) \geq F\left(\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_n}\right)$$

is also valid in the particular case

$$a_1, a_2, \dots, a_n \geq 1.$$

□

P 5.102. If $a_1, a_2, \dots, a_n$ ($n \geq 4$) are positive real numbers such that

$$a_1 + a_2 + \dots + a_n = n, \quad a_n = \max\{a_1, a_2, \dots, a_n\},$$

then

$$n \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_{n-1}} \right) \geq 4(a_1^2 + a_2^2 + \dots + a_n^2) + n(n-5).$$

(Vasile C., 2022)

Solution. Assume that a_n is fixed and $a_1 \leq a_2 \leq \dots \leq a_n$. According to Corollary 5 (case $k = 2$ and $m = -1$), we have:

• If $a_1, a_2, \dots, a_{n-1}$ are positive real numbers so that

$$a_1 + a_2 + \dots + a_{n-1} = \text{fixed}, \quad a_1^2 + a_2^2 + \dots + a_{n-1}^2 = \text{fixed}, \quad a_1 \leq a_2 \leq \dots \leq a_{n-1},$$

then the sum $\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_{n-1}}$ is minimum for $a_1 = a_2 = \dots = a_{n-2} \leq a_{n-1}$.

Therefore, it suffices to consider the case $a_1 = a_2 = \dots = a_{n-2}$, that is to show that $F(a, b) \geq 0$, where

$$F(a, b) = n \left(\frac{n-2}{a} + \frac{1}{b} \right) - 4(n-2)a^2 - 4b^2 - 4c^2 - n(n-5), \quad c = n - (n-2)a - b,$$

with a, b positive real numbers such that $a \leq b \leq c$. From $c \geq b$, we get

$$(n-2)a + 2b \leq n.$$

We will show that

$$F(a, b) \geq F(t, t) \geq 0,$$

where

$$t = \frac{(n-2)a + b}{n-1}, \quad t \leq 1.$$

Since

$$\begin{aligned} F(a, b) - F(t, t) &= n \left(\frac{n-2}{a} + \frac{1}{b} - \frac{n-1}{t} \right) - 4[(n-2)a^2 + b^2 - (n-1)t^2] \\ &= \frac{n(n-2)(a-b)^2}{(n-1)abt} - \frac{4(n-2)(a-b)^2}{n-1} \\ &\geq \frac{n(n-2)(a-b)^2}{(n-1)ab} - \frac{4(n-2)(a-b)^2}{n-1} \\ &= \frac{(n-2)(a-b)^2(n-4ab)}{(n-1)ab}, \end{aligned}$$

it suffices to show that $4ab \leq n$. From

$$n \geq (n-2)a + 2b \geq 2\sqrt{2(n-2)ab},$$

we get

$$4ab - n \leq \frac{n^2}{2(n-2)} - n = \frac{n(4-n)}{n-2} \leq 0.$$

In addition,

$$\begin{aligned} F(t, t) &= \frac{n(n-1)}{t} - 4(n-1)t^2 - 4[n - (n-1)t]^2 - n(n-5) \\ &= \frac{n(n-1)(1-t)(1-2t)^2}{t} \geq 0. \end{aligned}$$

The equality occurs for $a_1 = a_2 = \dots = a_n = 1$, and also for

$$a_1 = a_2 = \dots = a_{n-1} = \frac{1}{2}, \quad a_n = \frac{n+1}{2}.$$

□

P 5.103. If $a_1, a_2, \dots, a_n$ are positive real numbers such that

$$a_1 + a_2 + \dots + a_n = n,$$

then

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} + n - 2 \geq \frac{n(n-1)^2}{\sum_{1 \leq i < j \leq n} a_i a_j}.$$

(Vasile C., 2022)

Solution. For $n = 2$, the inequality is an identity. Consider further that $n \geq 3$ and $a_1 \leq a_2 \leq \dots \leq a_n$. By Corollary 5 (case $k = 2$ and $m = -1$), if $0 < a_1 \leq a_2 \leq \dots \leq a_n$ and

$$a_1 + a_2 + \dots + a_n = n, \quad \sum_{1 \leq i < j \leq n} a_i a_j = \text{fixed},$$

then the sum $\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}$ is minimum for $a_1 = \dots = a_{n-1} \leq a_n$. Therefore, we only need to show that $(n-1)x + y = n$ ($0 < x \leq y$) involves

$$\frac{n-1}{x} + \frac{1}{y} + n - 2 \geq \frac{2n(n-1)}{x[(n-2)x + 2y]}.$$

The inequality is equivalent to

$$\begin{aligned} \frac{n-1}{x} + \frac{1}{n - (n-1)x} + n - 2 &\geq \frac{2(n-1)}{x(2-x)}, \\ \frac{n-1}{x} + \frac{1}{n - (n-1)x} - n &\geq 2(n-1) \left[\frac{1}{x(2-x)} - 1 \right], \\ \frac{n(n-1)(x-1)^2}{n - (n-1)x} &\geq \frac{2(n-1)(x-1)^2}{2-x}, \\ (n-1)(n-2)x(x-1)^2 &\geq 0. \end{aligned}$$

For $n \geq 3$, the equality occurs for $a_1 = a_2 = \dots = a_n = 1$.

□

P 5.104. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers such that

$$a_1 + a_2 + \dots + a_n = n,$$

then

$$\sum_{i=1}^n \frac{1}{n - a_i} \leq \frac{n-2}{n-1} + \frac{n}{\sum_{1 \leq i < j \leq n} a_i a_j}.$$

(V. Cîrtoaje and L. Giugiuc, 2022)

Solution. For $n = 2$, the inequality is an identity. Consider further that $n \geq 3$ and $a_1 \leq a_2 \leq \dots \leq a_n$, and apply the EV-Theorem (Corollary 1) to the function $f(u) = \frac{1}{n-u}$ defined on $[0, n)$. We have

$$f'(u) = \frac{1}{(n-u)^2} := g(u), \quad g''(u) = \frac{6}{(n-u)^4} > 0.$$

According to Corollary 1, if $a_1 + a_2 + \dots + a_n = n$ (which implies $a_i < n$ for $i = 1, 2, \dots, n$) and

$\sum_{1 \leq i < j \leq n} a_i a_j = \text{fixed}$, the sum $\sum_{i=1}^n \frac{1}{n - a_i}$ is maximum when $a_1 = \dots = a_{n-1} \leq a_n$. Therefore,

we only need to show that $(n-1)x + y = n$ ($0 < x \leq y$) involves

$$\frac{n-1}{n-x} + \frac{1}{n-y} \leq \frac{n-2}{n-1} + \frac{2n}{(n-1)x[(n-2)x+2y]}.$$

The inequality is equivalent to

$$\begin{aligned} \frac{(n-1)^2}{n-x} + \frac{1}{x} &\leq n-2 + \frac{2}{x(2-x)}, \\ \frac{(n-1)^2}{n-x} + \frac{1}{x} - n &\leq -2 + \frac{2}{x(2-x)}, \\ \frac{n(x-1)^2}{n-x} &\leq \frac{2(x-1)^2}{2-x}, \\ (n-2)x(x-1)^2 &\geq 0. \end{aligned}$$

For $n \geq 3$, the equality occurs when $a_1 = a_2 = \dots = a_n = 1$.

□

P 5.105. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers such that

$$a_1 + a_2 + \dots + a_n = n,$$

then

$$\sum_{i=1}^n \frac{1}{n - a_i} \geq 1 + \frac{n}{2 \sum_{1 \leq i < j \leq n} a_i a_j}.$$

(V. Cîrtoaje and L. Giugiuc, 2022)

Solution. We use the induction method and EV-Theorem (Corollary 1). For $n = 2$, the inequality reduces to $(a_1 - a_2)^2 \geq 0$. Consider further that $n \geq 3$ and $a_1 \geq a_2 \geq \dots \geq a_n$. According to Corollary 1 applied to the function $f(u) = \frac{1}{n-u}$ (see the preceding problem), if $a_1 + a_2 + \dots + a_n = n$ and $\sum_{1 \leq i < j \leq n} a_i a_j = \text{fixed}$, the sum $\sum_{i=1}^n \frac{1}{n-a_i}$ is minimum when either $a_1 = \dots = a_{n-1} \geq a_n > 0$ or $a_n = 0$. Therefore, we only need to consider these cases.

Case 1: $a_1 = \dots = a_{n-1} \geq a_n > 0$. We need to show that

$$(n-1)x + y = n \quad (x \geq y > 0)$$

involves

$$\frac{n-1}{n-x} + \frac{1}{n-y} \geq 1 + \frac{n}{(n-1)x[(n-2)x+2y]}.$$

The inequality is equivalent to

$$\begin{aligned} \frac{(n-1)^2}{n-x} + \frac{1}{x} &\geq n-1 + \frac{1}{x(2-x)}, \\ \frac{(n-1)^2}{n-x} + \frac{1}{x} - n &\geq -1 + \frac{1}{x(2-x)}, \\ \frac{n(x-1)^2}{n-x} &\geq \frac{(x-1)^2}{2-x}, \\ (x-1)^2 y &\geq 0. \end{aligned}$$

Case 2: $a_n = 0$. We need to show that $a_1 + a_2 + \dots + a_{n-1} = n$ involves

$$\sum_{i=1}^{n-1} \frac{1}{n-a_i} + \frac{1}{n} \geq 1 + \frac{n}{2 \sum_{1 \leq i < j \leq n-1} a_i a_j}.$$

Using the substitution

$$a_i = \frac{n}{n-1} x_i, \quad i = 1, 2, \dots, n-1,$$

we need to show that $x_1 + x_2 + \dots + x_{n-1} = n-1$ implies

$$\frac{n-1}{n} \sum_{i=1}^{n-1} \frac{1}{n-1-x_i} + \frac{1}{n} \geq 1 + \frac{(n-1)^2}{2n \sum_{1 \leq i < j \leq n-1} x_i x_j},$$

which is equivalent to the induction hypothesis

$$\sum_{i=1}^{n-1} \frac{1}{n-1-x_i} \geq 1 + \frac{n-1}{2 \sum_{1 \leq i < j \leq n-1} x_i x_j}.$$

The proof is completed. The equality occurs for $a_1 = \dots = a_k = \frac{n}{k}$ and $a_{k+1} = \dots = a_n = 0$, where $k = 2, 3, \dots, n$. □

P 5.106. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers such that

$$\sum_{i=1}^n \frac{1}{2a_i + n - 2} \leq 1,$$

then

$$2 \sum_{1 \leq i < j \leq n} a_i a_j \geq (n-1)(a_1 + a_2 + \dots + a_n).$$

(L. Giugiuc and V. Cîrtoaje, 2022)

Solution. For $n = 2$, the conclusion coincides with the hypothesis. Consider now $n \geq 3$, and use the contradiction method and the EV-Theorem. By the contradiction method, it suffices to show that

$$2 \sum_{1 \leq i < j \leq n} a_i a_j < (n-1)(a_1 + a_2 + \dots + a_n)$$

involves

$$f(a_1) + f(a_2) + \dots + f(a_n) < -1,$$

where

$$f(u) = \frac{-1}{2u + n - 2}.$$

To prove this, we apply EV-Theorem (Corollary 1) to the function $f(u) = \frac{-1}{2u + n - 2}$, which satisfies

$$f'(u) = \frac{2}{(2u + n - 2)^2} := g(u), \quad g''(u) = \frac{48}{(2u + n - 2)^4} > 0.$$

According to EV-Theorem, for fixed $\sum_{i=1}^n a_i$ and $\sum_{1 \leq i < j \leq n} a_i a_j$, the sum $f(a_1) + f(a_2) + \dots + f(a_n)$ is maximum for $a_1 = a_2 = \dots = a_{n-1} \leq a_n$. Denoting the equal variables with x and a_n with y ($x \leq y$), we need to show that

$$(n-2)x^2 + 2xy < (n-1)x + y$$

involves

$$\frac{1}{2y + n - 2} + \frac{x - 1}{2x + n - 2} > 1.$$

Applying again the contradiction method, it suffices to show that

$$\frac{1}{2y + n - 2} + \frac{x - 1}{2x + n - 2} \leq 1$$

involves

$$(n-2)x^2 + (2x-1)y \geq (n-1)x.$$

From the hypothesis, we get

$$y \geq \frac{n-2-(n-3)x}{2x-1}, \quad x > \frac{1}{2}.$$

Thus, we have

$$\begin{aligned}(n-2)x^2 + (2x-1)y - (n-1)x &\geq (n-2)x^2 + n-2 - (n-3)x - (n-1)x \\ &= (n-2)(x-1)^2 \geq 0.\end{aligned}$$

The proof is completed. For $n \geq 3$, the equality occurs when $a_1 = a_2 = \cdots = a_n = 1$.

□

P 5.107. If $a_1, a_2, \dots, a_n$ are positive real numbers such that

$$\frac{1}{2a_1 + n - 2} + \frac{1}{2a_2 + n - 2} + \cdots + \frac{1}{2a_n + n - 2} = 1,$$

then

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} + \frac{n(n-2)}{a_1 + a_2 + \cdots + a_n} \geq 2(n-1).$$

(V. Cîrtoaje and L. Giugiuc, 2022)

Solution. We use the contradiction method and the EV-Theorem. For $n = 2$, the required inequality is an identity. Consider further $n \geq 3$ and $a_1 \leq a_2 \leq \cdots \leq a_n$. By the contradiction method, assume that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} + \frac{n(n-2)}{a_1 + a_2 + \cdots + a_n} < 2(n-1),$$

and show that

$$\frac{1}{2a_1 + n - 2} + \frac{1}{2a_2 + n - 2} + \cdots + \frac{1}{2a_n + n - 2} < 1.$$

To prove this, we apply EV-Theorem (Corollary 2) to the function $f(u) = \frac{1}{2u + n - 2}$, which satisfies

$$f'(u) = \frac{-2}{(2u + n - 2)^2}, \quad g(x) = f'\left(\frac{1}{\sqrt{x}}\right) = \frac{-2x}{[(n-2)\sqrt{x} + 2]^2},$$

$$g''(x) = \frac{12}{\sqrt{x} [(n-2)\sqrt{x} + 2]^4} > 0.$$

According to the EV-Theorem, for fixed $a_1 + a_2 + \cdots + a_n$ and $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$, the sum

$\sum_{i=1}^n \frac{1}{2a_i + n - 2}$ is maximum when $a_1 = a_2 = \cdots = a_{n-1} \leq a_n$. Denoting by x the equal variables and by y the remaining variable ($x \leq y$), we need to show that

$$\frac{n-1}{x} + \frac{1}{y} + \frac{n(n-2)}{(n-1)x + y} < 2(n-1)$$

involves

$$\frac{n-1}{2x + n - 2} + \frac{1}{2y + n - 2} < 1.$$

Using again the contradiction method, we need to show that

$$\frac{n-1}{2x+n-2} + \frac{1}{2y+n-2} \geq 1$$

involves

$$\frac{n-1}{x} + \frac{1}{y} + \frac{n(n-2)}{(n-1)x+y} \geq 2(n-1).$$

From the hypothesis, we get

$$y \leq \frac{n-2-(n-3)x}{2x-1}, \quad \frac{1}{2} < x < \frac{n-2}{n-3}.$$

Clearly, it suffices to prove the required inequality

$$\frac{n-1}{x} + \frac{1}{y} - n \geq (n-2) \left[1 - \frac{n}{(n-1)x+y} \right]$$

for $y = \frac{n-2-(n-3)x}{2x-1}$. The inequality is equivalent to

$$\frac{(n-1)(n-2)(x-1)^2}{x[n-2-(n-3)x]} \geq \frac{2(n-1)(n-2)(x-1)^2}{2(n-1)x^2-2(n-2)x+n-2}.$$

This is true if

$$\frac{1}{x[n-2-(n-3)x]} \geq \frac{2}{2(n-1)x^2-2(n-2)x+n-2},$$

which reduces to

$$(n-2)(2x-1)^2 \geq 0.$$

The proof is completed. For $n \geq 3$, the equality occurs when $a_1 = a_2 = \cdots = a_n = 1$.

□

P 5.108. If $a_1, a_2, \dots, a_n$ are positive real numbers such that

$$\frac{1}{a_1+n-1} + \frac{1}{a_2+n-1} + \cdots + \frac{1}{a_n+n-1} = 1,$$

then

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} + \frac{n(3n-4)}{a_1+a_2+\cdots+a_n} \geq 4(n-1).$$

(V. Cîrtoaje and L. Giugiuc, 2022)

Solution. The proof is similar to the one of the preceding problem. For $n = 2$, the required inequality reduces to $(a+b-2)^2 \geq 0$. For

$$a_1 = a_2 = \cdots = a_{n-1} = x, \quad a_n = y,$$

we need to show that

$$\frac{n-1}{x+n-1} + \frac{1}{y+n-1} = 1$$

involves

$$\frac{n-1}{x} + \frac{1}{y} + \frac{n(3n-4)}{(n-1)x+y} \geq 4(n-1).$$

So, we need to show that

$$y = \frac{n-1-(n-2)x}{x}, \quad 0 < x < \frac{n-1}{n-2},$$

involves

$$\frac{n-1}{x} + \frac{1}{y} - n \geq (3n-4) \left[1 - \frac{n}{(n-1)x+y} \right].$$

The inequality is equivalent to

$$\frac{(n-1)^2(x-1)^2}{x[n-1-(n-2)x]} \geq \frac{(n-1)(3n-4)(x-1)^2}{(n-1)x^2-(n-2)x+n-1}.$$

This is true if

$$\frac{n-1}{x[n-1-(n-2)x]} \geq \frac{3n-4}{(n-1)x^2-(n-2)x+n-1}.$$

which reduces to

$$[(2n-3)x-n+1]^2 \geq 0.$$

The proof is completed. The equality occurs when $a_1 = a_2 = \dots = a_n = 1$, and also for $a_1 = a_2 = \dots = a_{n-1} = \frac{n-1}{2n-3}$ and $a_n = n-1$ (or any cyclic permutation). □

P 5.109. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers such that

$$\sum_{i=1}^n \frac{1}{a_i + 1} \geq n-1,$$

then

$$(n-2) \sum_{i=1}^n a_i + \frac{n}{2 \sum_{1 \leq i < j \leq n} a_i a_j} \geq \frac{2n^2 - 4n + 1}{n-1}.$$

(V. Cîrtoaje and L. Giugiuc, *Mathproblems*, 3, 2023)

Solution. We use the contradiction method and the EV-Theorem. For $n = 2$, the required inequality coincides with the hypothesis $\frac{1}{a_1+1} + \frac{1}{a_2+1} \geq 1$. Consider further $n \geq 3$. By the contradiction method, it suffices to show that

$$(n-2) \sum_{i=1}^n a_i + \frac{n}{2 \sum_{1 \leq i < j \leq n} a_i a_j} < \frac{2n^2 - 4n + 1}{n-1}$$

involves

$$f(a_1) + f(a_2) + \dots + f(a_n) > 1 - n,$$

where

$$f(u) = \frac{-1}{u+1}.$$

To prove this, we apply EV-Theorem (Corollary 1) to the function f , which satisfies

$$f'(u) = \frac{1}{(u+1)^2} := g(u), \quad g''(u) = \frac{6}{(u+1)^4} > 0.$$

According to EV-Theorem, for fixed $\sum_{i=1}^n a_i$ and $\sum_{1 \leq i < j \leq n} a_i a_j$, the sum $\sum_{i=1}^n \frac{1}{a_i + 1}$ is maximum for either $a_1 = a_2 = \dots = a_{n-1} \geq a_n > 0$ or $a_n = 0$. Thus, it suffices to consider these cases. Applying again the contradiction method (only for these cases), it suffices to show that

$$\sum_{i=1}^n \frac{1}{a_i + 1} \geq n - 1$$

involves

$$(n-2) \sum_{i=1}^n a_i + \frac{n}{2 \sum_{1 \leq i < j \leq n} a_i a_j} \geq \frac{2n^2 - 4n + 1}{n-1}$$

(which is just the original statement).

Case 1: $a_1 = a_2 = \dots = a_{n-1} \geq a_n > 0$. We need to show that

$$\frac{n-1}{x+1} + \frac{1}{y+1} \geq n-1, \quad x \geq y > 0$$

involves

$$(n-2)[(n-1)x + y] + \frac{n}{(n-1)x[(n-2)x + 2y]} \geq \frac{2n^2 - 4n + 1}{n-1}.$$

From the hypothesis, we have

$$y \leq \frac{1 - (n-2)x}{(n-1)x}, \quad x < \frac{1}{n-2}.$$

We claim that for fixed x , the function

$$F(y) = (n-2)[(n-1)x + y] + \frac{n}{(n-1)x[(n-2)x + 2y]}$$

is decreasing. We need to show that $F'(y) \leq 0$, which is equivalent to

$$(n-1)(n-2)x[(n-2)x + 2y]^2 \leq 2n.$$

Since $(n-2)x < 1$, it suffices to show that

$$(n-1)[(n-2)x + 2y]^2 \leq 2n.$$

According to Lemma below, we have

$$(n-1)[(n-2)x + 2y]^2 \leq \frac{n^2}{n-1} < 2n.$$

Since $F(y)$ is decreasing, we have

$$F(y) \geq F\left(\frac{1 - (n-2)x}{(n-1)x}\right),$$

and it suffices to show that

$$F\left(\frac{1 - (n-2)x}{(n-1)x}\right) \geq \frac{2n^2 - 4n + 1}{n-1}.$$

This inequality is equivalent to

$$[1 - (n-1)x]^2[1 - (n-2)x][2 - (n-1)x] \geq 0,$$

which is true for $(n-2)x < 1$.

Case 2: $a_n = 0$. We need to prove that

$$\sum_{i=1}^{n-1} \frac{1}{a_i + 1} \geq n-2,$$

involves

$$(n-2)S + \frac{n}{2 \sum_{1 \leq i < j \leq n-1} a_i a_j} \geq \frac{2n^2 - 4n + 1}{n-1},$$

where

$$S = \sum_{i=1}^{n-1} a_i.$$

Case 2a: $S \geq \frac{n-1}{n-2}$. Write the hypothesis as

$$\sum_{i=1}^{n-1} \frac{a_i}{a_i + 1} \leq 1.$$

From the Cauchy-Schwarz inequality

$$\left(\sum_{i=1}^{n-1} \frac{a_i}{a_i + 1}\right) \left[\sum_{i=1}^{n-1} a_i(a_i + 1)\right] \geq \left(\sum_{i=1}^{n-1} a_i\right)^2,$$

we get

$$\sum_{i=1}^{n-1} a_i^2 + S \geq S^2,$$

$$S^2 - 2 \sum_{1 \leq i < j \leq n-1} a_i a_j + S \geq S^2, \quad 2 \sum_{1 \leq i < j \leq n-1} a_i a_j \leq S.$$

Thus, it suffices to show that

$$(n-2)S + \frac{n}{S} \geq \frac{2n^2 - 4n + 1}{n-1},$$

which is equivalent to the obvious inequality

$$\left(S - \frac{n-1}{n-2}\right) \left(S - \frac{n}{n-1}\right) \geq 0.$$

Case 2b: $S \leq \frac{n-1}{n-2}$. Since

$$S^2 \geq \frac{2(n-1)}{n-2} \sum_{1 \leq i < j \leq n-1} a_i a_j,$$

it suffices to show that

$$(n-2)S + \frac{n(n-1)}{(n-2)S^2} \geq \frac{2n^2 - 4n + 1}{n-1},$$

which is equivalent to

$$\left(\frac{n-1}{n-2} - S\right) \left(1 + \frac{n-2}{n-1}S - \frac{n-2}{n}S^2\right) \geq 0.$$

This inequality is true because

$$1 + \frac{n-2}{n-1}S - \frac{n-2}{n}S^2 \geq 1 + \frac{n-2}{n-1}S - \frac{n-1}{n}S = 1 - \frac{1}{n(n-1)}S > 0.$$

The proof is completed. For $n \geq 3$, the equality occurs when $a_1 = a_2 = \dots = a_n = \frac{1}{n-1}$, and also for $a_1 = a_2 = \dots = a_{n-1} = \frac{1}{n-2}$ and $a_n = 0$ (or any cyclic permutation). □

P 5.110. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers such that

$$\sum_{i=1}^n \frac{1}{a_i + n-1} = 1,$$

then

$$2 \sum_{1 \leq i < j \leq n} a_i a_j \geq \frac{n^2(n-1)}{a_1 + a_2 + \dots + a_n}.$$

(V. Cîrtoaje, 2022)

Solution. We use the contradiction method, the induction method and the EV-Theorem. For $n = 2$, we need to prove that $a_1 a_2 = 1$ implies $a_1 + a_2 \geq 2$, which is true. Consider further $n \geq 3$. By the contradiction method, it suffices to show that

$$2 \sum_{1 \leq i < j \leq n} a_i a_j < \frac{n^2(n-1)}{a_1 + a_2 + \dots + a_n}$$

involves

$$f(a_1) + f(a_2) + \cdots + f(a_n) < -1,$$

where

$$f(u) = \frac{-1}{u+n-1}.$$

To prove this, we apply EV-Theorem (Corollary 1) to the function f , which satisfies

$$f'(u) = \frac{1}{(u+n-1)^2} := g(u), \quad g''(u) = \frac{6}{(u+n-1)^4} > 0.$$

According to EV-Theorem, for fixed $\sum_{i=1}^n a_i$ and $\sum_{1 \leq i < j \leq n} a_i a_j$, the sum $\sum_{i=1}^n \frac{1}{a_i + n - 1}$ is maximum for either $a_1 = a_2 = \cdots = a_{n-1} \geq a_n > 0$ or $a_n = 0$. Thus, it suffices to consider these cases. Applying again the contradiction method (only for these cases), it suffices to show that

$$\sum_{i=1}^n \frac{1}{a_i + n - 1} \leq 1$$

involves

$$2 \sum_{1 \leq i < j \leq n} a_i a_j \geq \frac{n^2(n-1)}{a_1 + a_2 + \cdots + a_n}$$

Case 1: $a_1 = a_2 = \cdots = a_{n-1} \geq a_n > 0$. We need to show that

$$\frac{n-1}{x+n-1} + \frac{1}{y+n-1} \leq 1, \quad x \geq y > 0$$

involves

$$(n-2)x^2 + 2xy - n \geq \frac{n^2}{(n-1)x+y} - n.$$

From the hypothesis, we have

$$y \geq \frac{n-1-(n-2)x}{x}, \quad x < \frac{n-1}{n-2}.$$

Thus, it is sufficient to show that

$$(n-2)x^2 + 2[n-1-(n-2)x] - n \geq \frac{n^2}{(n-1)x + \frac{n-1-(n-2)x}{x}} - n,$$

that is

$$(n-2)(x-1)^2 + \frac{n(x-1)^2}{(n-1)x^2 - (n-2)x + n-1} \geq 0.$$

Case 2: $a_n = 0$. We need to prove that

$$\sum_{i=1}^{n-1} \frac{1}{a_i + n - 1} \leq \frac{n-2}{n-1}$$

involves

$$2 \sum_{1 \leq i < j \leq n-1} a_i a_j \geq \frac{n^2(n-1)}{a_1 + a_2 + \cdots + a_{n-1}}.$$

Using the substitution

$$a_i = \frac{n-1}{n-2} x_i, \quad i = 1, 2, \dots, n-1,$$

we need to show that

$$\sum_{i=1}^{n-1} \frac{1}{x_i + n-2} \leq 1$$

implies

$$2 \sum_{1 \leq i < j \leq n-1} x_i x_j \geq \frac{n^2(n-2)^3}{(n-1)^2(x_1 + x_2 + \cdots + x_{n-1})}.$$

By the induction hypothesis, we have

$$2 \sum_{1 \leq i < j \leq n-1} x_i x_j \geq \frac{(n-1)^2(n-2)}{x_1 + x_2 + \cdots + x_{n-1}}.$$

Thus, it suffices to show that

$$\frac{(n-1)^2(n-2)}{x_1 + x_2 + \cdots + x_{n-1}} \geq \frac{n^2(n-2)^3}{(n-1)^2(x_1 + x_2 + \cdots + x_{n-1})},$$

This is true if $(n-1)^4 \geq n^2(n-2)^2$, which is true if $(n-1)^2 \geq n(n-2)$. The proof is completed. The equality occurs for $a_1 = a_2 = \cdots = a_n = 1$. □

P 5.111. If $a_1, a_2, \dots, a_n$ ($n \geq 3$) are nonnegative real numbers such that

$$\sum_{i=1}^n \frac{1}{a_i + n-1} \leq 1,$$

then

$$\frac{n}{a_1 + a_2 + \cdots + a_n} - 1 + k \left[\frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} a_i a_j - 1 \right] \geq 0,$$

where

$$k = \frac{4(n-1)^2}{(n-2)(3n-4)}.$$

(L. Giugiuc and V. Cîrtoaje, *Recreatii Matematice*, no. 1, 2022)

Solution. By the contradiction method, it suffices to show that

$$\frac{n}{a_1 + a_2 + \cdots + a_n} - 1 + k \left[\frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} a_i a_j - 1 \right] < 0$$

involves

$$\sum_{i=1}^n \frac{1}{a_i + n - 1} > 1.$$

To prove this, we apply EV-Theorem (Corollary 1) to the function $f(u) = \frac{-1}{u + n - 1}$, which satisfies

$$f'(u) = \frac{1}{(u + n - 1)^2} := g(u), \quad g''(u) = \frac{6}{(u + n - 1)^4} > 0.$$

According to EV-Theorem, for fixed $\sum_{i=1}^n a_i$ and $\sum_{1 \leq i < j \leq n} a_i a_j$, the sum $\sum_{i=1}^n \frac{1}{a_i + n - 1}$ is minimum for $a_1 = a_2 = \dots = a_{n-1} \leq a_n$. Thus, it suffices to consider $a_1 = a_2 = \dots = a_{n-1} = x$ and $a_n = y$, $x \leq y$. We need to prove that

$$\frac{n}{(n-1)x + y} - 1 + \frac{k}{n}[(n-2)x^2 + 2xy - n] < 0$$

involves

$$\frac{n-1}{x+n-1} + \frac{1}{y+n-1} > 1.$$

Applying again the contradiction method, it suffices to show that

$$\frac{n-1}{x+n-1} + \frac{1}{y+n-1} \leq 1$$

involves

$$\frac{n}{(n-1)x + y} - 1 + \frac{k}{n}[(n-2)x^2 + 2xy - n] \geq 0.$$

From the hypothesis, we get

$$y \geq \frac{n-1}{x} - n + 2.$$

Having in view Lemma below, we only need to prove the inequality for $y = \frac{n-1}{x} - n + 2$. Since

$$\frac{n}{(n-1)x + y} - 1 = \frac{-(n-1)(x-1)^2}{(n-1)(x^2+1) - (n-2)x}$$

and

$$(n-2)x^2 + 2xy - n = (n-2)(x-1)^2,$$

we need to show that

$$\frac{-(n-1)}{(n-1)(x^2+1) - (n-2)x} + \frac{k(n-2)}{n} \geq 0,$$

which is equivalent to

$$\frac{4(n-1)}{n(3n-4)} \geq \frac{1}{(n-1)(x^2+1) - (n-2)x},$$

$$[2(n-1)x - n + 2]^2 \geq 0.$$

The proof is completed. The equality occurs when $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = a_2 = \cdots = a_{n-1} = \frac{n-2}{2(n-1)}$ and $a_n = \frac{n^2-2}{n-2}$ (or any cyclic permutation)

Lemma. For fixed $x > 0$, define the function

$$f(y) = \frac{n}{(n-1)x+y} + \frac{2k}{n}xy,$$

where $n \geq 3$ and $k = \frac{4(n-1)^2}{(n-2)(3n-4)}$. The function f is strictly increasing for

$$y \geq \frac{n-1}{x} - n + 2.$$

Proof. We need to show that $f'(y) \geq 0$, that is

$$2kx[(n-1)x+y]^2 \geq n^2.$$

Since

$$(n-1)x+y \geq (n-1)x + \frac{n-1}{x} - n + 2 = (n-1)\left(x + \frac{1}{x}\right) - n + 2,$$

we need to prove that

$$2kx \left[(n-1)\left(x + \frac{1}{x}\right) - n + 2 \right]^2 \geq n^2.$$

In addition, since

$$(n-1)\left(x + \frac{1}{x}\right) - n + 2 \geq 2(n-1) - n + 2 = n,$$

it is sufficient to show that

$$2kx \left[(n-1)\left(x + \frac{1}{x}\right) - n + 2 \right] \geq n,$$

which is equivalent to

$$\frac{8(n-1)^2}{(n-2)(3n-4)} [(n-1)(x^2+1) - (n-2)x] \geq n.$$

Since

$$\frac{n-1}{n-2} > 1, \quad \frac{3(n-1)}{3n-4} > 1,$$

it suffices to show that

$$(n-1)(x^2+1) - (n-2)x \geq \frac{3}{8}n.$$

For $n = 3$, we have

$$(n-1)(x^2+1) - (n-2)x - \frac{3}{8}n = 2(x^2+1) - x - \frac{9}{8} > 2(x^2+1) - x - \frac{15}{8} = \frac{1}{8}(4x-1)^2 \geq 0.$$

Also, for $n \geq 4$, we have

$$(n-1)(x^2+1) - (n-2)x - \frac{3}{8}n > (n-2)(x^2-x+1) - \frac{3}{8}n \geq \frac{3}{4}(n-2) - \frac{3}{8}n = \frac{3}{8}(n-4) \geq 0.$$

□

P 5.112. If $a_1, a_2, \dots, a_n$ are positive real numbers such that $\sum_{1 \leq i < j \leq n} a_i a_j = \frac{n(n-1)}{2}$, then

$$n \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right) + n(n-2) \geq 2(n-1)(a_1 + a_2 + \dots + a_n).$$

(V. Cîrtoaje and L. Giugiuc, 2022)

Solution. For $n = 2$, we need to show that $a_1 a_2 = 1$ involves $a_1 a_2 \geq 1$, which is true. Consider further $n \geq 3$ and apply EV-Theorem (Corollary 5, case $k = 2$ and $m = -1$): For fixed $a_1 + a_2 + \dots + a_n$ and $\sum_{1 \leq i < j \leq n} a_i a_j = \frac{n(n-1)}{2}$, the sum $\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}$ is minimum when $a_1 = a_2 = \dots = a_{n-1} \leq a_n$. Thus, it suffices to consider $a_1 = a_2 = \dots = a_{n-1} = x$ and $a_n = y$ ($0 < x \leq 1 \leq y$). Thus, we need to show that

$$n \left(\frac{n-1}{x} + \frac{1}{y} \right) + n(n-2) \geq 2(n-1)[(n-1)x + y]$$

for

$$(n-2)x^2 + 2xy = n, \quad y = \frac{n - (n-2)x^2}{2x}.$$

Write the inequality to be proved as follows:

$$\begin{aligned} n \left(\frac{n-1}{x} + \frac{1}{y} - n \right) &\geq 2(n-1)[(n-1)x + y - n], \\ \frac{n^2(x-1)^2[(n-2)x + n-1]}{x[n - (n-2)x^2]} &\geq \frac{n(n-1)(x-1)^2}{x}. \end{aligned}$$

This is true if

$$\frac{n[(n-2)x + n-1]}{n - (n-2)x^2} \geq n-1,$$

which is equivalent to

$$(n-2)x[(n-1)x + n] \geq 0.$$

The proof is completed. The equality occurs for $a_1 = a_2 = \dots = a_n = 1$.

Remark 1. Since $a_1 + a_2 + \dots + a_n \geq n$, from P 5.112 we get

$$n \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right) + (n-2)(a_1 + a_2 + \dots + a_n) \geq 2(n-1)(a_1 + a_2 + \dots + a_n).$$

Thus, the following statement is true:

- If $a_1, a_2, \dots, a_n$ are positive real numbers such that $\sum_{1 \leq i < j \leq n} a_i a_j = \frac{n(n-1)}{2}$, then

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \geq a_1 + a_2 + \dots + a_n.$$

Remark 2. Since

$$a_1 + a_2 + \cdots + a_n \geq 2n - \frac{n^2}{a_1 + a_2 + \cdots + a_n},$$

from P 5.112 we get

- If $a_1, a_2, \dots, a_n$ are positive real numbers such that $\sum_{1 \leq i < j \leq n} a_i a_j = \frac{n(n-1)}{2}$, then

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} + \frac{2n(n-1)}{a_1 + a_2 + \cdots + a_n} \geq 3n - 2.$$

□

P 5.113. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers such that $\sum_{1 \leq i < j \leq n} a_i a_j = \frac{n(n-1)}{2}$, then

$$\frac{1}{a_1 + 1} + \frac{1}{a_2 + 1} + \cdots + \frac{1}{a_n + 1} - \frac{n}{2} + k \left(\frac{n}{a_1 + a_2 + \cdots + a_n} - 1 \right) \leq 0,$$

where

$$k = \frac{1}{2}(n-1)\sqrt{n(n-2)}.$$

(V. Cîrtoaje and L. Giugiuc, 2022)

Solution. For $n = 2$, the inequality is an identity. Consider further $n \geq 3$ and apply EV-Theorem (Corollary 1) to the function $f(u) = \frac{-1}{u+1}$, which satisfies

$$f'(u) = \frac{1}{(u+1)^2} := g(u), \quad g''(u) = \frac{6}{(u+1)^4} > 0.$$

According to EV-Theorem, for fixed $\sum_{i=1}^n a_i$ and $\sum_{1 \leq i < j \leq n} a_i a_j$, the sum $\sum_{i=1}^n \frac{1}{a_i + 1}$ is maximum for either $a_1 = a_2 = \cdots = a_{n-1} \geq a_n > 0$ or $a_n = 0$. Thus, it suffices to consider these cases.

Case 1: $a_1 = a_2 = \cdots = a_{n-1} \geq a_n > 0$. We need to show that

$$y = \frac{n - (n-2)x^2}{2x}, \quad 0 < x < \sqrt{\frac{n}{n-2}},$$

involves

$$\frac{n-1}{x+1} + \frac{1}{y+1} - \frac{n}{2} + k \left[\frac{n}{(n-1)x+y} - 1 \right] \leq 0,$$

which is equivalent to

$$\frac{n(n-2)(x-1)^2}{2[n+2x-(n-2)x^2]} - \frac{k(x-1)^2}{x^2+1} \leq 0.$$

This is true if

$$\frac{n(n-2)}{n+2x-(n-2)x^2} - \frac{2k}{x^2+1} \leq 0,$$

that is

$$\sqrt{n-2} [(n-1)\sqrt{n-2} + \sqrt{n}] x^2 - 2(n-1)x - \sqrt{n} [(n-1)\sqrt{n} - \sqrt{n-2}] \leq 0,$$

$$(x\sqrt{n-2} - \sqrt{n})(ax + b) \leq 0,$$

where

$$a = (n-1)\sqrt{n-2} + \sqrt{n}, \quad b = (n-1)\sqrt{n} - \sqrt{n-2}.$$

Case 2: $a_n = 0$. We need to prove that

$$\sum_{1 \leq i < j \leq n-1} a_i a_j = \frac{n(n-1)}{2}$$

involves

$$\frac{1}{a_1+1} + \frac{1}{a_2+1} + \cdots + \frac{1}{a_{n-1}+1} - \frac{n-2}{2} + k \left(\frac{n}{a_1+a_2+\cdots+a_{n-1}} - 1 \right) \leq 0,$$

which is equivalent to

$$\frac{a_1}{a_1+1} + \frac{a_2}{a_2+1} + \cdots + \frac{a_{n-1}}{a_{n-1}+1} - \frac{n}{2} \geq k \left(\frac{n}{a_1+a_2+\cdots+a_{n-1}} - 1 \right).$$

Denoting

$$a_1 + a_2 + \cdots + a_{n-1} = (n-1)S,$$

from the known inequality

$$(a_1 + a_2 + \cdots + a_{n-1})^2 \geq \frac{2(n-1)}{n-2} \sum_{1 \leq i < j \leq n-1} a_i a_j,$$

we get

$$S \geq \sqrt{\frac{n}{n-2}}.$$

On the other hand, by the Cauchy-Schwarz inequality, we have

$$\left[\sum_{i=1}^{n-1} a_i(a_i+1) \right] \left(\sum_{i=1}^{n-1} \frac{a_i}{a_i+1} \right) \geq \left(\sum_{i=1}^{n-1} a_i \right)^2,$$

hence

$$\begin{aligned} \sum_{i=1}^{n-1} \frac{a_i}{a_i+1} &\geq \frac{(n-1)^2 S^2}{\sum_{i=1}^{n-1} a_i^2 + (n-1)S} \\ &= \frac{(n-1)^2 S^2}{(n-1)^2 S^2 - 2 \sum_{1 \leq i < j \leq n-1} a_i a_j + (n-1)S} = \frac{(n-1)S^2}{(n-1)S^2 + S - n}. \end{aligned}$$

Therefore, it suffices to show that

$$\frac{(n-1)S^2}{(n-1)S^2 + S - n} - \frac{n}{2} \geq k \left[\frac{n}{(n-1)S} - 1 \right],$$

which is equivalent to

$$\frac{2k[(n-1)S - n]}{(n-1)S} \geq \frac{[(n-1)S - n][(n-2)S + n]}{(n-1)S^2 + S - n}.$$

Since

$$(n-1)S - n \geq (n-1)\sqrt{\frac{n}{n-2}} - n > 0,$$

we need to show that

$$\frac{2k}{(n-1)S} \geq \frac{(n-2)S + n}{(n-1)S^2 + S - n},$$

which is equivalent to

$$\begin{aligned} \sqrt{n(n-2)} &\geq \frac{(n-2)S^2 + nS}{(n-1)S^2 + S - n}, \\ (S\sqrt{n-2} - \sqrt{n})(aS + b) &\geq 0, \end{aligned}$$

where

$$a = (n-1)\sqrt{n} - \sqrt{n-2}, \quad b = n\sqrt{n-2}.$$

Clearly, the last inequality is true.

The proof is completed. The equality occurs for $a_1 = a_2 = \dots = a_n = 1$, and also for $a_1 = a_2 = \dots = a_{n-1} = \sqrt{\frac{n}{n-2}}$ and $a_n = 0$ (or any cyclic permutation).

□

P 5.114. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers such that $\sum_{1 \leq i < j \leq n} a_i a_j = \frac{n(n-1)}{2}$, then

$$\frac{1}{a_1 + 1} + \frac{1}{a_2 + 1} + \dots + \frac{1}{a_n + 1} - \frac{n}{2} + k \left(\frac{n}{a_1 + a_2 + \dots + a_n} - 1 \right) \geq 0,$$

where

$$k = \frac{n(n-2)}{2(n-1)^2} \left(\sqrt{n^2 - 2n + 2} - 1 \right).$$

(V. Cîrtoaje and L. Giugiuc, 2022)

Solution. For $n = 2$, we need to show that $a_1 a_2 = 1$ involves $a_1 a_2 \leq 1$, which is true. Consider further $n \geq 3$ and apply the EV-Theorem (Corollary 1) to the function $f(x) = \frac{-1}{x+1}$, which satisfies

$$f'(x) = \frac{1}{(x+1)^2} := g(x), \quad g''(x) = \frac{6}{(x+1)^4} > 0.$$

According to the EV-Theorem, for fixed $\sum_{i=1}^n a_i$ and $\sum_{1 \leq i < j \leq n} a_i a_j$, the sum $\sum_{i=1}^n \frac{1}{a_i + 1}$ is minimum for $a_1 = a_2 = \dots = a_{n-1} \leq a_n$. Thus, it suffices to consider that $a_1 = a_2 = \dots = a_{n-1} = x$ and $a_n = y$. So, we need to show that

$$\frac{(n-1)(n-2)}{2}x^2 + (n-1)xy = \frac{n(n-1)}{2},$$

that is

$$y = \frac{n - (n-2)x^2}{2x}, \quad 0 < x < \sqrt{\frac{n}{n-2}},$$

involves

$$\frac{n-1}{x+1} + \frac{1}{y+1} - \frac{n}{2} + k \left[\frac{n}{(n-1)x+y} - 1 \right] \geq 0,$$

which is equivalent to

$$\frac{n(n-2)(x-1)^2}{2[n+2x-(n-2)x^2]} - \frac{k(x-1)^2}{x^2+1} \geq 0.$$

This is true if

$$\frac{n(n-2)}{n+2x-(n-2)x^2} - \frac{2k}{x^2+1} \geq 0,$$

that is

$$\begin{aligned} (n-2)(2k+n)x^2 - 4kx + n(n-2-2k) &\geq 0, \\ \left[x - \frac{2k}{(n-2)(2k+n)} \right]^2 &\geq 0, \\ \left(x + n - 1 - \sqrt{n^2 - 2n + 2} \right)^2 &\geq 0. \end{aligned}$$

The proof is completed. The equality occurs for $a_1 = a_2 = \dots = a_n = 1$, and also for $a_1 = a_2 = \dots = a_{n-1} = \sqrt{n^2 - 2n + 2} - n + 1$ and $a_n = \sqrt{n^2 - 2n + 2} + (n-1)^2$ (or any cyclic permutation).

□

P 5.115. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers such that $\sum_{1 \leq i < j \leq n} a_i a_j = \frac{n(n-1)}{2}$, then

$$(n-2)(a_1 + a_2 + \dots + a_n) \geq 2 \left(\frac{1}{a_1+1} + \frac{1}{a_2+1} + \dots + \frac{1}{a_n+1} \right) + n(n-3).$$

(V. Cîrtoaje and L. Giugiuc, 2022)

Solution. For $n = 2$, we need to show that $a_1 a_2 = 1$ involves $a_1 a_2 \leq 1$, which is true. Consider further $n \geq 3$ and apply EV-Theorem (Corollary 1) to the function $f(u) = \frac{-1}{u+1}$, which satisfies

$$f'(u) = \frac{1}{(u+1)^2} := g(u), \quad g''(u) = \frac{6}{(u+1)^4} > 0.$$

According to EV-Theorem, for fixed $\sum_{i=1}^n a_i$ and $\sum_{1 \leq i < j \leq n} a_i a_j$, the sum $\sum_{i=1}^n \frac{1}{a_i + 1}$ is maximum for either $a_1 = a_2 = \cdots = a_{n-1} \geq a_n > 0$ or $a_n = 0$. Thus, it suffices to consider these cases.

Case 1: $a_1 = a_2 = \cdots = a_{n-1} \geq a_n > 0$. We need to show that

$$y = \frac{n - (n-2)x^2}{2x}, \quad 0 < x < \sqrt{\frac{n}{n-2}}$$

involves

$$(n-2)[(n-1)x + y] \geq 2 \left(\frac{n-1}{x+1} + \frac{1}{y+1} \right) + n(n-3),$$

which is equivalent to

$$\begin{aligned} (n-2)[(n-1)x + y - n] &\geq \frac{2(n-1)}{x+1} + \frac{2}{y+1} - n, \\ \frac{n(n-2)(x-1)^2}{2x} &\geq \frac{n(n-2)(x-1)^2}{n+2x-(n-2)x^2}, \\ n(n-2)(x-1)^2[n-(n-2)x^2] &\geq 0. \end{aligned}$$

Clearly, the last inequality is true.

Case 2: $a_n = 0$. We need to prove that

$$\sum_{1 \leq i < j \leq n-1} a_i a_j = \frac{n(n-1)}{2}$$

involves

$$(n-2) \sum_{i=1}^{n-1} a_i \geq 2 \sum_{i=1}^{n-1} \frac{1}{a_i + 1} + n^2 - 3n + 2,$$

which is equivalent to

$$(n-2) \sum_{i=1}^{n-1} a_i + 2 \sum_{i=1}^{n-1} \frac{a_i}{a_i + 1} \geq n(n-1).$$

Denoting

$$a_1 + a_2 + \cdots + a_{n-1} = (n-1)S,$$

from the known inequality

$$(a_1 + a_2 + \cdots + a_{n-1})^2 \geq \frac{2(n-1)}{n-2} \sum_{1 \leq i < j \leq n-1} a_i a_j,$$

we get

$$S \geq \sqrt{\frac{n}{n-2}}.$$

On the other hand, by the Cauchy-Schwarz inequality, we have

$$\left[\sum_{i=1}^{n-1} a_i(a_i + 1) \right] \left(\sum_{i=1}^{n-1} \frac{a_i}{a_i + 1} \right) \geq \left(\sum_{i=1}^{n-1} a_i \right)^2,$$

hence

$$\begin{aligned} \sum_{i=1}^{n-1} \frac{a_i}{a_i + 1} &\geq \frac{(n-1)^2 S^2}{\sum_{i=1}^{n-1} a_i^2 + (n-1)S} \\ &= \frac{(n-1)^2 S^2}{(n-1)^2 S^2 - 2 \sum_{1 \leq i < j \leq n-1} a_i a_j + (n-1)S} = \frac{(n-1)S^2}{(n-1)S^2 + S - n}. \end{aligned}$$

Therefore, it suffices to show that

$$(n-1)(n-2)S + \frac{2(n-1)S^2}{(n-1)S^2 + S - n} \geq n(n-1),$$

which is equivalent to

$$(n-1)(n-2)S^3 - n(n-2)S^2 - n(n-1)S + n^2 \geq 0,$$

$$[(n-1)S - n][(n-2)S^2 - n] \geq 0.$$

Since

$$(n-1)S - n \geq (n-1)\sqrt{\frac{n}{n-2}} - n > 0,$$

the last inequality is clearly true.

The proof is completed. The equality occurs for $a_1 = a_2 = \dots = a_n = 1$, and also for $a_1 = a_2 = \dots = a_{n-1} = \sqrt{\frac{n}{n-2}}$ and $a_n = 0$ (or any cyclic permutation).

□

P 5.116. If $a_1, a_2, \dots, a_n$ are positive real numbers such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} = n,$$

then

$$2(n+1) \left(\frac{1}{a_1+1} + \frac{1}{a_2+1} + \dots + \frac{1}{a_n+1} \right) \geq \frac{n^2(n-1)}{a_1+a_2+\dots+a_n} + 2n.$$

(V. Cîrtoaje, 2022)

Solution. For $n = 2$, we need to show that $a_1 + a_2 = 2a_1a_2$ involves $a_1a_2 \geq 1$. This follows from

$$0 \leq (a_1 - a_2)^2 = (a_1 + a_2)^2 - 4a_1a_2 = 4a_1^2a_2^2 - 4a_1a_2 = 4a_1a_2(a_1a_2 - 1).$$

Consider further $n \geq 3$ and apply the EV-Theorem (Corollary 2) to the function $f(x) = \frac{1}{x+1}$, which satisfies

$$f'(x) = \frac{-1}{(x+1)^2}, \quad g(x) = f'\left(\frac{1}{\sqrt{x}}\right) = \frac{-x}{(\sqrt{x}+1)^2},$$

$$g''(x) = \frac{3}{2\sqrt{x}(\sqrt{x}+1)^4} > 0.$$

According to the EV-Theorem, for fixed $a_1 + a_2 + \cdots + a_n$ and $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = n$, the sum $\sum_{i=1}^n \frac{1}{a_i + 1}$ is minimum when $a_1 \leq a_2 = a_3 = \cdots = a_n$. Thus, it suffices to consider this case. We only need to show that

$$\frac{1}{x} + \frac{n-1}{y} = n, \quad x \leq y,$$

that is

$$y = \frac{(n-1)x}{nx-1}, \quad x > \frac{1}{n},$$

involves

$$2(n+1) \left(\frac{1}{x+1} + \frac{n-1}{y+1} \right) \geq \frac{n^2(n-1)}{x+(n-1)y} + 2n,$$

that is

$$n(n-1) \left[1 - \frac{n}{x+(n-1)y} \right] \geq (n+1) \left(n - \frac{2}{x+1} - \frac{2n-2}{y+1} \right).$$

Since

$$1 - \frac{n}{x+(n-1)y} = \frac{(x-1)^2}{x(x+n-2)}$$

and

$$n - \frac{2}{x+1} - \frac{2n-2}{y+1} = \frac{n(x-1)^2}{(x+1)[(2n-1)x-1]},$$

the inequality becomes

$$\frac{n(n-1)(x-1)^2}{x(x+n-2)} \geq \frac{n(n+1)(x-1)^2}{(x+1)[(2n-1)x-1]}.$$

This inequality holds if

$$\frac{n-1}{x(x+n-2)} \geq \frac{n+1}{(x+1)[(2n-1)x-1]},$$

which is equivalent to

$$(nx-1)[2(n-2)x+n-1] \geq 0.$$

This inequality is clearly true.

The proof is completed. The equality occurs for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 5.117. If $a_1, a_2, \dots, a_n$ are positive real numbers such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = n,$$

then

$$a_1^2 + a_2^2 + \cdots + a_n^2 + \frac{2n^2}{a_1 + a_2 + \cdots + a_n} \geq 3n.$$

(V. Cîrtoaje, 2022)

Solution. For $n = 2$, we need to show that $a_1 + a_2 = 2a_1a_2$ involves

$$(a_1a_2 - 1)(2a_1^2a_2^2 + a_1a_2 - 2) \geq 0.$$

This is true if $a_1a_2 \geq 1$. Indeed, we have

$$0 \leq (a_1 - a_2)^2 = (a_1 + a_2)^2 - 4a_1a_2 = 4a_1^2a_2^2 - 4a_1a_2 = 4a_1a_2(a_1a_2 - 1),$$

hence $a_1a_2 \geq 1$. Consider further $n \geq 3$ and apply the EV-Theorem (Corollary 2) to the function $f(x) = x^2$, which satisfies

$$f'(x) = 2x, \quad g(x) = f'\left(\frac{1}{\sqrt{x}}\right) = \frac{2}{\sqrt{x}}, \quad g''(x) = \frac{3}{2x^2\sqrt{x}} > 0.$$

According to the EV-Theorem, for fixed $a_1 + a_2 + \cdots + a_n$ and $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = n$, the sum $a_1^2 + a_2^2 + \cdots + a_n^2$ is minimum when $a_1 \leq a_2 = a_3 = \cdots = a_n$. Thus, it suffices to consider this case. We only need to show that

$$\frac{1}{x} + \frac{n-1}{y} = n, \quad x \leq y,$$

that is

$$y = \frac{(n-1)x}{nx-1}, \quad \frac{1}{n} < x \leq 1,$$

involves

$$x^2 + (n-1)y^2 + \frac{2n^2}{x + (n-1)y} \geq 3n,$$

that is

$$x^2 + (n-1)y^2 - n \geq 2n \left[1 - \frac{n}{x + (n-1)y} \right].$$

Since

$$x^2 + (n-1)y^2 - n = \frac{n(x-1)^2[nx^2 + 2(n-1)x - 1]}{(nx-1)^2}$$

and

$$1 - \frac{n}{x + (n-1)y} = \frac{(x-1)^2}{x(x+n-2)},$$

the inequality becomes

$$\frac{n(x-1)^2[nx^2 + 2(n-1)x - 1]}{(nx-1)^2} \geq \frac{2n(x-1)^2}{x(x+n-2)}.$$

This inequality holds if

$$\frac{nx^2 + 2(n-1)x - 1}{(nx-1)^2} \geq \frac{2}{x(x+n-2)}.$$

Since $nx > 1$, we have

$$nx^2 + 2(n-1)x - 1 > x + 2(n-1)x - 1 = (2n-1)x - 1.$$

Thus, it suffices to show that

$$\frac{(2n-1)x-1}{(nx-1)^2} \geq \frac{2}{x(x+n-2)}.$$

which is equivalent to

$$(1-x)[3nx - (2n-1)x^2 - 2] \geq 0.$$

This inequality is true because

$$3nx - (2n-1)x^2 - 2 = nx(3-2x) + x^2 - 2 > (3-2x) + x^2 - 2 = (x-1)^2 \geq 0.$$

The proof is completed. The equality occurs for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 5.118. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$, then

$$\sum \frac{1}{a_1^2 + (n^2 - n + 1)(a_2^2 + \cdots + a_n^2)} \leq \frac{1}{n^2 - 2n + 2}.$$

(V. Cîrtoaje, 2022)

Solution. For $n = 2$, we need to show that $a_1 + a_2 = 2$ involves

$$3(a_1^4 + a_2^4) + 10a_1^2a_2^2 \geq 8(a_1^2 + a_2^2),$$

which is equivalent to

$$\begin{aligned} 3(a_1^4 + a_2^4) + 10a_1^2a_2^2 &\geq 2(a_1 + a_2)^2(a_1^2 + a_2^2), \\ (a_1 - a_2)^4 &\geq 0. \end{aligned}$$

Consider further $n \geq 3$ and write the inequality in the homogeneous form

$$\sum \frac{1}{(n^2 - n + 1)(a_1^2 + a_2^2 + \cdots + a_n^2) - n(n-1)a_1^2} \leq \frac{n^2}{(n^2 - 2n + 2)(a_1 + a_2 + \cdots + a_n)^2}.$$

Without loss of generality, assume that

$$a_1^2 + a_2^2 + \cdots + a_n^2 = \frac{n^2 - n}{n^2 - n + 1}.$$

We need to show that

$$f(a_1) + f(a_2) + \cdots + f(a_n) \leq \frac{n^3(n-1)}{(n^2 - 2n + 2)(a_1 + a_2 + \cdots + a_n)^2},$$

where

$$f(x) = \frac{1}{1-x^2}, \quad 0 \leq x < 1.$$

Since

$$g(x) = f'(x) = \frac{2x}{(1-x^2)^2}, \quad g''(x) = \frac{24x(x^2+1)}{(1-x^2)^4} \geq 0,$$

according to the EV-Theorem, for fixed $a_1 + a_2 + \cdots + a_n$ and $a_1^2 + a_2^2 + \cdots + a_n^2$, the sum $f(a_1) + f(a_2) + \cdots + f(a_n)$ is maximum when $a_1 = a_2 = \cdots = a_{n-1} \leq a_n$. Thus, it suffices to prove the homogeneous inequality for $a_1 = a_2 = \cdots = a_{n-1} = 0$ and $a_1 = a_2 = \cdots = a_{n-1} = 1$. In the first case, the inequality becomes

$$\frac{n-1}{n^2-n+1} + 1 \leq \frac{n^2}{n^2-2n+2},$$

$$n \geq 1.$$

In the second case, when $a_1^2 + a_2^2 + \cdots + a_n^2 = a_n^2 + n - 1$, we need to show that

$$\frac{n-1}{(n^2-n+1)a_n^2 + (n-1)^3} + \frac{1}{a_n^2 + (n-1)(n^2-n+1)} \leq \frac{n^2}{(n^2-2n+2)(a_n+n-1)^2}.$$

Using the substitutions $a_n = x$ and $n = k + 1$, the inequality can be written as follows:

$$\frac{k}{(k^2+k+1)x^2+k^3} + \frac{1}{x^2+k(k^2+k+1)} \leq \frac{(k+1)^2}{(k^2+1)(x+k)^2},$$

$$\frac{x^2+k^2}{[(k^2+k+1)x^2+k^3][x^2+k(k^2+k+1)]} \leq \frac{1}{(k^2+1)(x+k)^2},$$

$$x^4 - 2(k^2+1)x^3 + (k^4+4k^2+1)x^2 - 2k^2(k^2+1)x + k^4 \geq 0,$$

$$(x-1)^2(x-k^2)^2 \geq 0.$$

The proof is completed. The equality occurs for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = a_2 = \cdots = a_{n-1} = \frac{1}{n-1}$ and $a_n = n-1$ (or any cyclic permutation). □

P 5.119. If a, b, c are nonnegative real numbers such that $ab + bc + ca = 3$, then

$$\frac{1}{(5a+2b+2c)^2} + \frac{1}{(2a+5b+2c)^2} + \frac{1}{(2a+2b+5c)^2} \leq \frac{1}{27}.$$

(V. Cîrtoaje, 2022)

Solution. Write the inequality in the homogeneous form

$$\frac{1}{(5a+2b+2c)^2} + \frac{1}{(2a+5b+2c)^2} + \frac{1}{(2a+2b+5c)^2} \leq \frac{1}{9(ab+bc+ca)}.$$

Due to homogeneity, we may consider $a+b+c=3$, $0 \leq a \leq b \leq c$, when the inequality becomes

$$f(a) + f(b) + f(c) + \frac{1}{ab+bc+ca} \geq 0,$$

where

$$f(x) = \frac{-1}{(x+2)^2}, \quad x \geq 0.$$

To prove it, we apply Corollary 1 to the function f . We have

$$g(x) = f'(x) = \frac{2}{(x+2)^3}, \quad g''(x) = \frac{24}{(x+2)^5}.$$

Since $g''(x) > 0$, g is strictly convex. According to Corollary 1, if

$$a + b + c = 3, \quad ab + bc + ca = \text{fixed}, \quad 0 \leq a \leq b \leq c,$$

then the sum $f(a) + f(b) + f(c)$ is minimum for either $a = 0$ or $0 < a \leq b = c$. Thus, we only need to consider these cases.

Case 1: $a = 0$. The homogeneous inequality becomes

$$\frac{1}{4(b+c)^2} + \frac{1}{(5b+2c)^2} + \frac{1}{(5c+2b)^2} \leq \frac{1}{9bc}.$$

Since

$$\frac{1}{9bc} - \frac{1}{4(b+c)^2} \geq \frac{1}{9bc} - \frac{1}{16bc} > \frac{2}{48bc} > \frac{2}{49bc},$$

it suffices to show that

$$\frac{2}{49bc} \geq \frac{1}{(5b+2c)^2} + \frac{1}{(5c+2b)^2},$$

which is equivalent to

$$200(b^4 + c^4) - 261bc(b^2 + c^2) + 122b^2c^2 \geq 0,$$

$$200(b^2 + c^2)^2 - 261bc(b^2 + c^2) - 278b^2c^2 \geq 0,$$

$$(b^2 + c^2 - 2bc)(200b^2 + 200c^2 + 139bc) \geq 0.$$

Case 2: $b = c$. For $b = c = 1$, the homogeneous inequality becomes

$$\frac{1}{(5a+4)^2} + \frac{2}{(2a+7)^2} \leq \frac{1}{9(2a+1)},$$

$$100a^4 - 112a^3 - 21a^2 - 22a + 55 \geq 0,$$

$$(a-1)^2(100a^2 + 88a + 55) \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 5.120. If a, b, c are nonnegative real numbers such that $ab + bc + ca = 3$, then

$$\frac{1}{(3a+b+c)^2} + \frac{1}{(a+3b+c)^2} + \frac{1}{(a+b+3c)^2} \leq \frac{1}{8}.$$

(V. Cîrtoaje, 2022)

Solution. Write the inequality in the homogeneous form

$$\frac{1}{(3a+b+c)^2} + \frac{1}{(a+3b+c)^2} + \frac{1}{(a+b+3c)^2} \leq \frac{3}{8(ab+bc+ca)}.$$

As we showed in the previous P 5.119, it is sufficient to consider the cases $a = 0$ and $0 < a \leq b = c$.

Case 1: $a = 0$. The homogeneous inequality becomes

$$\frac{1}{(b+c)^2} + \frac{1}{(3b+c)^2} + \frac{1}{(3c+b)^2} \leq \frac{3}{8bc}.$$

Since

$$\frac{3}{8bc} - \frac{1}{(b+c)^2} \geq \frac{3}{8bc} - \frac{1}{4bc} = \frac{1}{8bc},$$

it suffices to show that

$$\frac{1}{8bc} \geq \frac{1}{(3b+c)^2} + \frac{1}{(3c+b)^2},$$

which is equivalent to

$$\begin{aligned} 9(b^2 + c^2)^2 - 20bc(b^2 + c^2) + 4b^2c^2 &\geq 0, \\ (b^2 + c^2 - 2bc)(9b^2 + 9c^2 - 2bc) &\geq 0. \end{aligned}$$

Case 2: $b = c$. For $b = c = 1$, the homogeneous inequality becomes

$$\begin{aligned} \frac{1}{(3a+2)^2} + \frac{2}{(a+4)^2} &\leq \frac{3}{8(2a+1)}. \\ a(27a^3 - 52a^2 + 68a + 32) &\geq 0. \end{aligned}$$

It is true since

$$27a^3 - 52a^2 + 68a + 32 > 27a^3 - 54a^2 + 27a = 27a(a-1)^2 \geq 0.$$

The equality holds for $a = 0$ and $b = c = \sqrt{3}$ (or any cyclic permutation).

□

P 5.121. If a, b, c are nonnegative real numbers such that $ab + bc + ca = 3$, then

$$\frac{1}{(a+4b+4c)^2} + \frac{1}{(4a+b+4c)^2} + \frac{1}{(4a+4b+c)^2} \leq \frac{1}{27}.$$

(V. Cîrtoaje, 2022)

Solution. Write the inequality in the homogeneous form

$$\frac{1}{(a+4b+4c)^2} + \frac{1}{(4a+b+4c)^2} + \frac{1}{(4a+4b+c)^2} \leq \frac{1}{9(ab+bc+ca)}.$$

Due to homogeneity, we may consider $a + b + c = 3$, $0 \leq a \leq b \leq c$, when the inequality becomes

$$f(a) + f(b) + f(c) \leq \frac{1}{ab + bc + ca},$$

where

$$f(x) = \frac{1}{(4-x)^2}, \quad x \in [0, 1].$$

To prove it, we apply Corollary 1 to the function f . We have

$$g(x) = f'(x) = \frac{2}{(4-x)^3}, \quad g''(x) = \frac{24}{(4-x)^5}.$$

Since $g''(x) > 0$, g is strictly convex. According to Corollary 1, if

$$a + b + c = 3, \quad ab + bc + ca = \text{fixed}, \quad 0 \leq a \leq b \leq c,$$

then the sum $f(a) + f(b) + f(c)$ is maximum for $a = b \leq c$. Thus, we only need to prove the homogeneous inequality for $a = b = 1$, that is

$$\frac{2}{4c+1} + \frac{1}{c+8} \leq \frac{1}{9(2c+1)},$$

$$16c^4 - 28c^3 + 231c^2 - 442c + 223 \geq 0,$$

$$(c-1)^2(16c^2 + 4c + 223) \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 5.122. If $n \geq 3$, $1 < k \leq n$ and $a_1, a_2, \dots, a_n$ are nonnegative real numbers, then

$$(n-1)^{k-1}(a_1^k + \dots + a_n^k) + n[n^{k-1} - (n-1)^{k-1}](a_1 \dots a_n)^{k/n} \geq (a_1 + a_2 + \dots + a_n)^k.$$

(Petru Mironescu, 2022)

Solution. According to Corollary 4, if $0 \leq a_1 \leq a_2 \leq \dots \leq a_n$ and

$$a_1 + a_2 + \dots + a_n = n, \quad a_1^k + a_2^k + \dots + a_n^k = \text{fixed},$$

then the product $P = a_1 a_2 \dots a_n$ has the minimum value for either $a_n = 0$ or $a_1 = \dots = a_{n-1} \geq a_n > 0$. Thus, it suffices to prove the desired inequality for these two cases.

Case 1: $a_n = 0$. We need to show that

$$(n-1)^{k-1}(a_1^k + \dots + a_{n-1}^k) \geq (a_1 + a_2 + \dots + a_{n-1})^k,$$

which is a consequence of the power mean inequality.

Case 2: $a_1 = \dots = a_{n-1} \geq a_n > 0$. Due to homogeneity, we need to prove that $f(x) \geq 0$ for $x \geq 1$, where

$$f(x) = (n-1)^{k-1}[(n-1)x^k + 1] + nAx^{k(n-1)/n} - [(n-1)x + 1]^k, \quad A = n^{k-1} - (n-1)^{k-1} > 0.$$

We have

$$f'(x) = k(n-1)^k x^{k-1} + k(n-1)Ax^{k(n-1)/n-1} - k(n-1)[(n-1)x+1]^{k-1} = k(n-1)x^{k-1}g(x),$$

where

$$g(x) = (n-1)^{k-1} + Ax^{-k/n} - \left(n-1 + \frac{1}{x}\right)^{k-1},$$

with

$$g'(x) = \frac{k-1}{x^2} \left(n-1 + \frac{1}{x}\right)^{k-2} - \frac{k}{n}Ax^{-k/n-1} = \frac{1}{nx^2} \left(n-1 + \frac{1}{x}\right)^{k-2} h(x),$$

where

$$h(x) = (k-1)n - kAh_1(x), \quad h_1(x) = x^{1-k/n} \left(n-1 + \frac{1}{x}\right)^{2-k}.$$

We will show that $h_1(x)$ is strictly increasing for $x \geq 1$. If $k \geq 2$, this is obvious. For $1 < k < 2$, we have

$$\begin{aligned} h'_1(x) &= \left(1 - \frac{k}{n}\right) x^{-k/n} \left(n-1 + \frac{1}{x}\right)^{2-k} - (2-k)x^{-1-k/n} \left(n-1 + \frac{1}{x}\right)^{1-k} \\ &= x^{-1-k/n} \left(n-1 + \frac{1}{x}\right)^{1-k} h_2(x), \end{aligned}$$

where

$$h_2(x) = \left(1 - \frac{k}{n}\right) [(n-1)x+1] - 2 + k.$$

Since $h_2(x) \geq h_2(1) = n-2 > 0$, we have $h'_1(x) > 0$, $h_1(x)$ is strictly increasing and $h(x)$ is strictly decreasing. We will show that $h(1) > 0$. We have

$$\frac{h(1)}{n} = k \left(1 - \frac{1}{n}\right)^{k-1} - 1, \quad \frac{n-1}{n^2}h(1) = k \left(1 - \frac{1}{n}\right)^k - 1 + \frac{1}{n}.$$

If $2 \leq k \leq n$, by Bernoulli's inequality, we get

$$\frac{h(1)}{n} > k \left(1 - \frac{k-1}{n}\right) - 1 = \frac{(k-1)(n-k)}{n} \geq 0.$$

Also, for $1 < k < 2$, we have

$$\frac{n-1}{n^2}h(1) > k \left(1 - \frac{k}{n}\right) - 1 + \frac{1}{n} = \frac{(k-1)(n-k-1)}{n} > \frac{(k-1)(n-3)}{n} \geq 0.$$

Since $h(x)$ is strictly decreasing, $h(1) > 0$, $\lim_{x \rightarrow \infty} h(x) < 0$, there is $x_1 > 1$ such that $h(x_1) = 0$, $h(x) > 0$ for $x \in [1, x_1]$, and $h(x) < 0$ for $x \in (x_1, \infty)$. Since $g'(x)$ has the same sign as $h(x)$, $g(x)$ is increasing on $[1, x_1]$ and decreasing on $[x_1, \infty)$. So, from

$$g(1) = 0, \quad \lim_{x \rightarrow \infty} g(x) = 0,$$

it follows that $g(x) \geq 0$, therefore $f(x)$ is increasing and $f(x) \geq f(1) = 0$.

The proof is completed. The equality occurs for $a_1 = a_2 = \cdots = a_n$ and also for $a_1 = \cdots = a_{n-1}$ and $a_n = 0$ (or any cyclic permutation).

Remark 1. The inequality is also true for $n = 2$, when it becomes as follows:

$$\frac{x^k + x^{-k}}{2} + k - 1 \geq k \left(\frac{x + x^{-1}}{2} \right)^k,$$

where $k \geq 2$ and $x > 0$.

Remark 2. For $k = 2$, we get Shleifer's inequality:

$$(n-1)(a_1^2 + a_2^2 + \cdots + a_n^2) + n(a_1 a_2 \cdots a_n)^{2/n} \geq (a_1 + a_2 + \cdots + a_n)^2.$$

For $n = 4$, Shleifer's inequality turns into Turkevich's inequality:

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 + 2\sqrt{a_1 a_2 a_3 a_4} \geq a_1 a_2 + a_1 a_3 + a_1 a_4 + a_2 a_3 + a_2 a_4 + a_3 a_4.$$

Remark 3. In the same conditions, the following inequality (Petru Mironescu) is also true:

$$(n-1)^{1-1/k}(a_1^k + \cdots + a_n^k)^{1/k} + n^{1/k}[n^{1-1/k} - (n-1)^{1-1/k}](a_1 \cdots a_n)^{1/n} \geq a_1 + a_2 + \cdots + a_n,$$

with equality for $a_1 = a_2 = \cdots = a_n$ and also for $a_1 = \cdots = a_{n-1}$ and $a_n = 0$ (or any cyclic permutation).

For $a_1 a_2 \cdots a_n = 0$, the inequality is a consequence of the power mean inequality. For $a_1 a_2 \cdots a_n > 0$, the inequality follows from P 5.122. Due to homogeneity, we may assume $a_1 a_2 \cdots a_n = 1$. Denoting $m = \frac{n-1}{n} < 1$ and $x = \frac{a_1 + a_2 + \cdots + a_n}{n} \geq 1$, the inequality in P 5.122 and the desired inequality become

$$(n-1)^{k-1}(a_1^k + \cdots + a_n^k) \geq n^k(x^k - 1 + m^{k-1})$$

and

$$(n-1)^{k-1}(a_1^k + \cdots + a_n^k) \geq n^k(x - 1 - m^{1-1/k})^k,$$

respectively. So, it suffices to show that

$$x^k - 1 + m^{k-1} \geq (x - 1 + m^{1-1/k})^k,$$

which is equivalent to $f(x) \geq 0$, where

$$f(x) = x^k - 1 + m^{k-1} - (x - 1 + m^{1-1/k})^k.$$

Since $k > 1$, $m < 1$ and $x \geq 1$, we have

$$\frac{1}{k}f'(x) = x^{k-1} - (x - 1 + m^{1-1/k})^{k-1} > x^{k-1} - x^{k-1} = 0,$$

$f(x)$ is increasing, therefore $f(x) \geq f(1) = 0$.

□

P 5.123. If $a_1, a_2, \dots, a_n$ ($n \geq 3$) are nonnegative real numbers such that

$$\sum_{i=1}^n \frac{1}{(n-1)a_i + 1} = 1.$$

then

$$a_1 + a_2 + \dots + a_n \geq n \sqrt[n-2]{a_1 a_2 \dots a_n}.$$

(V. Cîrtoaje, 2022)

Solution. Consider $a_1 \leq a_2 \leq \dots \leq a_n$. If $n \geq 4$, we may apply Theorem 2 for $m = 1$. So, since for fixed $a_1 + a_2 + \dots + a_n$, the product $a_1 a_2 \dots a_n$ has the maximum value when $a_1 \leq a_2 = a_3 = \dots = a_n$, we only need to show that if

$$\frac{1}{(n-1)x + 1} + \frac{n-1}{(n-1)y + 1} = 1, \quad 0 < x \leq y,$$

that is

$$y = \frac{(n-2)x + 1}{(n-1)x},$$

then

$$x + (n-1)y \geq n \sqrt[n-2]{xy^{n-1}},$$

that is

$$x + (n-1)y \geq ny \sqrt[n-2]{xy},$$

$$x^2 + (n-2)x + 1 \geq \frac{n}{n-1} [(n-2)x + 1] \sqrt[n-2]{\frac{(n-2)x + 1}{n-1}}.$$

By Bernoulli's inequality, we have

$$\sqrt[n-2]{\frac{(n-2)x + 1}{n-1}} = \sqrt[n-2]{1 + \frac{(n-2)(x-1)}{n-1}} \leq 1 + \frac{x-1}{n-1} = \frac{x+n-2}{n-1}.$$

So, it suffices to show that

$$x^2 + (n-2)x + 1 \geq \frac{n}{(n-1)^2} [(n-2)x + 1](x+n-2),$$

which is equivalent to $(x-1)^2 \geq 0$.

For $n = 3$, we need to show that

$$a_1 + a_2 + a_3 \geq 3a_1 a_2 a_3$$

for

$$\frac{1}{2a_1 + 1} + \frac{1}{2a_2 + 1} + \frac{1}{2a_3 + 1} = 1,$$

that is

$$4a_1 a_2 a_3 = a_1 + a_2 + a_3 + 1.$$

Denote $t = \sqrt[3]{a_1 a_2 a_3}$. From the AM-GM inequality, we have

$$4t^3 = a_1 + a_2 + a_3 + 1 \geq 3t + 1,$$

$$(t-1)(2t+1)^2 \geq 0,$$

hence $a_1 a_2 a_3 \geq 1$. Finally, we get

$$a_1 + a_2 + a_3 - 3a_1 a_2 a_3 = a_1 a_2 a_3 - 1 \geq 0.$$

The equality occurs for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 5.124. If $a_1, a_2, \dots, a_n$ ($n \geq 3$) are nonnegative real numbers such that

$$\sum_{i=1}^n \frac{1}{(n-1)a_i + 1} = 1,$$

then

$$a_1 + a_2 + \cdots + a_n - n \leq k(a_1 a_2 \cdots a_n - 1),$$

where $k = \left(\frac{n-1}{n-2}\right)^{n-1}$.

(V. Cîrtoaje and L. Giugiuc, *Creat. Math. Inform.*, 2, 2022)

Solution. Consider $a_1 \leq a_2 \leq \cdots \leq a_n$. For $n = 3$, the inequality is an identity. Consider further $n \geq 4$. According to Theorem 2 (case $m = 1$), for fixed $a_1 + a_2 + \cdots + a_n$, the product $a_1 a_2 \cdots a_n$ is minimum when $a_1 = a_2 = \cdots = a_{n-1} \leq a_n$. We need to show that if

$$\frac{n-1}{(n-1)x+1} + \frac{1}{(n-1)y+1} = 1,$$

which leads to

$$y = \frac{1}{(n-1)x - n + 2}, \quad \frac{n-2}{n-1} < x \leq y,$$

then

$$(n-1)x + y - n \leq k(x^{n-1}y - 1),$$

which is equivalent to

$$1 + [(n-1)x - n + 2][(n-1)x - n] \leq k[x^{n-1} - (n-1)x + n - 2], \quad (*)$$

or

$$(n-1)^2(x-1)^2 \leq kf(x), \quad f(x) = x^{n-1} - 1 - (n-1)(x-1).$$

Since

$$f(x) = (x-1)(x^{n-2} + x^{n-3} + \cdots + x - n + 2) = (x-1)^2 g(x),$$

where

$$g(x) = x^{n-3} + 2x^{n-4} + \cdots + (n-2),$$

we only need to show that

$$(n-1)^2 \leq kg(x).$$

Since g is an increasing function, it suffices to show that

$$(n-1)^2 \leq kg \left(\frac{n-2}{n-1} \right).$$

This inequality is true if the inequality (*) holds for $x = \frac{n-2}{n-1}$. Indeed, in this case, (**) is an identity.

For $n \geq 4$, the equality occurs when $a_1 = a_2 = \dots = a_n = 1$.

Remark. From P 5.123 and P 5.124, we get the following nice double inequality:

- If $a_1, a_2, \dots, a_n$ ($n \geq 4$) are nonnegative real numbers such that

$$\sum_{i=1}^n \frac{1}{(n-1)a_i + 1} = 1,$$

then

$$\sqrt[n-2]{a_1 a_2 \dots a_n} \leq \frac{a_1 + a_2 + \dots + a_n}{n} \leq a_1 a_2 \dots a_n.$$

By the AM-HM inequality

$$\left[\sum_{i=1}^{n-1} ((n-1)a_i + 1) \right] \left(\sum_{i=1}^{n-1} \frac{1}{(n-1)a_i + 1} \right) \geq n^2,$$

we get $a_1 + a_2 + \dots + a_n \geq n$. As a consequence, the inequality in P 5.124 involves

$$a_1 a_2 \dots a_n \geq 1.$$

So, denoting $p = a_1 a_2 \dots a_n$ ($p \geq 1$), the inequality in P 5.124 leads to

$$na_1 a_2 \dots a_n - (a_1 + a_2 + \dots + a_n) \geq np - k(p-1) - n = (n-k)(p-1) \geq 0.$$

□

P 5.125. If $a_1, a_2, \dots, a_n$ ($n \geq 3$) are nonnegative real numbers such that

$$\sum_{i=1}^n \frac{1}{(n-1)a_i + 1} = 1,$$

then

$$(n-2)(a_1 + a_2 + \dots + a_n) + a_1 a_2 \dots a_n \geq (n-1)^2.$$

(V. Cîrtoaje and L. Giugiuc, *Creat. Math. Inform.*, 2, 2022)

Solution. Consider $a_1 \geq a_2 \geq \cdots \geq a_n$. According to Theorem 2 (case $m = n - 1$), for fixed $a_1 + a_2 + \cdots + a_n$, the product $a_1 a_2 \cdots a_n$ has the minimum value for either $a_1 = a_2 = \cdots = a_{n-1} \geq a_n > 0$ or $a_n = 0$. Thus, it suffices to consider these cases.

Case 1: $a_1 = a_2 = \cdots = a_{n-1} \geq a_n > 0$. We need to show that if

$$\frac{n-1}{x+1} + \frac{1}{y+1} = 1,$$

which leads to

$$y = \frac{n-1-(n-2)x}{x}, \quad 0 < y \leq x < \frac{n-1}{n-2},$$

then

$$(n-2)[(n-1)x + y] + x^{n-1}y \geq (n-1)^2,$$

which is equivalent to

$$(n-2)y + x^{n-1}y \geq (n-1)[n-1-(n-2)x].$$

Since $n-1-(n-2)x = xy$, we only need to show that

$$n-2+x^{n-1} \geq (n-1)x,$$

which is just the AM-GM inequality.

Case 2: $a_n = 0$. We need to show that

$$\sum_{i=1}^{n-1} \frac{1}{a_i + n-1} = \frac{n-2}{n-1}$$

involves

$$(n-2)(a_1 + a_2 + \cdots + a_{n-1}) \geq (n-1)^2.$$

This follows immediately from the AM-HM inequality

$$\left[\sum_{i=1}^{n-1} (a_i + n-1) \right] \left(\sum_{i=1}^{n-1} \frac{1}{a_i + n-1} \right) \geq (n-1)^2.$$

The proof is completed. The equality occurs for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = a_2 = \cdots = a_{n-1} = \frac{n-1}{n-2}$ and $a_n = 0$ (or any cyclic permutation). □

P 5.126. If $a_1, a_2, \dots, a_n$ ($n \geq 3$) are nonnegative real numbers such that

$$\sum_{i=1}^n \frac{1}{2a_i + n-2} = 1,$$

then

$$a_1 + a_2 + \cdots + a_n - n \geq 2^{n-1}(a_1 a_2 \cdots a_n - 1).$$

(V. Cîrtoaje and L. Giugiuc, *Creat. Math. Inform.*, 2, 2022)

Solution. Consider $a_1 \geq a_2 \geq \cdots \geq a_n$. For $n = 3$, the inequality is an identity. For $n \geq 4$, according to Theorem 1 (case $m = n - 2$), for fixed $a_1 + a_2 + \cdots + a_n$, the product $a_1 a_2 \cdots a_n$ attains its maximum value when $a_1 \geq a_2 = a_3 = \cdots = a_n$. Thus, we only need to show that

$$y + (n - 1)x - n \geq 2^{n-1}(yx^{n-1} - 1)$$

for

$$\frac{1}{2y + n - 2} + \frac{n - 1}{2x + n - 2} = 1,$$

which implies

$$y = \frac{n - 2 - (n - 3)x}{2x - 1}, \quad \frac{1}{2} < x \leq y.$$

The required inequality is equivalent to

$$\begin{aligned} & \frac{n - 2 - (n - 3)x + (2x - 1)[(n - 1)x - n]}{2^{n-1}} \geq \\ & \geq (n - 2)(x^{n-1} - 1) - (n - 3)(x^n - 1) - 2(x - 1), \end{aligned} \quad (*)$$

or

$$\frac{(n - 1)(x - 1)^2}{2^{n-2}} \geq (x - 1)f(x),$$

where

$$\begin{aligned} f(x) &= (n - 2)(x^{n-2} + x^{n-3} + \cdots + x + 1) - (n - 3)(x^{n-1} + x^{n-2} + \cdots + x + 1) - 2 \\ &= (n - 2)[(x^{n-2} - 1) + (x^{n-3} - 1) + \cdots + (x - 1)] - (n - 3)[(x^{n-1} - 1) + (x^{n-2} - 1) + \cdots + (x - 1)] \\ &= (x - 1)g(x), \\ g(x) &= (n - 2)[x^{n-3} + 2x^{n-4} + \cdots + (n - 2)] - (n - 3)[x^{n-2} + 2x^{n-3} + \cdots + (n - 1)] \\ &= -(n - 3)x^{n-2} - (n - 4)x^{n-3} - \cdots - x^2 + 1. \end{aligned}$$

So, we only need to show that

$$\frac{n - 1}{2^{n-2}} \geq g(x).$$

Since g is a decreasing function, it suffices to show that

$$\frac{n - 1}{2^{n-2}} \geq g\left(\frac{1}{2}\right).$$

This is true if the inequality $(*)$ holds for $x = \frac{1}{2}$. It is easy to show that $(*)$ is an identity.

For $n \geq 4$, the equality occurs when $a_1 = a_2 = \cdots = a_n = 1$.

□

P 5.127. Let $n \geq 3$, and let $a_1 \geq a_2 \geq \cdots \geq a_n > 0$ such that $\sum_{1 \leq i < j \leq n} a_i a_j = \frac{n(n-1)}{2}$. Then

$$(a) \quad \frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2} - n \geq \left(3 - \frac{4}{n}\right) \left(\frac{a_{n-1}}{a_n} + \frac{a_n}{a_{n-1}} - 2\right);$$

$$(b) \quad \frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2} - n \geq \left(3 - \frac{4}{n}\right) (a_{n-1} - a_n)^2.$$

(V. Cîrtoaje, 2024)

Solution. Since $a_{n-1}a_n \leq 1$, the inequality (b) follows from the inequality (a). We write now the equality constraint as follows:

$$2 \sum_{1 \leq i < j \leq n-1} a_i a_j + 2a_n \sum_{i=1}^{n-1} a_i = n(n-1),$$

$$\left(\sum_{i=1}^{n-1} a_i\right)^2 - \sum_{i=1}^{n-1} a_i^2 + 2a_n \sum_{i=1}^{n-1} a_i = n(n-1),$$

and fix a_n , $\sum_{i=1}^{n-1} a_i$ and $\sum_{i=1}^{n-1} a_i^2$. To prove the inequality (a), we denote

$$E(a_{n-1}) = \frac{a_{n-1}}{a_n} + \frac{a_n}{a_{n-1}} - 2,$$

will use Lemma below and Corollary 5 for $a_1 \geq a_2 \geq \cdots \geq a_{n-1} > 0$, $k = 2$ and $m = -2$:

- For $a_1 \geq a_2 \geq \cdots \geq a_{n-1} > 0$, if $S_1 = \sum_{i=1}^{n-1} a_i$ and $S_2 = \sum_{i=1}^{n-1} a_i^2$ are fixed, then the sum

$$\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_{n-1}^2}$$

has the minimum value when $a_1 \geq a_2 = \cdots = a_{n-1}$.

On the other hand,

$$E'(a_{n-1}) = \frac{1}{a_n} - \frac{a_n}{a_{n-1}^2} \geq 0,$$

$E(a_{n-1})$ is increasing and has the maximum value when a_{n-1} is maximum, hence when $a_1 \geq a_2 = \cdots = a_{n-1}$ (see Lemma below). So, it suffices to prove the inequality (a) for $a_1 \geq a_2 = \cdots = a_{n-1}$. To do it, write the inequality in the homogeneous form

$$2 \left(\sum_{1 \leq i < j \leq n} a_i a_j \right) \left(\sum_{i=1}^n \frac{1}{a_i^2} \right) - n^2(n-1) \geq \frac{(n-1)(3n-4)(a_{n-1} - a_n)^2}{a_{n-1}a_n}.$$

Due to homogeneity, we may set $a_n = 1$. Denoting $a_1 := x$ and $a_2 = \cdots = a_{n-1} := y$, we need to prove that $x \geq y \geq 1$ implies $f(x) \geq 0$, where

$$f(x) = [2(n-2)xy + 2x + (n-2)(n-3)y^2 + 2(n-2)y] \left(\frac{1}{x^2} + \frac{n-2}{y^2} + 1 \right)$$

$$-n^2(n-1) - \frac{(n-1)(3n-4)(y-1)^2}{y}.$$

We have

$$\begin{aligned} \frac{f'(x)}{2} &= [(n-2)y+1] \left(\frac{1}{x^2} + \frac{n-2}{y^2} + 1 \right) - \frac{2(n-2)xy + 2x + (n-2)(n-3)y^2 + 2(n-2)y}{x^3}, \\ \frac{f''(x)}{2} &= \frac{-2[(n-1)y+1]}{x^3} + \frac{4(n-2)y+4}{x^3} + \frac{3(n-2)[(n-3)y^2+2y]}{x^4} \\ &= \frac{2x[(n-2)y+1] + 3(n-2)[(n-3)y^2+2y]}{x^4}. \end{aligned}$$

Since $f''(x) > 0$, $f'(x)$ is increasing, therefore

$$\begin{aligned} \frac{f'(x)}{2} &\geq \frac{f'(y)}{2} = [(n-2)y+1] \left(\frac{n-1}{y^2} + 1 \right) - \frac{2(n-2)y^2 + 2y + (n-2)(n-3)y^2 + 2(n-2)y}{y^3} \\ &= \frac{(n-2)y^3 + y^2 - n + 1}{y^2} = \frac{(y-1)[(n-2)y^2 + (n-1)y + n-1]}{y^2} \geq 0. \end{aligned}$$

Since $f'(x) \geq 0$, $f(x)$ is increasing, therefore

$$\begin{aligned} \frac{f(x)}{n-1} &\geq \frac{f(y)}{n-1} = [(n-2)y^2+2y] \left(\frac{n-1}{y^2} + 1 \right) - n^2 - \frac{(3n-4)(y-1)^2}{y} \\ &= \frac{[(n-2)y+2n-2](y-1)^2 - (3n-4)(y-1)^2}{y} = \frac{(n-2)(y-1)^3}{y} \geq 0. \end{aligned}$$

The proof is completed. The inequalities (a) and (b) become equalities for $a_1 = a_2 = \dots = a_n = 1$.

Lemma. Let $n \geq 3$, and let the fixed sums $S_1 = a_1 + a_2 + \dots + a_{n-1}$ and $S_2 = a_1^2 + a_2^2 + \dots + a_{n-1}^2$, where $a_1, a_2, \dots, a_{n-1}$ are positive real numbers satisfying $a_1 \geq a_2 \geq \dots \geq a_{n-1}$. Then the number a_{n-1} has the maximum value when $a_1 \geq a_2 = \dots = a_{n-1}$.

Proof. Let

$$S = \frac{a_2 + \dots + a_{n-1}}{n-2},$$

hence

$$S_1 = a_1 + (n-2)S.$$

From the known inequality

$$a_2^2 + \dots + a_{n-1}^2 \geq (n-2)S^2$$

(with equality for $a_2 = \dots = a_{n-1}$), we have

$$S_2 \geq a_1^2 + (n-2)S^2 = [S_1 - (n-2)S]^2 + (n-2)S^2,$$

therefore

$$\begin{aligned} (n-1)(n-2)S^2 - 2(n-2)S_1S + S_1^2 - S_2 &\leq 0, \\ (n-1)S &\leq S_1 - \sqrt{S_1^2 - (n-1)/(n-2) \cdot (S_1^2 - S_2)}. \end{aligned}$$

Since $a_{n-1} \leq S$ (with equality for $a_2 = \cdots = a_{n-1}$), we obtain

$$(n-1)a_{n-1} \leq S_1 - \sqrt{S_1^2 - (n-1)/(n-2) \cdot (S_1^2 - S_2)},$$

with equality for $a_2 = \cdots = a_{n-1}$.

□

Chapter 6

EV Method for Real Variables

6.1 Theoretical Basis

The Equal Variables Method may be extended to solve some difficult symmetric inequalities in real variables.

EV-Theorem (*Vasile Cîrtoaje*, 2010). *Let $x_1, x_2, \dots, x_n$ ($n \geq 3$) be real numbers such that $x_1 \leq x_2 \leq \dots \leq x_n$, let k be an even positive integer, and let f be a differentiable function on $\mathbb{R}$ so that the joined function $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by*

$$g(x) = f' \left(\sqrt[k-1]{x} \right)$$

is strictly convex on $\mathbb{R}$. For fixed $x_1 + x_2 + \dots + x_n$ and $x_1^k + x_2^k + \dots + x_n^k$, the sum

$$S_n = f(x_1) + f(x_2) + \dots + f(x_n)$$

has the minimum value for $x_2 = x_3 = \dots = x_n$, and the maximum value for $x_1 = x_2 = \dots = x_{n-1}$.

To prove this theorem, we will use EV-Lemma and EV-Proposition below.

EV-Lemma. *Let x, y, z be real numbers such that $x \leq y \leq z$. For fixed $x+y+z$ and $x^k+y^k+z^k$, where k is an even positive integer, there exist two real numbers m and M so that*

- (1) $y \in [m, M]$;
- (2) $y = m$ if and only if $x = y$;
- (3) $y = M$ if and only if $y = z$.

Proof. We may consider x and z as functions of y . From

$$x' + z' = -1, \quad x^{k-1}x' + z^{k-1}z' = -y^{k-1},$$

we get

$$x' = \frac{y^{k-1} - z^{k-1}}{z^{k-1} - x^{k-1}}, \quad z' = \frac{y^{k-1} - x^{k-1}}{x^{k-1} - z^{k-1}}. \quad (*)$$

The two-sided inequality

$$x(y) \leq y \leq z(y)$$

is equivalent to the inequalities $f_1(y) \leq 0$ and $f_2(y) \geq 0$, where

$$f_1(y) = x(y) - y, \quad f_2(y) = z(y) - y.$$

Using (*), we get

$$f'_1(y) = \frac{y^{k-1} - z^{k-1}}{z^{k-1} - x^{k-1}} - 1$$

and

$$f'_2(y) = \frac{y^{k-1} - x^{k-1}}{x^{k-1} - z^{k-1}} - 1.$$

Since $f'_1(y) \leq -1$ and $f'_2(y) \leq -1$, f_1 and f_2 are strictly decreasing. Thus, the inequality $f_1(y) \leq 0$ involves $y \geq m$, where m is the root of the equation $x(y) = y$, while the inequality $f_2(y) \geq 0$ involves $y \leq M$, where M is the root of the equation $z(y) = y$. Moreover, $y = m$ if and only if $x = y$, and $y = M$ if and only if $y = z$.

EV-Proposition. *Let x, y, z be real numbers such that $x \leq y \leq z$, let k be an even positive integer, and let f be a differentiable function on $\mathbb{R}$ so that the joined function $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by*

$$g(x) = f' \left(\sqrt[k]{x} \right)$$

is strictly convex on $\mathbb{R}$. For fixed $x + y + z$ and $x^k + y^k + z^k$, the sum

$$S = f(x) + f(y) + f(z)$$

has the minimum value for $y = z$, and the maximum value for $x = y$.

Proof. Assume that x and z are functions of y . Thus, we have

$$S = f(x(y)) + f(y) + f(z(y)) := F(y).$$

According to EV-Lemma, it suffices to show that $F(y)$ is maximum for $y = m$ and is minimum for $y = M$. Using (*), we have

$$\begin{aligned} F'(y) &= x'f'(x) + f'(y) + z'f'(z) \\ &= \frac{y^{k-1} - z^{k-1}}{z^{k-1} - x^{k-1}}g(x^{k-1}) + g(y^{k-1}) + \frac{y^{k-1} - x^{k-1}}{x^{k-1} - z^{k-1}}g(z^{k-1}), \end{aligned}$$

hence

$$\begin{aligned} \frac{F'(y)}{(y^{k-1} - x^{k-1})(y^{k-1} - z^{k-1})} &= \frac{g(x^{k-1})}{(x^{k-1} - y^{k-1})(x^{k-1} - z^{k-1})} \\ &\quad + \frac{g(y^{k-1})}{(y^{k-1} - z^{k-1})(y^{k-1} - x^{k-1})} + \frac{g(z^{k-1})}{(z^{k-1} - x^{k-1})(z^{k-1} - y^{k-1})}. \end{aligned}$$

Since g is strictly convex, the right hand side is positive. Moreover, since

$$(y^{k-1} - x^{k-1})(y^{k-1} - z^{k-1}) < 0,$$

we have $F'(y) < 0$ for $y \in (m, M)$, hence F is strictly decreasing on $[m, M]$. Therefore, F is maximum for $y = m$ and is minimum for $y = M$.

Proof of EV-Theorem. For $n = 3$, EV-Theorem follows immediately from EV-Proposition. Consider next that $n \geq 4$. Since $X = (x_1, x_2, \dots, x_n)$ is defined in EV-Theorem as a compact set in $\mathbb{R}^n$, S_n attains its minimum and maximum. Using this property and EV-Proposition, we can prove EV-Theorem via contradiction. Thus, for the sake of contradiction, assume that S_n attains its maximum at $(b_1, b_2, \dots, b_n)$, where $b_1 \leq b_2 \leq \dots \leq b_n$ and $b_1 < b_{n-1}$. Let x_1, x_{n-1} and x_n be real numbers so that

$$x_1 = x_{n-1} \leq x_n, \quad x_1 + x_{n-1} + x_n = b_1 + b_{n-1} + b_n, \quad x_1^k + x_{n-1}^k + x_n^k = b_1^k + b_{n-1}^k + b_n^k.$$

According to EV-Proposition, we have

$$f(x_1) + f(x_{n-1}) + f(x_n) > f(b_1) + f(b_{n-1}) + f(b_n),$$

which is a contradiction. Similarly, we can prove that S_n is minimum for $x_2 = x_3 = \dots = x_n$.

Corollary 1. Let $x_1, x_2, \dots, x_n$ ($n \geq 3$) be real variables such that

$$x_1 \leq x_2 \leq \dots \leq x_n,$$

and let f be a differentiable function on $\mathbb{R}$ so that the derivative f' is strictly convex on $\mathbb{R}$. For fixed $x_1 + x_2 + \dots + x_n$ and $x_1^2 + x_2^2 + \dots + x_n^2$, the sum

$$S_n = f(x_1) + f(x_2) + \dots + f(x_n)$$

has the minimum value for $x_2 = x_3 = \dots = x_n$, and the maximum value for $x_1 = x_2 = \dots = x_{n-1}$.

Corollary 2. Let $x_1, x_2, \dots, x_n$ ($n \geq 3$) be real variables such that

$$x_1 \leq x_2 \leq \dots \leq x_n,$$

and let k be an even positive integer and m an odd integer such that $m > k$. For fixed $x_1 + x_2 + \dots + x_n$ and $x_1^k + x_2^k + \dots + x_n^k$, the power sum

$$S_n = x_1^m + x_2^m + \dots + x_n^m$$

has the minimum value for $x_2 = x_3 = \dots = x_n$, and the maximum value for $x_1 = x_2 = \dots = x_{n-1}$.

Proof. We apply the EV-Theorem the function $f(u) = u^m$. The joined function

$$g(x) = f' \left({}^{k-1}\sqrt{x} \right) = m {}^{k-1}\sqrt{x^{m-1}}$$

is strictly convex on $\mathbb{R}$ because its derivative

$$g'(x) = \frac{m(m-1)}{k-1} {}^{k-1}\sqrt{x^{m-k}}$$

is strictly increasing on $\mathbb{R}$.

Theorem 1. Let $x_1, x_2, \dots, x_n$ ($n \geq 3$) be real numbers such that $x_1 + x_2 + \dots + x_n$ and $x_1^2 + x_2^2 + \dots + x_n^2$ are fixed.

(a) The power sum

$$A_4 = x_1^4 + x_2^4 + \dots + x_n^4$$

has the minimum and maximum values when the set $(x_1, x_2, \dots, x_n)$ has at most two distinct values.

(b) For $m = 6$ and $m = 8$, the power sum

$$A_m = x_1^m + x_2^m + \dots + x_n^m$$

has the maximum values when the set $(x_1, x_2, \dots, x_n)$ has at most two distinct values.

The proof of Theorem 1 is based on the following Proposition 1 below.

Proposition 1. Let x, y, z be real numbers such that $x + y + z$ and $x^2 + y^2 + z^2$ are fixed.

(a) The power sum

$$A_4 = x^4 + y^4 + z^4$$

has the minimum and maximum values when two of x, y, z are equal to each other.

(b) For $m = 6$ and $m = 8$, the power sum

$$A_m = x_1^m + x_2^m + \dots + x_n^m$$

has the maximum values when two of x, y, z are equal to each other.

Proof. Let

$$p = x + y + z, \quad q = xy + yz + zx, \quad r = xyz.$$

From

$$(x - y)^2(y - z)^2(z - x)^2 \geq 0,$$

which is equivalent to

$$-27r^2 - 2(2p^3 - 9pq)r + p^2q^2 - 4q^3 \geq 0,$$

we get $r \in [r_1, r_2]$, where

$$r_1 = \frac{9pq - 2p^3 - 2(p^2 - 3q)\sqrt{p^2 - 3q}}{27}, \quad r_2 = \frac{9pq - 2p^3 + 2(p^2 - 3q)\sqrt{p^2 - 3q}}{27}.$$

From

$$-27(r - r_1)(r - r_2) = (x - y)^2(y - z)^2(z - x)^2 \geq 0,$$

it follows that the product $r = xyz$ attains its minimum r_1 and its maximum r_2 when two of x, y, z are equal to each other. For fixed p and q , we have

$$A_4 = 4pr + h_4(p, q) := g_4(r),$$

$$A_6 = 3r^2 + f_6(p, q)r + h_6(p, q) := g_6(r),$$

$$A_8 = 4(3p^2 - 2q)r^2 + f_8(p, q)r + h_8(p, q) := g_8(r).$$

(a) Since $g_4(r)$ is monotone, A_4 has the minimum and maximum values for $r = r_1$ or $r = r_2$; that is, when two of x, y, z are equal to each other.

(b) Since

$$3p^2 - 2q = \frac{7}{3}p^2 + \frac{2}{3}(p^2 - 3q) > 0,$$

the functions g_6 and g_8 are strictly convex, hence A_6 and A_8 has the maximum value only for $r = r_1$ or $r = r_2$; that is, when two of x, y, z are equal.

Open problem. *The point (b) in Theorem 1 is valid for any integer number $m \geq 3$.*

Note. The EV-Theorem for real variables and Corollary 1 are also valid under the conditions in Note 2 and Note 3 from the preceding chapter 5, where $m, M \in \mathbb{R}$.

6.2 Applications

6.1. If a, b, c, d are real numbers so that $a + b + c + d = 4$, then

$$\left(a^2 + b^2 + c^2 + d^2 + \frac{8}{3}\right)^2 \geq 4 \left(a^3 + b^3 + c^3 + d^3 + \frac{64}{9}\right).$$

6.2. If a, b, c, d are real numbers so that $a + b + c + d = 4$, then

$$(a^2 + b^2 + c^2 + d^2 - 4) \left(a^2 + b^2 + c^2 + d^2 + \frac{76}{3}\right) \geq 8(a^3 + b^3 + c^3 + d^3 - 4).$$

6.3. If a, b, c are real numbers so that $a + b + c = 3$, then

$$(a^2 + b^2 + c^2 - 3)(a^2 + b^2 + c^2 + 93) \geq 24(a^3 + b^3 + c^3 - 3).$$

6.4. If a, b, c, d are real numbers so that $a + b + c + d = 4$, then

$$(a^2 + b^2 + c^2 + d^2 - 4)(a^2 + b^2 + c^2 + d^2 + 116) \geq 24(a^3 + b^3 + c^3 + d^3 - 4).$$

6.5. Let a, b, c, d be real numbers so that $a + b + c + d = 4$, and let

$$E = a^2 + b^2 + c^2 + d^2 - 4, \quad F = a^3 + b^3 + c^3 + d^3 - 4.$$

Prove that

$$E \left(\sqrt{\frac{E}{3}} + 3 \right) \geq F.$$

6.6. Let $a_1, a_2, \dots, a_n$ be real numbers so that

$$a_1 + a_2 + \dots + a_n = 0, \quad a_1^2 + a_2^2 + \dots + a_n^2 = n(n-1).$$

If m is an odd number ($m \geq 3$), then

$$n-1 - (n-1)^m \leq a_1^m + a_2^m + \dots + a_n^m \leq (n-1)^m - n + 1.$$

6.7. Let $a_1, a_2, \dots, a_n$ be real numbers so that

$$a_1 + a_2 + \dots + a_n = 1, \quad a_1^2 + a_2^2 + \dots + a_n^2 = n^2 + n - 1.$$

If m is an odd number ($m \geq 3$), then

$$(n-1) \left(1 + \frac{2}{n}\right)^m - \left(n - \frac{2}{n}\right)^m \leq a_1^m + a_2^m + \dots + a_n^m \leq n^m - n + 1.$$

6.8. Let $a_1, a_2, \dots, a_n$ be real numbers so that

$$a_1 + a_2 + \dots + a_n = 1, \quad a_1^2 + a_2^2 + \dots + a_n^2 = n^2 - 3n + 3.$$

If m is an odd number ($m \geq 3$), then

$$n - 1 - (n - 2)^m \leq a_1^m + a_2^m + \dots + a_n^m \leq \left(n - 2 + \frac{2}{n}\right)^m - (n - 1) \left(1 - \frac{2}{n}\right)^m.$$

6.9. Let $a_1, a_2, \dots, a_n$ be real numbers so that

$$a_1 + a_2 + \dots + a_n = a_1^2 + a_2^2 + \dots + a_n^2 = n - 1.$$

If m is an odd number ($m \geq 3$), then

$$n - 1 \leq a_1^m + a_2^m + \dots + a_n^m \leq (n - 1) \left(1 - \frac{2}{n}\right)^m + \left(2 - \frac{2}{n}\right)^m.$$

6.10. Let $a_1, a_2, \dots, a_n$ be real numbers so that

$$a_1 + a_2 + \dots + a_n = n + 1, \quad a_1^2 + a_2^2 + \dots + a_n^2 = n + 3.$$

If m is an odd number ($m \geq 3$), then

$$\left(\frac{2}{n}\right)^m + (n - 1) \left(1 + \frac{2}{n}\right)^m \leq a_1^m + a_2^m + \dots + a_n^m \leq 2^m + n - 1.$$

6.11. If $a_1, a_2, \dots, a_n$ are real numbers so that

$$a_1 + a_2 + \dots + a_n = a_1^4 + a_2^4 + \dots + a_n^4 = n - 1,$$

then

$$a_1^5 + a_2^5 + \dots + a_n^5 \geq n - 1.$$

6.12. If a, b, c are real numbers so that $a^2 + b^2 + c^2 = 3$, then

$$a^3 + b^3 + c^3 + 3 \geq 2(a + b + c).$$

6.13. If $a_1, a_2, \dots, a_n$ are real numbers so that

$$a_1 + a_2 + \dots + a_n = 0, \quad a_1^2 + a_2^2 + \dots + a_n^2 = n(n - 1),$$

then

$$a_1^4 + a_2^4 + \dots + a_n^4 \leq n(n - 1)(n^2 - 3n + 3).$$

6.14. If $a_1, a_2, \dots, a_n$ are real numbers so that

$$a_1 + a_2 + \dots + a_n = n + 1, \quad a_1^2 + a_2^2 + \dots + a_n^2 = 4n^2 + n - 1,$$

then

$$a_1^4 + a_2^4 + \dots + a_n^4 \leq 16n^4 + n - 1.$$

6.15. If n is an odd number and $a_1, a_2, \dots, a_n$ are real numbers so that

$$a_1 + a_2 + \dots + a_n = 0, \quad a_1^2 + a_2^2 + \dots + a_n^2 = n(n^2 - 1),$$

then

$$a_1^4 + a_2^4 + \dots + a_n^4 \geq n(n^2 - 1)(n^2 + 3).$$

6.16. If $a_1, a_2, \dots, a_n$ are real numbers so that

$$a_1 + a_2 + \dots + a_n = n^2 - n - 1, \quad a_1^2 + a_2^2 + \dots + a_n^2 = n^3 + 2n^2 - n - 1,$$

then

$$a_1^4 + a_2^4 + \dots + a_n^4 \geq n^4 + (n - 1)(n + 1)^4.$$

6.17. If $a_1, a_2, \dots, a_n$ are real numbers so that

$$a_1 + a_2 + \dots + a_n = n^2 - 2n - 1, \quad a_1^2 + a_2^2 + \dots + a_n^2 = n^3 + 2n + 1,$$

then

$$a_1^4 + a_2^4 + \dots + a_n^4 \geq (n + 1)^4 + (n - 1)n^4.$$

6.18. If $a_1, a_2, \dots, a_n$ are real numbers so that

$$a_1 + a_2 + \dots + a_n = n^2 - 3n - 2, \quad a_1^2 + a_2^2 + \dots + a_n^2 = n^3 + 2n^2 - 3n - 2,$$

then

$$a_1^4 + a_2^4 + \dots + a_n^4 \geq 2n^4 + (n - 2)(n + 1)^4.$$

6.19. If a, b, c, d are real numbers so that $a + b + c + d = 4$, then

$$(a^2 + b^2 + c^2 + d^2 - 4)(a^2 + b^2 + c^2 + d^2 + 36) \leq 12(a^4 + b^4 + c^4 + d^4 - 4).$$

6.20. If $a_1, a_2, \dots, a_n$ are real numbers so that

$$a_1 + a_2 + \dots + a_n = 0, \quad a_1^2 + a_2^2 + \dots + a_n^2 = n(n - 1),$$

then

$$a_1^6 + a_2^6 + \dots + a_n^6 \leq (n - 1)^6 + n - 1.$$

6.21. If $a_1, a_2, \dots, a_n$ are real numbers so that

$$a_1 + a_2 + \dots + a_n = 1, \quad a_1^2 + a_2^2 + \dots + a_n^2 = n^2 + n - 1,$$

then

$$a_1^6 + a_2^6 + \dots + a_n^6 \leq n^6 + n - 1.$$

6.22. If $a_1, a_2, \dots, a_n$ are real numbers so that

$$a_1 + a_2 + \dots + a_n = 0, \quad a_1^2 + a_2^2 + \dots + a_n^2 = n(n - 1),$$

then

$$a_1^8 + a_2^8 + \dots + a_n^8 \leq (n - 1)^8 + n - 1.$$

6.23. If $a_1, a_2, \dots, a_n$ are real numbers so that

$$a_1 + a_2 + \dots + a_n = 1, \quad a_1^2 + a_2^2 + \dots + a_n^2 = n^2 + n - 1,$$

then

$$a_1^8 + a_2^8 + \dots + a_n^8 \leq n^8 + n - 1.$$

6.24. Let $a_1, a_2, \dots, a_n$ ($n \geq 2$) be real numbers (not all equal), and let

$$A = \frac{a_1 + a_2 + \dots + a_n}{n}, \quad B = \frac{a_1^2 + a_2^2 + \dots + a_n^2}{n}, \quad C = \frac{a_1^3 + a_2^3 + \dots + a_n^3}{n}.$$

Then,

$$\frac{1}{4} \left(1 - \sqrt{1 + \frac{2n^2}{n-1}} \right) \leq \frac{B^2 - AC}{B^2 - A^4} \leq \frac{1}{4} \left(1 + \sqrt{1 + \frac{2n^2}{n-1}} \right).$$

6.25. If a, b, c, d are real numbers so that

$$a + b + c + d = 2,$$

then

$$a^4 + b^4 + c^4 + d^4 \leq 40 + \frac{3}{4}(a^2 + b^2 + c^2 + d^2)^2.$$

6.26. If a, b, c, d, e are real numbers, then

$$a^4 + b^4 + c^4 + d^4 + e^4 \leq \frac{31 + 18\sqrt{3}}{8}(a + b + c + d + e)^4 + \frac{3}{4}(a^2 + b^2 + c^2 + d^2 + e^2)^2.$$

6.27. Let $a, b, c, d, e \neq \frac{-5}{4}$ be real numbers so that $a + b + c + d + e = 5$. Then,

$$\frac{a(a-1)}{(4a+5)^2} + \frac{b(b-1)}{(4b+5)^2} + \frac{c(c-1)}{(4c+5)^2} + \frac{d(d-1)}{(4d+5)^2} + \frac{e(e-1)}{(4e+5)^2} \geq 0.$$

6.28. If a, b, c are real numbers so that

$$a + b + c = 9, \quad ab + bc + ca = 15,$$

then

$$\frac{19}{175} \leq \frac{1}{b^2 + bc + c^2} + \frac{1}{c^2 + ca + a^2} + \frac{1}{a^2 + ab + b^2} \leq \frac{7}{19}.$$

6.29. If a, b, c are real numbers so that

$$8(a^2 + b^2 + c^2) = 9(ab + bc + ca),$$

then

$$\frac{419}{175} \leq \frac{a^2}{b^2 + bc + c^2} + \frac{b^2}{c^2 + ca + a^2} + \frac{c^2}{a^2 + ab + b^2} \leq \frac{311}{19}.$$

6.30. Let $a_1, a_2, \dots, a_n$ be real numbers such that $a_1 + a_2 + \dots + a_n = n$. If $n \leq 10$, then

$$2(a_1^2 + a_2^2 + \dots + a_n^2)^2 - n(a_1^3 + a_2^3 + \dots + a_n^3) \geq n^2.$$

6.3 Solutions

P 6.1. If a, b, c, d are real numbers so that $a + b + c + d = 4$, then

$$\left(a^2 + b^2 + c^2 + d^2 + \frac{8}{3}\right)^2 \geq 4 \left(a^3 + b^3 + c^3 + d^3 + \frac{64}{9}\right).$$

(Vasile Cîrtoaje, 2010)

Solution. Apply Corollary 2 for $n = 4$, $k = 2$, $m = 3$:

- If a, b, c, d are real numbers so that $a \leq b \leq c \leq d$ and

$$a + b + c + d = 4, \quad a^2 + b^2 + c^2 + d^2 = \text{fixed},$$

then

$$S_4 = a^3 + b^3 + c^3 + d^3$$

is maximum for $a = b = c \leq d$.

Thus, we only need to show that $3a + d = 4$ involves

$$\left(3a^2 + d^2 + \frac{8}{3}\right)^2 \geq 4 \left(3a^3 + d^3 + \frac{64}{9}\right).$$

This inequality is equivalent to

$$(a - 1)^2(3a - 2)^2 \geq 0.$$

The equality holds for $a = b = c = d = 1$, and also for

$$a = b = c = \frac{2}{3}, \quad d = 2$$

(or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- If $a_1, a_2, \dots, a_n$ are real numbers so that

$$a_1 + a_2 + \dots + a_n = n,$$

then

$$\left(a_1^2 + a_2^2 + \dots + a_n^2 + \frac{n^3}{8n - 8}\right)^2 \geq n \left(a_1^3 + a_2^3 + \dots + a_n^3\right) + \frac{n^4(n^2 + 16n - 16)}{64(n - 1)^2},$$

with equality for $a_1 = a_2 = \dots = a_n = 1$, and also for

$$a_1 = a_2 = \dots = a_{n-1} = \frac{n}{2n - 2}, \quad a_n = \frac{n}{2}$$

(or any cyclic permutation).

□

P 6.2. If a, b, c, d are real numbers so that $a + b + c + d = 4$, then

$$(a^2 + b^2 + c^2 + d^2 - 4) \left(a^2 + b^2 + c^2 + d^2 + \frac{76}{3} \right) \geq 8(a^3 + b^3 + c^3 + d^3 - 4).$$

(Vasile Cîrtoaje, 2010)

Solution. As shown in the preceding P 6.1, we only need to show that

$$3a + d = 4$$

involves

$$(3a^2 + d^2 - 4) \left(3a^2 + d^2 + \frac{76}{3} \right) \geq 8(3a^3 + d^3 - 4).$$

This inequality is equivalent to

$$(a - 1)^2(3a - 1)^2 \geq 0.$$

The equality holds for $a = b = c = d = 1$, and also for

$$a = b = c = \frac{1}{3}, \quad d = 3$$

(or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- If $a_1, a_2, \dots, a_n$ are real numbers so that

$$a_1 + a_2 + \dots + a_n = n,$$

then

$$(a_1^2 + \dots + a_n^2 - n) \left[a_1^2 + \dots + a_n^2 + \frac{n(n^2 + n - 1)}{n - 1} \right] \geq 2n(a_1^3 + \dots + a_n^3 - n),$$

with equality for $a_1 = a_2 = \dots = a_n = 1$, and also for

$$a_1 = a_2 = \dots = a_{n-1} = \frac{1}{n-1}, \quad a_n = n-1$$

(or any cyclic permutation).

□

P 6.3. If a, b, c are real numbers so that $a + b + c = 3$, then

$$(a^2 + b^2 + c^2 - 3)(a^2 + b^2 + c^2 + 93) \geq 24(a^3 + b^3 + c^3 - 3).$$

(Vasile Cîrtoaje, 2010)

Solution. As shown in the proof of P 6.1, we only need to show that

$$2a + c = 3$$

involves

$$(2a^2 + c^2 - 3)(2a^2 + c^2 + 93) \geq 24(2a^3 + c^3 - 3).$$

This inequality is equivalent to

$$(a^2 - 1)^2 \geq 0.$$

The equality holds for $a = b = c = 1$, and also for

$$a = b = -1, \quad c = 5$$

(or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

• Let a, b, c be real numbers so that $a + b + c = 3$. For any real k , the following inequality holds

$$(a^2 + b^2 + c^2 - 3)(a^2 + b^2 + c^2 + 6k^2 + 36k - 3) \geq 12k(a^3 + b^3 + c^3 - 3),$$

with equality for $a = b = c = 1$, and also for

$$a = b = 1 - k, \quad c = 1 + 2k$$

(or any cyclic permutation).

□

P 6.4. If a, b, c, d are real numbers so that $a + b + c + d = 4$, then

$$(a^2 + b^2 + c^2 + d^2 - 4)(a^2 + b^2 + c^2 + d^2 + 116) \geq 24(a^3 + b^3 + c^3 + d^3 - 4).$$

(Vasile Cîrtoaje, 2010)

Solution. As shown in the proof of P 6.1, we only need to show that

$$3a + d = 4$$

involves

$$(3a^2 + d^2 - 4)(3a^2 + d^2 + 116) \geq 24(3a^3 + d^3 - 4).$$

This inequality is equivalent to

$$(a^2 - 1)^2 \geq 0.$$

The equality holds for $a = b = c = d = 1$, and also for

$$a = b = c = -1, \quad d = 7$$

(or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- Let $a_1, a_2, \dots, a_n$ be real numbers so that

$$a_1 + a_2 + \dots + a_n = n.$$

If k is a real number, then

$$\frac{k(a_1^3 + \dots + a_n^3 - n)}{a_1^2 + \dots + a_n^2 - n} \leq \frac{a_1^2 + \dots + a_n^2 + n(n-1)(n-2)^2k^2 + 6n(n-1)k - n}{2n(n-1)},$$

with equality for

$$a_1 = \dots = a_{n-1} = 1 - (n-2)k, \quad a_n = 1 + (n-1)(n-2)k$$

(or any cyclic permutation).

For $k = \frac{-6}{n-2}$, we get the following nice inequality

$$(a_1^2 + a_2^2 + \dots + a_n^2 - n)^2 + \frac{12n(n-1)}{n-2} (a_1^3 + a_2^3 + \dots + a_n^3 - n) \geq 0,$$

with equality for $a_1 = a_2 = \dots = a_n = 1$, and also for

$$a_1 = \dots = a_{n-1} = 7, \quad a_n = 7 - 6n$$

(or any cyclic permutation).

□

P 6.5. Let a, b, c, d be real numbers so that $a + b + c + d = 4$, and let

$$E = a^2 + b^2 + c^2 + d^2 - 4, \quad F = a^3 + b^3 + c^3 + d^3 - 4.$$

Prove that

$$E \left(\sqrt{\frac{E}{3}} + 3 \right) \geq F.$$

(Vasile Cîrtoaje, 2016)

Solution. As shown in the proof of P 6.1, we only need to prove the desired inequality for $3a + d = 4$ and

$$E = 3a^2 + d^2 - 4, \quad F = 3a^3 + d^3 - 4.$$

Since

$$E = 12(1-a)^2, \quad F = 12(5-2a)(1-a)^2,$$

we get

$$\begin{aligned} E \left(\sqrt{\frac{E}{3}} + 3 \right) - F &= 12(1-a)^2(2|1-a| + 3) - 12(5-2a)(1-a)^2 \\ &= 24(1-a)^2[|1-a| - (1-a)] \geq 0. \end{aligned}$$

The equality holds for

$$a = b = c = \frac{4-d}{3} \leq 1$$

(or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- Let $a_1, a_2, \dots, a_n$ be real numbers so that $a_1 + a_2 + \dots + a_n = n$, and let

$$E = a_1^2 + a_2^2 + \dots + a_n^2 - n, \quad F = a_1^3 + a_2^3 + \dots + a_n^3 - n.$$

Then,

$$E \left[(n-2) \sqrt{\frac{E}{n(n-1)}} + 3 \right] \geq F,$$

with equality for

$$a_1 = \dots = a_{n-1} = \frac{n - a_n}{n-1} \leq 1$$

(or any cyclic permutation).

□

P 6.6. Let $a_1, a_2, \dots, a_n$ be real numbers so that

$$a_1 + a_2 + \dots + a_n = 0, \quad a_1^2 + a_2^2 + \dots + a_n^2 = n(n-1).$$

If m is an odd number ($m \geq 3$), then

$$n-1 - (n-1)^m \leq a_1^m + a_2^m + \dots + a_n^m \leq (n-1)^m - n + 1.$$

(Vasile Cîrtoaje, 2010)

Solution. Without loss of generality, assume that

$$a_1 \leq a_2 \leq \dots \leq a_n.$$

(a) Consider the right inequality. For $n = 2$, we need to show that

$$a_1 + a_2 = 0, \quad a_1^2 + a_2^2 = 2$$

implies

$$a_1^m + a_2^m \leq 0.$$

We have

$$a_1 = -1, \quad a_2 = 1,$$

therefore $a_1^m + a_2^m = 0$. Assume now that $n \geq 3$. According to Corollary 2, the sum

$$S_n = a_1^m + a_2^m + \dots + a_n^m$$

is maximum for $a_1 = a_2 = \cdots = a_{n-1}$. Thus, we only need to show that

$$(n-1)a + b = 0, \quad (n-1)a^2 + b^2 = n(n-1), \quad a \leq b$$

involve

$$(n-1)a^m + b^m \leq (n-1)^m - n + 1.$$

From the equations above, we get

$$a = -1, \quad b = n-1;$$

therefore,

$$(n-1)a^m + b^m = (n-1)(-1)^m + (n-1)^m = (n-1)^m - n + 1.$$

The equality holds for

$$a_1 = \cdots = a_{n-1} = -1, \quad a_n = n-1$$

(or any cyclic permutation).

(b) The left inequality follows from the right inequality by replacing $a_1, a_2, \dots, a_n$ with $-a_1, -a_2, \dots, -a_n$, respectively. The equality holds for

$$a_1 = -n+1, \quad a_2 = a_3 = \cdots = a_n = 1$$

(or any cyclic permutation).

□

P 6.7. Let $a_1, a_2, \dots, a_n$ be real numbers so that

$$a_1 + a_2 + \cdots + a_n = 1, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n^2 + n - 1.$$

If m is an odd number ($m \geq 3$), then

$$(n-1) \left(1 + \frac{2}{n}\right)^m - \left(n - \frac{2}{n}\right)^m \leq a_1^m + a_2^m + \cdots + a_n^m \leq n^m - n + 1.$$

(Vasile Cîrtoaje, 2010)

Solution. Without loss of generality, assume that

$$a_1 \leq a_2 \leq \cdots \leq a_n.$$

For $n = 2$, we need to show that

$$a_1 + a_2 = 1, \quad a_1^2 + a_2^2 = 5,$$

implies

$$2^m - 1 \leq a_1^m + a_2^m \leq 2^m - 1.$$

We have

$$a_1 = -1, \quad a_2 = 2,$$

for which $a_1^m + a_2^m = 2^m - 1$. Assume now that $n \geq 3$.

(a) Consider the right inequality. According to Corollary 2, the sum

$$S_n = a_1^m + a_2^m + \cdots + a_n^m$$

is maximum for $a_1 = a_2 = \cdots = a_{n-1}$. Thus, we only need to show that

$$(n-1)a + b = 1, \quad (n-1)a^2 + b^2 = n^2 + n - 1, \quad a \leq b$$

involve

$$(n-1)a^m + b^m \leq n^m - n + 1.$$

From the equations above, we get

$$a = -1, \quad b = n;$$

therefore,

$$(n-1)a^m + b^m = (n-1)(-1)^m + n^m = n^m - n + 1.$$

The equality holds for

$$a_1 = a_2 = \cdots = a_{n-1} = -1, \quad a_n = n$$

(or any cyclic permutation).

(b) Consider the left inequality. According to Corollary 2, the sum

$$S_n = a_1^m + a_2^m + \cdots + a_n^m$$

is minimum for $a_2 = a_3 = \cdots = a_n$. Thus, we only need to show that

$$a + (n-1)b = 1, \quad a^2 + (n-1)b^2 = n^2 + n - 1, \quad a \leq b$$

involve

$$a^m + (n-1)b^m \geq (n-1) \left(1 + \frac{2}{n}\right)^m - \left(n - \frac{2}{n}\right)^m.$$

From the equations above, we get

$$a = -n + \frac{2}{n}, \quad b = 1 + \frac{2}{n};$$

therefore,

$$\begin{aligned} a^m + (n-1)b^m &= \left(-n + \frac{2}{n}\right)^m + (n-1) \left(1 + \frac{2}{n}\right)^m \\ &= (n-1) \left(1 + \frac{2}{n}\right)^m - \left(n - \frac{2}{n}\right)^m. \end{aligned}$$

The equality holds for

$$a_1 = -n + \frac{2}{n}, \quad a_2 = a_3 = \cdots = a_n = 1 + \frac{2}{n}$$

(or any cyclic permutation).

□

P 6.8. Let $a_1, a_2, \dots, a_n$ be real numbers so that

$$a_1 + a_2 + \dots + a_n = 1, \quad a_1^2 + a_2^2 + \dots + a_n^2 = n^2 - 3n + 3.$$

If m is an odd number ($m \geq 3$), then

$$n - 1 - (n - 2)^m \leq a_1^m + a_2^m + \dots + a_n^m \leq \left(n - 2 + \frac{2}{n}\right)^m - (n - 1) \left(1 - \frac{2}{n}\right)^m.$$

(Vasile Cîrtoaje, 2010)

Solution. Without loss of generality, assume that

$$a_1 \leq a_2 \leq \dots \leq a_n.$$

For $n = 2$, we need to show that

$$a_1 + a_2 = 1, \quad a_1^2 + a_2^2 = 1,$$

implies

$$1 \leq a_1^m + a_2^m \leq 1.$$

We have

$$a_1 = 0, \quad a_2 = 1,$$

when $a_1^m + a_2^m = 1$. Assume now that $n \geq 3$.

(a) Consider the left inequality. According to Corollary 2, the sum

$$S_n = a_1^m + a_2^m + \dots + a_n^m$$

is minimum for $a_2 = a_3 = \dots = a_n$. Thus, we only need to show that

$$a + (n - 1)b = 1, \quad a^2 + (n - 1)b^2 = n^2 - 3n + 3, \quad a \leq b$$

involve

$$a^m + (n - 1)b^m \leq n - 1 - (n - 2)^m.$$

From the equations above, we get

$$a = 2 - n, \quad b = 1;$$

therefore,

$$a^m + (n - 1)b^m = (2 - n)^m + n - 1 = n - 1 - (n - 2)^m.$$

The equality holds for

$$a_1 = 2 - n, \quad a_2 = a_3 = \dots = a_n = 1$$

(or any cyclic permutation).

(b) Consider the right inequality. According to Corollary 2, the sum

$$S_n = a_1^m + a_2^m + \dots + a_n^m$$

is maximum for $a_1 = a_2 = \cdots = a_{n-1}$. Thus, we only need to show that

$$(n-1)a + b = 1, \quad (n-1)a^2 + b^2 = n^2 - 3n + 3, \quad a \leq b$$

involve

$$(n-1)a^m + b^m \leq \left(n-2 + \frac{2}{n}\right)^m - (n-1) \left(1 - \frac{2}{n}\right)^m.$$

From the equations above, we get

$$a = -1 + \frac{2}{n}, \quad b = n - 2 + \frac{2}{n};$$

therefore,

$$\begin{aligned} (n-1)a^m + b^m &= (n-1) \left(-1 + \frac{2}{n}\right)^m + \left(n-2 + \frac{2}{n}\right)^m \\ &= \left(n-2 + \frac{2}{n}\right)^m - (n-1) \left(1 - \frac{2}{n}\right)^m. \end{aligned}$$

The equality holds for

$$a_1 = \cdots = a_{n-1} = -1 + \frac{2}{n}, \quad a_n = n - 2 + \frac{2}{n}$$

(or any cyclic permutation).

□

P 6.9. Let $a_1, a_2, \dots, a_n$ be real numbers so that

$$a_1 + a_2 + \cdots + a_n = a_1^2 + a_2^2 + \cdots + a_n^2 = n - 1.$$

If m is an odd number ($m \geq 3$), then

$$n - 1 \leq a_1^m + a_2^m + \cdots + a_n^m \leq (n-1) \left(1 - \frac{2}{n}\right)^m + \left(2 - \frac{2}{n}\right)^m.$$

(Vasile Cîrtoaje, 2010)

Solution. Without loss of generality, assume that

$$a_1 \leq a_2 \leq \cdots \leq a_n.$$

For $n = 2$, we need to show that

$$a_1 + a_2 = 1, \quad a_1^2 + a_2^2 = 1,$$

implies

$$1 \leq a_1^m + a_2^m \leq 1.$$

The above equations involve

$$a_1 = 0, \quad a_2 = 1,$$

hence $a_1^m + a_2^m = 1$. Assume now that $n \geq 3$.

(a) Consider the left inequality. According to Corollary 2, the sum

$$S_n = a_1^m + a_2^m + \cdots + a_n^m$$

is minimum for $a_2 = a_3 = \cdots = a_n$. Thus, we only need to show that

$$a + (n-1)b = n-1, \quad a^2 + (n-1)b^2 = n-1, \quad a \leq b$$

involve

$$a^m + (n-1)b^m \geq n-1.$$

From the equations above, we get

$$a = 0, \quad b = 1;$$

therefore,

$$a^m + (n-1)b^m = n-1.$$

The equality holds for

$$a_1 = 0, \quad a_2 = \cdots = a_n = 1$$

(or any cyclic permutation).

(b) Consider the right inequality. According to Corollary 2, the sum

$$S_n = a_1^m + a_2^m + \cdots + a_n^m$$

is maximum for $a_1 = a_2 = \cdots = a_{n-1}$. Thus, we only need to show that

$$(n-1)a + b = n-1, \quad (n-1)a^2 + b^2 = n-1, \quad a \leq b$$

involve

$$(n-1)a^m + b^m \leq (n-1) \left(1 - \frac{2}{n}\right)^m + \left(2 - \frac{2}{n}\right)^m.$$

From the equations above, we get

$$a = 1 - \frac{2}{n}, \quad b = 2 - \frac{2}{n},$$

when

$$(n-1)a^m + b^m = (n-1) \left(1 - \frac{2}{n}\right)^m + \left(2 - \frac{2}{n}\right)^m.$$

The equality holds for

$$a_1 = a_2 = \cdots = a_{n-1} = 1 - \frac{2}{n}, \quad a_n = 2 - \frac{2}{n}$$

(or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- Let $a_1, a_2, \dots, a_n$ be real numbers so that

$$a_1 + a_2 + \dots + a_n = k, \quad a_1^2 + a_2^2 + \dots + a_n^2 = n^2 + (2k - 1)n + k(k - 2),$$

where k is a real number, $k \geq -n$. If m is an odd number ($m \geq 3$), then

$$\left(\frac{2k}{n} + 1 - n - k\right)^m + (n - 1) \left(\frac{2k}{n} + 1\right)^m \leq a_1^m + a_2^m + \dots + a_n^m \leq (n + k - 1)^m - n + 1.$$

The left inequality is an equality for

$$a_1 = \frac{2k}{n} + 1 - n - k, \quad a_2 = \dots = a_n = \frac{2k}{n} + 1$$

(or any cyclic permutation). The right inequality is an equality for

$$a_1 = \dots = a_{n-1} = -1, \quad a_n = n + k - 1$$

(or any cyclic permutation).

For $k = 0$ and $k = 1$, we get the inequalities in P 6.6 and P 6.7, respectively. For $k = -1$ and $k = -n + 1$, by replacing k with $-k$ and $a_1, a_2, \dots, a_n$ with $-a_1, -a_2, \dots, -a_n$, we get the inequalities in P 6.8 and P 6.9, respectively.

□

P 6.10. Let $a_1, a_2, \dots, a_n$ be real numbers so that

$$a_1 + a_2 + \dots + a_n = n + 1, \quad a_1^2 + a_2^2 + \dots + a_n^2 = n + 3.$$

If m is an odd number ($m \geq 3$), then

$$\left(\frac{2}{n}\right)^m + (n - 1) \left(1 + \frac{2}{n}\right)^m \leq a_1^m + a_2^m + \dots + a_n^m \leq 2^m + n - 1.$$

(Vasile Cîrtoaje, 2010)

Solution. Without loss of generality, assume that

$$a_1 \leq a_2 \leq \dots \leq a_n.$$

For $n = 2$, we need to show that

$$a_1 + a_2 = 3, \quad a_1^2 + a_2^2 = 5,$$

implies

$$2^m + 1 \leq a_1^m + a_2^m \leq 2^m + 1.$$

We get

$$a_1 = 1, \quad a_2 = 2,$$

when $a_1^m + a_2^m = 2^m + 1$. Assume now that $n \geq 3$.

(a) Consider the left inequality. According to Corollary 2, the sum

$$S_n = a_1^m + a_2^m + \cdots + a_n^m$$

is minimum for $a_2 = a_3 = \cdots = a_n$. Thus, we only need to show that

$$a + (n-1)b = n+1, \quad a^2 + (n-1)b^2 = n+3, \quad a \leq b$$

involve

$$a^m + (n-1)b^m \geq \left(\frac{2}{n}\right)^m + (n-1) \left(1 + \frac{2}{n}\right)^m.$$

From the equations

$$a + (n-1)b = n+1, \quad a^2 + (n-1)b^2 = n+3,$$

we get

$$a = \frac{2}{n}, \quad b = 1 + \frac{2}{n};$$

therefore,

$$a^m + (n-1)b^m = \left(\frac{2}{n}\right)^m + (n-1) \left(1 + \frac{2}{n}\right)^m.$$

The equality holds for

$$a_1 = \frac{2}{n}, \quad a_2 = \cdots = a_n = 1 + \frac{2}{n}$$

(or any cyclic permutation).

(b) Consider the right inequality. According to Corollary 2, the sum

$$S_n = a_1^m + a_2^m + \cdots + a_n^m$$

is maximum for $a_1 = a_2 = \cdots = a_{n-1}$. Thus, we only need to show that

$$(n-1)a + b = n+1, \quad (n-1)a^2 + b^2 = n+3, \quad a \leq b$$

involve

$$(n-1)a^m + b^m \leq 2^m + n-1.$$

From the equations

$$(n-1)a + b = n+1, \quad (n-1)a^2 + b^2 = n+3,$$

we get

$$a = 1, \quad b = 2;$$

therefore,

$$(n-1)a^m + b^m = n-1 + 2^m.$$

The equality holds for

$$a_1 = \cdots = a_{n-1} = 1, \quad a_n = 2$$

(or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- Let $a_1, a_2, \dots, a_n$ be real numbers so that

$$a_1 + a_2 + \dots + a_n = k, \quad a_1^2 + a_2^2 + \dots + a_n^2 = n^2 - (2k + 1)n + k(k + 2),$$

where k is a positive number, $k > n$. If m is an odd number ($m \geq 3$), then

$$\left(\frac{2k}{n} - 1 + n - k\right)^m + (n - 1) \left(\frac{2k}{n} - 1\right)^m \leq a_1^m + a_2^m + \dots + a_n^m \leq (k - n + 1)^m + n - 1.$$

The left inequality is an equality for

$$a_1 = \frac{2k}{n} - 1 + n - k, \quad a_2 = \dots = a_n = \frac{2k}{n} - 1$$

(or any cyclic permutation). The right inequality is an equality for

$$a_1 = \dots = a_{n-1} = 1, \quad a_n = k - n + 1$$

(or any cyclic permutation).

For $k = n + 1$, we get the inequalities in P 6.10.

□

P 6.11. If $a_1, a_2, \dots, a_n$ are real numbers so that

$$a_1 + a_2 + \dots + a_n = a_1^4 + a_2^4 + \dots + a_n^4 = n - 1,$$

then

$$a_1^5 + a_2^5 + \dots + a_n^5 \geq n - 1.$$

(Vasile Cîrtoaje, 2010)

Solution. For $n = 2$, we need to show that

$$a_1 + a_2 = 1, \quad a_1^4 + a_2^4 = 1,$$

implies

$$a_1^5 + a_2^5 \geq 1.$$

We have

$$a_1 = 0, \quad a_2 = 1,$$

or

$$a_1 = 1, \quad a_2 = 0.$$

For each of these cases, the inequality is an equality. Assume now that $n \geq 3$ and

$$a_1 \leq a_2 \leq \dots \leq a_n.$$

According to Corollary 2, the sum

$$S_n = a_1^5 + a_2^5 + \dots + a_n^5$$

is minimum for $a_2 = a_3 = \cdots = a_n$. Thus, we only need to show that

$$a + (n-1)b = a^4 + (n-1)b^4 = n-1, \quad a \leq b$$

involve

$$a^5 + (n-1)b^5 \geq n-1.$$

The equations

$$a + (n-1)b = n-1, \quad a^4 + (n-1)b^4 = n-1,$$

are equivalent to

$$(1-b)[(n-1)^3(1-b)^3 - 1 - b - b^2 - b^3] = 0, \quad a = (n-1)(1-b);$$

that is,

$$b = 1, \quad a = 0,$$

and

$$a^3 = 1 + b + b^2 + b^3, \quad a = (n-1)(1-b).$$

For the second case, the condition $a \leq b$ involves

$$b^3 \geq 1 + b + b^2 + b^3,$$

which is not possible. Therefore, it suffices to show that

$$a^5 + (n-1)b^5 \geq n-1$$

for $a = 0$ and $b = 1$, that is clearly true. Thus, the proof is completed. The equality holds for

$$a_1 = 0, \quad a_2 = \cdots = a_n = 1$$

(or any cyclic permutation).

□

P 6.12. If a, b, c are real numbers so that

$$a^2 + b^2 + c^2 = 3,$$

then

$$a^3 + b^3 + c^3 + 3 \geq 2(a + b + c).$$

(Vasile Cîrtoaje, 2010)

Solution. Assume that

$$a \leq b \leq c.$$

According to Corollary 2, for $a \leq b \leq c$ and

$$a + b + c = \text{fixed}, \quad a^2 + b^2 + c^2 = 3,$$

the sum

$$S_3 = a^3 + b^3 + c^3$$

is minimum for $a \leq b = c$. Thus, we only need to show that

$$a^2 + 2b^2 = 3, \quad a \leq b,$$

involves

$$a^3 + 2b^3 + 3 \geq 2(a + 2b).$$

We will show this by two methods. From $a^2 + 2b^2 = 3$ and $a \leq b$, it follows that

$$-\sqrt{3} \leq a \leq 1, \quad -\sqrt{\frac{3}{2}} < b \leq \sqrt{\frac{3}{2}}.$$

Method 1. Write the desired inequality as

$$a^3 + b(3 - a^2) + 3 \geq 2(a + 2b),$$

$$a^3 - 2a + 3 \geq b(a^2 + 1).$$

For $a \geq 0$, we have

$$a^3 - 2a + 3 \geq -2a + 3 > 0,$$

and for $a \leq 0$, we have

$$a^3 - 2a + 3 = a(a^2 - 3) + a + 3 = -2ab^2 + a + 3 \geq a + 3 > 0.$$

Thus, it suffices to show that

$$(a^3 - 2a + 3)^2 \geq b^2(a^2 + 1)^2,$$

which is equivalent to

$$2(a^3 - 2a + 3)^2 \geq (3 - a^2)(a^2 + 1)^2,$$

$$(a - 1)^2 f(a) \geq 0,$$

where

$$f(a) = a^4 + 2a^3 + 2a + 5.$$

We need to prove that $f(a) \geq 0$. For $a \geq -1$, we have

$$f(a) = (a + 2)(a^3 + 2) + 1 > 0.$$

For $a \leq -1$, we have

$$f(a) = (a + 1)^2(a + 2)^2 + g(a), \quad g(a) = -4a^3 - 13a^2 - 10a + 1.$$

It suffices to show that $g(a) \geq 0$. Since

$$g(a) = -(a + 1) \left(2a + \frac{7}{2} \right)^2 + 5h(a), \quad h(a) = a^2 + \frac{13}{4}a + \frac{53}{20}$$

and

$$h(a) = \left(a + \frac{13}{8}\right)^2 + \frac{3}{320} > 0,$$

the conclusion follows. The equality holds for $a = b = c = 1$.

Method 2. Write the desired inequality as follows:

$$\begin{aligned} 2(a^3 - 2a + 1) + 4(b^3 - 2b + 1) &\geq 0, \\ 2(a^3 - 2a + 1) + 4(b^3 - 2b + 1) &\geq a^2 + 2b^2 - 3, \\ (2a^3 - a^2 - 4a + 3) + 2(b^3 - b^2 - 4b + 3) &\geq 0, \\ (a - 1)^2(2a + 3) + 2(b - 1)^2(2b + 3) &\geq 0. \end{aligned}$$

Since $2b + 3 > 0$, the inequality is true for $a \geq -3/2$. Consider further that

$$-\sqrt{3} \leq a \leq \frac{-3}{2},$$

and rewrite the desired inequality as follows:

$$\begin{aligned} 2(a^3 - 2a + 1) + 4(b^3 - 2b + 1) + 4(a^2 + 2b^2 - 3) &\geq 0, \\ (2a^3 + 4a^2 - 4a - 2) + 2(2b^3 + 4b^2 - 4b - 2) &\geq 0, \\ \left(2a^3 + 4a^2 - 4a - \frac{33}{4}\right) + \left(4b^3 + 8b^2 - 8b + \frac{9}{4}\right) &\geq 0, \\ (2a + 3) \left(a^2 + \frac{1}{2}a - \frac{11}{4}\right) + f(b) &\geq 0, \end{aligned}$$

where

$$f(b) = 4b^3 + 8b^2 - 8b + \frac{9}{4}.$$

Since $2a + 3 \leq 0$ and

$$a^2 + \frac{1}{2}a - \frac{11}{4} \leq 3 + \frac{1}{2}a - \frac{11}{4} = \frac{1}{4}(2a + 1) < 0,$$

it suffices to show that $f(b) \geq 0$. For $b \geq 0$, we have

$$f(b) > 8b^2 - 8b + 2 = 2(2b - 1)^2 \geq 0,$$

and for $b \leq 0$, we have

$$f(b) > 4b^3 + 8b^2 = 4b^2(b + 2) \geq 0.$$

□

P 6.13. If $a_1, a_2, \dots, a_n$ are real numbers so that

$$a_1 + a_2 + \dots + a_n = 0, \quad a_1^2 + a_2^2 + \dots + a_n^2 = n(n - 1),$$

then

$$a_1^4 + a_2^4 + \dots + a_n^4 \leq n(n - 1)(n^2 - 3n + 3).$$

(Vasile Cîrtoaje, 2010)

Solution. For $n = 2$, we need to show that

$$a_1 + a_2 = 0, \quad a_1^2 + a_2^2 = 2,$$

implies

$$a_1^4 + a_2^4 \leq 2.$$

We have

$$a_1 = -1, \quad a_2 = 1,$$

or

$$a_1 = 1, \quad a_2 = -1.$$

For each of these cases, the desired inequality is an equality. Assume now that $n \geq 3$. According to Theorem 1, the sum

$$S_n = a_1^4 + a_2^4 + \cdots + a_n^4$$

is maximum for

$$a_1 = \cdots = a_j, \quad a_{j+1} = \cdots = a_n,$$

where $j \in \{1, 2, \dots, n-1\}$. Thus, we only need to show that

$$ja_1 + (n-j)a_n = 0, \quad ja_1^2 + (n-j)a_n^2 = n(n-1)$$

involve

$$ja_1^4 + (n-j)a_n^4 \leq n(n-1)(n^2 - 3n + 3).$$

From the equations above, we get

$$a_1^2 = \frac{(n-j)(n-1)}{j}, \quad a_n^2 = \frac{j(n-1)}{n-j};$$

therefore,

$$ja_1^4 + (n-j)a_n^4 = \frac{(n-j)^3 + j^3}{j(n-j)}(n-1)^2 = \left[\frac{n^2}{j(n-j)} - 3 \right] n(n-1)^2.$$

Since

$$j(n-j) - (n-1) = (j-1)(n-j-1) \geq 0,$$

we get

$$ja_1^4 + (n-j)a_n^4 \leq \left[\frac{n^2}{n-1} - 3 \right] n(n-1)^2 = n(n-1)(n^2 - 3n + 3).$$

The equality holds for

$$a_1 = -n+1, \quad a_2 = \cdots = a_n = 1$$

and for

$$a_1 = n-1, \quad a_2 = \cdots = a_n = -1$$

(or any cyclic permutation).

□

P 6.14. If $a_1, a_2, \dots, a_n$ are real numbers so that

$$a_1 + a_2 + \dots + a_n = n + 1, \quad a_1^2 + a_2^2 + \dots + a_n^2 = 4n^2 + n - 1,$$

then

$$a_1^4 + a_2^4 + \dots + a_n^4 \leq 16n^4 + n - 1.$$

(Vasile Cîrtoaje, 2010)

Solution. Replacing n by $2n + 1$ in the preceding P 6.13, we get the following statement:

- If $a_1, a_2, \dots, a_{2n+1}$ are real numbers so that

$$a_1 + a_2 + \dots + a_{2n+1} = 0, \quad a_1^2 + a_2^2 + \dots + a_{2n+1}^2 = 2n(2n + 1),$$

then

$$a_1^4 + a_2^4 + \dots + a_{2n+1}^4 \leq 2n(2n + 1)(4n^2 - 2n + 1),$$

with equality for

$$a_1 = -2n, \quad a_2 = \dots = a_{2n+1} = 1$$

and for

$$a_1 = 2n, \quad a_2 = \dots = a_{2n+1} = -1$$

(or any cyclic permutation).

Putting

$$a_{n+1} = \dots = a_{2n+1} = -1,$$

it follows that

$$a_1 + a_2 + \dots + a_n - n - 1 = 0, \quad a_1^2 + a_2^2 + \dots + a_n^2 + n + 1 = 2n(2n + 1)$$

involve

$$a_1^4 + a_2^4 + \dots + a_n^4 + n + 1 \leq 2n(2n + 1)(4n^2 - 2n + 1).$$

This is equivalent to the desired statement. The equality holds for

$$a_1 = 2n, \quad a_2 = \dots = a_n = -1$$

(or any cyclic permutation).

□

P 6.15. If n is an odd number and $a_1, a_2, \dots, a_n$ are real numbers so that

$$a_1 + a_2 + \dots + a_n = 0, \quad a_1^2 + a_2^2 + \dots + a_n^2 = n(n^2 - 1),$$

then

$$a_1^4 + a_2^4 + \dots + a_n^4 \geq n(n^2 - 1)(n^2 + 3).$$

(Vasile Cîrtoaje, 2010)

Solution. According to Theorem 1, the sum

$$S_n = a_1^4 + a_2^4 + \cdots + a_n^4$$

is minimum for

$$a_1 = \cdots = a_j, \quad a_{j+1} = \cdots = a_n,$$

where $j \in \{1, 2, \dots, n-1\}$. Thus, we only need to show that

$$ja_1 + (n-j)a_n = 0, \quad ja_1^2 + (n-j)a_n^2 = n(n^2 - 1)$$

involve

$$ja_1^4 + (n-j)a_n^4 \leq n(n^2 - 1)(n^2 + 3).$$

From the equations above, we get

$$a_1^2 = \frac{(n-j)(n^2 - 1)}{j}, \quad a_n^2 = \frac{j(n^2 - 1)}{n-j};$$

therefore,

$$ja_1^4 + (n-j)a_n^4 = \frac{(n-j)^3 + j^3}{j(n-j)}(n^2 - 1)^2 = \left[\frac{n^2}{j(n-j)} - 3 \right] n(n^2 - 1)^2.$$

Since

$$\frac{n^2 - 1}{4} - j(n-j) = \frac{(n-2j)^2 - 1}{4} \geq 0,$$

we get

$$ja_1^4 + (n-j)a_n^4 \geq \left(\frac{4n^2}{n^2 - 1} - 3 \right) n(n^2 - 1)^2 = n(n^2 - 1)(n^2 + 3).$$

The equality holds when $\frac{n-1}{2}$ of $a_1, a_2, \dots, a_n$ are equal to $-n-1$ and the other $\frac{n+1}{2}$ are equal to $n-1$, and also when $\frac{n-1}{2}$ of $a_1, a_2, \dots, a_n$ are equal to $n+1$ and the other $\frac{n+1}{2}$ are equal to $-n+1$.

□

P 6.16. If $a_1, a_2, \dots, a_n$ are real numbers so that

$$a_1 + a_2 + \cdots + a_n = n^2 - n - 1, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n^3 + 2n^2 - n - 1,$$

then

$$a_1^4 + a_2^4 + \cdots + a_n^4 \geq n^4 + (n-1)(n+1)^4.$$

(Vasile Cîrtoaje, 2010)

Solution. Replacing $a_1, a_2, \dots, a_n$ by $2a_1, 2a_2, \dots, 2a_n$ and then n by $2n + 1$, the preceding P 6.15 becomes as follows:

- If $a_1, a_2, \dots, a_{2n+1}$ are real numbers so that

$$a_1 + a_2 + \dots + a_{2n+1} = 0, \quad a_1^2 + a_2^2 + \dots + a_{2n+1}^2 = n(n+1)(2n+1),$$

then

$$a_1^4 + a_2^4 + \dots + a_{2n+1}^4 \geq n(n+1)(2n+1)(n^2 + n + 1),$$

with equality when n of $a_1, a_2, \dots, a_{2n+1}$ are equal to $-n - 1$ and the other $n + 1$ are equal to n , and also when n of $a_1, a_2, \dots, a_{2n+1}$ are equal to $n + 1$ and the other $n + 1$ are equal to $-n$.

Putting

$$a_{n+1} = \dots = a_{2n} = -n, \quad a_{2n+1} = n + 1,$$

it follows that

$$a_1 + a_2 + \dots + a_n + n(-n) + (n + 1) = 0$$

and

$$a_1^2 + a_2^2 + \dots + a_n^2 + n(-n)^2 + (n + 1)^2 = n(n + 1)(2n + 1)$$

involve

$$a_1^4 + a_2^4 + \dots + a_n^4 + n(-n)^4 + (n + 1)^4 \leq n(n + 1)(2n + 1)(n^2 + n + 1).$$

This is equivalent to the desired statement. The equality holds for

$$a_1 = \dots = a_{n-1} = n + 1, \quad a_n = -n$$

(or any cyclic permutation).

□

P 6.17. If $a_1, a_2, \dots, a_n$ are real numbers so that

$$a_1 + a_2 + \dots + a_n = n^2 - 2n - 1, \quad a_1^2 + a_2^2 + \dots + a_n^2 = n^3 + 2n + 1,$$

then

$$a_1^4 + a_2^4 + \dots + a_n^4 \geq (n + 1)^4 + (n - 1)n^4.$$

(Vasile Cîrtoaje, 2010)

Solution. As shown in the proof of the preceding P 6.16, the following statement holds:

- If $a_1, a_2, \dots, a_{2n+1}$ are real numbers so that

$$a_1 + a_2 + \dots + a_{2n+1} = 0, \quad a_1^2 + a_2^2 + \dots + a_{2n+1}^2 = n(n+1)(2n+1),$$

then

$$a_1^4 + a_2^4 + \dots + a_{2n+1}^4 \geq n(n+1)(2n+1)(n^2 + n + 1),$$

with equality when n of $a_1, a_2, \dots, a_{2n+1}$ are equal to $-n - 1$ and the other $n + 1$ are equal to n , and also when n of $a_1, a_2, \dots, a_{2n+1}$ are equal to $n + 1$ and the other $n + 1$ are equal to $-n$.

Putting

$$a_{n+1} = \cdots = a_{2n-1} = -n - 1, \quad a_{2n} = a_{2n+1} = n,$$

it follows that

$$a_1 + a_2 + \cdots + a_n + (n-1)(-n-1) + 2n = 0$$

and

$$a_1^2 + a_2^2 + \cdots + a_n^2 + (n-1)(-n-1)^2 + 2n^2 = n(n+1)(2n+1)$$

involve

$$a_1^4 + a_2^4 + \cdots + a_n^4 + (n-1)(-n-1)^4 + 2n^4 \leq n(n+1)(2n+1)(n^2+n+1),$$

which is equivalent to the desired statement. The equality holds for

$$a_1 = -n - 1, \quad a_2 = \cdots = a_n = n$$

(or any cyclic permutation).

□

P 6.18. If $a_1, a_2, \dots, a_n$ are real numbers so that

$$a_1 + a_2 + \cdots + a_n = n^2 - 3n - 2, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n^3 + 2n^2 - 3n - 2,$$

then

$$a_1^4 + a_2^4 + \cdots + a_n^4 \geq 2n^4 + (n-2)(n+1)^4.$$

(Vasile Cîrtoaje, 2010)

Solution. As shown in the proof of P 6.16, the following statement holds:

- If $a_1, a_2, \dots, a_{2n+1}$ are real numbers so that

$$a_1 + a_2 + \cdots + a_{2n+1} = 0, \quad a_1^2 + a_2^2 + \cdots + a_{2n+1}^2 = n(n+1)(2n+1),$$

then

$$a_1^4 + a_2^4 + \cdots + a_{2n+1}^4 \geq n(n+1)(2n+1)(n^2+n+1),$$

with equality when n of $a_1, a_2, \dots, a_{2n+1}$ are equal to $-n-1$ and the other $n+1$ are equal to n , and also when n of $a_1, a_2, \dots, a_{2n+1}$ are equal to $n+1$ and the other $n+1$ are equal to $-n$.

Putting

$$a_{n+1} = \cdots = a_{2n-1} = -n, \quad a_{2n} = a_{2n+1} = n+1,$$

it follows that

$$a_1 + a_2 + \cdots + a_n + (n-1)(-n) + 2(n+1) = 0$$

and

$$a_1^2 + a_2^2 + \cdots + a_n^2 + (n-1)(-n)^2 + 2(n+1)^2 = n(n+1)(2n+1)$$

involve

$$a_1^4 + a_2^4 + \cdots + a_n^4 + (n-1)(-n)^4 + 2(n+1)^4 \leq n(n+1)(2n+1)(n^2+n+1),$$

which is equivalent to the desired statement. The equality holds for

$$a_1 = a_2 = -n, \quad a_3 = \cdots = a_n = n + 1$$

(or any permutation).

□

P 6.19. If a, b, c, d are real numbers so that $a + b + c + d = 4$, then

$$(a^2 + b^2 + c^2 + d^2 - 4)(a^2 + b^2 + c^2 + d^2 + 36) \leq 12(a^4 + b^4 + c^4 + d^4 - 4).$$

(Vasile Cîrtoaje, 2010)

Solution. By Theorem 1, for $a + b + c + d = 4$ and $a^2 + b^2 + c^2 + d^2 = \text{fixed}$, the sum $a^4 + b^4 + c^4 + d^4$ is maximum when the set (a, b, c, d) has at most two distinct values. Therefore, it suffices to consider the following two cases.

Case 1: $a = b$ and $c = d$. We need to show that $a + c = 2$ involves

$$(a^2 + c^2 - 2)(a^2 + c^2 + 18) \leq 6(a^4 + c^4 - 2).$$

Since

$$\begin{aligned} a^2 + c^2 - 2 &= (a + c)^2 - 2ac - 2 = 2(1 - ac), & a^2 + c^2 + 18 &= 2(11 - ac), \\ a^4 + c^4 - 2 &= (a^2 + c^2)^2 - 2a^2c^2 - 2 = 2(1 - ac)(7 - ac), \end{aligned}$$

the inequality becomes

$$\begin{aligned} (1 - ac)(11 - ac) &\leq 3(1 - ac)(7 - ac), \\ (1 - ac)(5 - ac) &\geq 0. \end{aligned}$$

It is true because

$$ac \leq \frac{1}{4}(a + c)^2 = 1.$$

Case 2: $b = c = d$. We need to show that $a + 3b = 4$ involves

$$(a^2 + 3b^2 - 4)(a^2 + 3b^2 + 36) \leq 12(a^4 + 3b^4 - 4).$$

Since

$$\begin{aligned} a^2 + 3b^2 - 4 &= 12(b - 1)^2, & a^2 + 3b^2 + 36 &= 4(3b^2 - 6b + 13), \\ a^4 + 3b^4 - 4 &= (4 - 3b)^4 + 3b^4 - 4 = 12(b - 1)^2(7b^2 - 22b + 21), \end{aligned}$$

the inequality becomes

$$\begin{aligned} (b - 1)^2[(3b^2 - 6b + 13)] &\leq 3(b - 1)^2(7b^2 - 22b + 21), \\ (b - 1)^2(3b - 5)^2 &\geq 0. \end{aligned}$$

The equality holds for $a = b = c = d = 1$, and also for

$$a = -1, \quad b = c = d = \frac{5}{3}$$

(or any cyclic permutation).

□

P 6.20. If $a_1, a_2, \dots, a_n$ are real numbers so that

$$a_1 + a_2 + \dots + a_n = 0, \quad a_1^2 + a_2^2 + \dots + a_n^2 = n(n-1),$$

then

$$a_1^6 + a_2^6 + \dots + a_n^6 \leq (n-1)^6 + n - 1.$$

(Vasile Cîrtoaje, 2010)

Solution. For $n = 2$, we need to show that

$$a_1 + a_2 = 0, \quad a_1^2 + a_2^2 = 2,$$

implies

$$a_1^6 + a_2^6 \leq 2.$$

We have

$$a_1 = -1, \quad a_2 = 1,$$

or

$$a_1 = 1, \quad a_2 = -1.$$

For each of these cases, the desired inequality is an equality. According to Theorem 1, the sum

$$S_n = a_1^6 + a_2^6 + \dots + a_n^6$$

is maximum for

$$a_1 = \dots = a_j, \quad a_{j+1} = \dots = a_n,$$

where $j \in \{1, 2, \dots, n-1\}$. Thus, we only need to show that

$$ja_1 + (n-j)a_n = 0, \quad ja_1^2 + (n-j)a_n^2 = n(n-1)$$

involve

$$ja_1^6 + (n-j)a_n^6 \leq (n-1)^6 + n - 1.$$

From the equations above, we get

$$a_1^2 = \frac{(n-j)(n-1)}{j}, \quad a_n^2 = \frac{j(n-1)}{n-j}.$$

Thus, the desired inequality becomes

$$\begin{aligned} \frac{(n-j)^5 + j^5}{j^2(n-j)^2} &\leq \frac{(n-1)^5 + 1}{(n-1)^2}, \\ \frac{(n-j)^4 - (n-j)^3j + (n-j)^2j^2 - (n-j)j^3 + j^4}{j^2(n-j)^2} &\leq \\ &\leq \frac{(n-1)^4 - (n-1)^3 + (n-1)^2 - (n-1) + 1}{(n-1)^2}, \\ \frac{(n-j)^2}{j^2} - \frac{n-j}{j} - \frac{j}{n-j} + \frac{j^2}{(n-j)^2} &\leq (n-1)^2 - (n-1) - \frac{1}{n-1} + \frac{1}{(n-1)^2}, \end{aligned}$$

which can be written as

$$f(a) \geq f(b),$$

where

$$f(x) = x^2 - x - \frac{1}{x} + \frac{1}{x^2},$$

$$a = n - 1, \quad b = \frac{n}{j} - 1.$$

Since $a \geq b$ and

$$ab - 1 = (n - 1) \left(\frac{n}{j} - 1 \right) - 1 = n \left(\frac{n - 1}{j} - 1 \right) \geq 0,$$

we have

$$\begin{aligned} f(a) - f(b) &= (a - b) \left(a + b - 1 + \frac{1}{ab} - \frac{a + b}{a^2 b^2} \right) \\ &= (a - b) \left(1 - \frac{1}{ab} \right) \left[(a + b) \left(1 + \frac{1}{ab} \right) - 1 \right] \geq 0. \end{aligned}$$

The equality holds for

$$a_1 = -n + 1, \quad a_2 = \cdots = a_n = 1,$$

and for

$$a_1 = n - 1, \quad a_2 = \cdots = a_n = -1$$

(or any cyclic permutation).

□

P 6.21. If $a_1, a_2, \dots, a_n$ are real numbers so that

$$a_1 + a_2 + \cdots + a_n = 1, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n^2 + n - 1,$$

then

$$a_1^6 + a_2^6 + \cdots + a_n^6 \leq n^6 + n - 1.$$

(Vasile Cîrtoaje, 2010)

Solution. The inequality follows from the preceding P 6.20 by replacing n with $n + 1$, and then making $a_{n+1} = -1$. The equality holds for

$$a_1 = n, \quad a_2 = \cdots = a_n = -1$$

(or any cyclic permutation).

□

P 6.22. If $a_1, a_2, \dots, a_n$ are real numbers so that

$$a_1 + a_2 + \cdots + a_n = 0, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n(n - 1),$$

then

$$a_1^8 + a_2^8 + \cdots + a_n^8 \leq (n - 1)^8 + n - 1.$$

(Vasile Cîrtoaje, 2010)

Solution. For $n = 2$, we need to show that

$$a_1 + a_2 = 0, \quad a_1^2 + a_2^2 = 2,$$

implies

$$a_1^8 + a_2^8 \leq 2.$$

We have

$$a_1 = -1, \quad a_2 = 1,$$

or

$$a_1 = 1, \quad a_2 = -1.$$

For each of these cases, the desired inequality is an equality. According to Theorem 1, the sum

$$S_n = a_1^8 + a_2^8 + \cdots + a_n^8$$

is maximum for

$$a_1 = \cdots = a_j, \quad a_{j+1} = \cdots = a_n,$$

where $j \in \{1, 2, \dots, n-1\}$. Thus, we only need to show that

$$ja_1 + (n-j)a_n = 0, \quad ja_1^2 + (n-j)a_n^2 = n(n-1)$$

involve

$$ja_1^8 + (n-j)a_n^8 \leq (n-1)^8 + n-1.$$

From the equations above, we get

$$a_1^2 = \frac{(n-j)(n-1)}{j}, \quad a_n^2 = \frac{j(n-1)}{n-j}.$$

Thus, the desired inequality becomes

$$\begin{aligned} \frac{(n-j)^7 + j^7}{j^3(n-j)^3} &\leq \frac{(n-1)^7 + 1}{(n-1)^4}, \\ \frac{(n-j)^3}{j^3} - \frac{(n-j)^2}{j^2} + \frac{n-j}{j} + \frac{j}{n-j} - \frac{j^2}{(n-j)^2} + \frac{j^3}{(n-j)^3} &\leq \\ &\leq (n-1)^3 - (n-1)^2 + (n-1) + \frac{1}{n-1} - \frac{1}{(n-1)^2} + \frac{1}{(n-1)^3}, \end{aligned}$$

$$f(a) \geq f(b),$$

where

$$a = n-1, \quad b = \frac{n}{j} - 1,$$

$$f(x) = x^3 - x^2 + x + \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3}, \quad x > 0.$$

Since

$$f(x) = (t-1)(t^2-2), \quad t = x + \frac{1}{x} \geq 2,$$

it suffices to show that

$$a + \frac{1}{a} \geq b + \frac{1}{b}.$$

We have $a \geq b$,

$$ab - 1 = (n-1) \left(\frac{n}{j} - 1 \right) - 1 = n \left(\frac{n-1}{j} - 1 \right) \geq 0,$$

therefore

$$a + \frac{1}{a} - b - \frac{1}{b} = (a-b) \left(1 - \frac{1}{ab} \right) \geq 0.$$

The equality holds for

$$a_1 = -n + 1, \quad a_2 = \cdots = a_n = 1$$

and for

$$a_1 = n - 1, \quad a_2 = \cdots = a_n = -1$$

(or any cyclic permutation).

□

P 6.23. If $a_1, a_2, \dots, a_n$ are real numbers so that

$$a_1 + a_2 + \cdots + a_n = 1, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n^2 + n - 1,$$

then

$$a_1^8 + a_2^8 + \cdots + a_n^8 \leq n^8 + n - 1.$$

(Vasile Cîrtoaje, 2010)

Solution. The inequality follows from the preceding P 6.22 by replacing n with $n + 1$, and making $a_{n+1} = -1$. The equality holds for

$$a_1 = n, \quad a_2 = \cdots = a_n = -1$$

(or any cyclic permutation).

□

P 6.24. Let $a_1, a_2, \dots, a_n$ ($n \geq 2$) be real numbers (not all equal), and let

$$A = \frac{a_1 + a_2 + \cdots + a_n}{n}, \quad B = \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}, \quad C = \frac{a_1^3 + a_2^3 + \cdots + a_n^3}{n}.$$

Then,

$$\frac{1}{4} \left(1 - \sqrt{1 + \frac{2n^2}{n-1}} \right) \leq \frac{B^2 - AC}{B^2 - A^4} \leq \frac{1}{4} \left(1 + \sqrt{1 + \frac{2n^2}{n-1}} \right).$$

(Vasile Cîrtoaje, 2010)

Solution. It is well-known that $B > A^2$, hence $B^2 > A^4$.

(a) For $n = 2$, the right inequality reduces to $(a_1^2 - a_2^2)^2 \geq 0$. Consider further that $n \geq 3$. Since the right inequality remains unchanged by replacing $a_1, a_2, \dots, a_n$ with $-a_1, -a_2, \dots, -a_n$, we may suppose that $A \geq 0$. Assuming that

$$A = \text{fixed}, \quad B = \text{fixed},$$

we only need to consider the case when C is minimum. Thus, according to Corollary 2, it suffices to prove the required inequality for $a_1 < a_2 = a_3 = \dots = a_n$. Setting

$$a_1 := a, \quad a_2 = a_3 = \dots = a_n := b, \quad a < b,$$

the inequality becomes

$$\frac{\left[\frac{a^2 + (n-1)b^2}{n}\right]^2 - \frac{a + (n-1)b}{n} \cdot \frac{a^3 + (n-1)b^3}{n}}{\left[\frac{a^2 + (n-1)b^2}{n}\right]^2 - \left[\frac{a + (n-1)b}{n}\right]^4} \leq \frac{1}{4} \left(1 + \sqrt{1 + \frac{2n^2}{n-1}}\right),$$

After dividing the numerator and denominator of the left fraction by $(a-b)^2$, the inequality reduces to

$$\begin{aligned} \frac{-4n^2ab}{(n+1)a^2 + 2(n-1)ab + (2n^2 - 3n + 1)b} &\leq 1 + \sqrt{1 + \frac{2n^2}{n-1}}, \\ \frac{-2ab}{(n+1)a^2 + 2(n-1)ab + (2n^2 - 3n + 1)b} &\leq \frac{1}{\sqrt{(n^2 - 1)(2n - 1)} - n + 1}, \\ \left(a + \sqrt{\frac{2n^2 - 3n + 1}{n+1}} b\right)^2 &\geq 0. \end{aligned}$$

The equality holds for

$$-\sqrt{\frac{n+1}{(n-1)(2n-1)}} a_1 = a_2 = \dots = a_n$$

(or any cyclic permutation).

(b) For $n = 2$, the left inequality reduces to $(a_1 - a_2)^4 \geq 0$. For $n \geq 3$, the proof is similar to the one of the right inequality. The equality holds for

$$\sqrt{\frac{n+1}{(n-1)(2n-1)}} a_1 = a_2 = \dots = a_n$$

(or any cyclic permutation).

□

P 6.25. If a, b, c, d are real numbers so that

$$a + b + c + d = 2,$$

then

$$a^4 + b^4 + c^4 + d^4 \leq 40 + \frac{3}{4}(a^2 + b^2 + c^2 + d^2)^2.$$

(Vasile Cîrtoaje, 2010)

Solution. Write the inequality in the homogeneous form

$$10(a + b + c + d)^4 + 3(a^2 + b^2 + c^2 + d^2)^2 \geq 4(a^4 + b^4 + c^4 + d^4).$$

By Theorem 1, for $a + b + c + d = \text{fixed}$ and $a^2 + b^2 + c^2 + d^2 = \text{fixed}$, the sum $a^4 + b^4 + c^4 + d^4$ is maximum when the set (a, b, c, d) has at most two distinct values. Therefore, it suffices to consider the following two cases.

Case 1: $a = b$ and $c = d$. The inequality reduces to

$$41(a^2 + c^2)^2 + 160ac(a^2 + c^2) + 164a^2c^2 \geq 0,$$

which can be written in the obvious form

$$(a^2 + c^2)^2 + 40(a^2 + c^2 + 2ac)^2 + 4a^2c^2 \geq 0.$$

Case 2: $b = c = d$. The inequality reduces to the obvious form

$$(a + 5b)^2(3a^2 + 10ab + 11b^2) \geq 0.$$

Since the homogeneous inequality becomes an equality for

$$\frac{-a}{5} = b = c = d$$

(or any cyclic permutation), the original inequality is an equality for

$$a = 5, \quad b = c = d = -1$$

(or any cyclic permutation).

□

P 6.26. If a, b, c, d, e are real numbers, then

$$a^4 + b^4 + c^4 + d^4 + e^4 \leq \frac{31 + 18\sqrt{3}}{8}(a + b + c + d + e)^4 + \frac{3}{4}(a^2 + b^2 + c^2 + d^2 + e^2)^2.$$

(Vasile Cîrtoaje, 2010)

Solution. We proceed as in the proof of the preceding P 6.25. Taking into account Theorem 1, it suffices to consider the cases $b = c = d = e$, and $a = b$ and $c = d = e$.

Case 1: $b = c = d = e$. Due to homogeneity, we may consider $b = c = d = e = 0$ and $b = c = d = e = 1$. The first case is trivial. In the second case, the inequality becomes

$$a^4 + 4 \leq \frac{31 + 18\sqrt{3}}{8}(a + 4)^4 + \frac{3}{4}(a^2 + 4)^2,$$

$$\left(a + 2 + 2\sqrt{3}\right)^2 \left[f(a) + 2\sqrt{3} g(a)\right] \geq 0,$$

where

$$f(a) = 29a^2 + 164a + 272, \quad g(a) = 9a^2 + 50a + 76.$$

It suffices to show that $f(a) \geq 0$ and $g(a) \geq 0$. Indeed, we have

$$f(a) > 25a^2 + 164a + 269 = \left(5a + \frac{82}{5}\right)^2 + \frac{1}{25} > 0,$$

$$g(a) > 9a^2 + 50a + 70 = \left(3a + \frac{25}{3}\right)^2 + \frac{5}{9} > 0.$$

Case 2: $a = b$ and $c = d = e$. It suffices to show that

$$a^4 + b^4 + c^4 + d^4 + e^4 \leq \frac{3}{4}(a^2 + b^2 + c^2 + d^2 + e^2)^2,$$

which reduces to

$$2a^4 + 3c^4 \leq \frac{3}{4}(2a^2 + 3c^2)^2,$$

$$3(2a^2 + 3c^2)^2 \geq 4(2a^4 + 3c^4),$$

$$4a^4 + 36a^2c^2 + 15c^4 \geq 0.$$

The equality holds for

$$\frac{-a}{2(1 + \sqrt{3})} = b = c = d = e$$

(or any cyclic permutation).

□

P 6.27. Let $a, b, c, d, e \neq \frac{-5}{4}$ be real numbers so that $a + b + c + d + e = 5$. Then,

$$\frac{a(a-1)}{(4a+5)^2} + \frac{b(b-1)}{(4b+5)^2} + \frac{c(c-1)}{(4c+5)^2} + \frac{d(d-1)}{(4d+5)^2} + \frac{e(e-1)}{(4e+5)^2} \geq 0.$$

(Vasile Cîrtoaje, 2010)

Solution. Write the inequality as

$$\sum \left[\frac{180a(a-1)}{(4a+5)^2} + 1 \right] \geq 5,$$

$$\sum \frac{(14a-5)^2}{(4a+5)^2} \geq 5.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{(14a-5)^2}{(4a+5)^2} \geq \frac{[\sum (4a+5)(14a-5)]^2}{\sum (4a+5)^4}.$$

Therefore, it suffices to show that

$$\left(56 \sum a^2 + 125 \right)^2 \geq 5 \sum (4a+5)^4.$$

Using the substitution

$$a_1 = \frac{4a+5}{9}, a_2 = \frac{4b+5}{9}, \dots, a_5 = \frac{4e+5}{9},$$

we need to prove that $a_1 + a_2 + a_3 + a_4 + a_5 = 5$ involves

$$\left(7 \sum_{i=1}^5 a_i^2 - 25 \right)^2 \geq 20 \sum_{i=1}^5 a_i^4.$$

Rewrite this inequality in the homogeneous form

$$\left[7 \sum_{i=1}^5 a_i^2 - \left(\sum_{i=1}^5 a_i \right)^2 \right]^2 \geq 20 \sum_{i=1}^5 a_i^4.$$

By Theorem 1, for $a_1 + a_2 + a_3 + a_4 + a_5 = 5$ and $a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 = \text{fixed}$, the sum $a_1^4 + a_2^4 + a_3^4 + a_4^4 + a_5^4$ is maximum when the set $(a_1, a_2, a_3, a_4, a_5)$ has at most two distinct values. Therefore, we need to consider the following two cases.

Case 1: $a_1 = x$ and $a_2 = a_3 = a_4 = a_5 = y$. The homogeneous inequality reduces to

$$(3x^2 + 6y^2 - 4xy)^2 \geq 5(x^4 + 4y^4),$$

which is equivalent to the obvious inequality

$$(x-y)^2(x-2y)^2 \geq 0.$$

Case 2: $a_1 = a_2 = x$ and $a_3 = a_4 = a_5 = y$. The homogeneous inequality becomes

$$(5x^2 + 6y^2 - 6xy)^2 \geq 5(2x^4 + 3y^4),$$

which is equivalent to the obvious inequality

$$(x-y)^2[5(x-y)^2 + 2y^2] \geq 0.$$

The equality holds for $a = b = c = d = e = 1$, and also for

$$a = \frac{5}{2}, \quad b = c = d = e = \frac{5}{8}$$

(or any cyclic permutation).

Remark. Similarly, we can prove the following generalization.

- Let $x_1, x_2, \dots, x_n \neq -k$ be real numbers so that $x_1 + x_2 + \dots + x_n = n$, where

$$k \geq \frac{n}{2\sqrt{n-1}}.$$

Then,

$$\frac{x_1(x_1 - 1)}{(x_1 + k)^2} + \frac{x_2(x_2 - 1)}{(x_2 + k)^2} + \dots + \frac{x_n(x_n - 1)}{(x_n + k)^2} \geq 0,$$

with equality for $x_1 = x_2 = \dots = x_n = 1$. If $k = \frac{n}{2\sqrt{n-1}}$, then the equality holds also for

$$x_1 = \frac{n}{2}, \quad x_2 = \dots = x_n = \frac{n}{2(n-1)}$$

(or any cyclic permutation).

□

P 6.28. If a, b, c are real numbers so that

$$a + b + c = 9, \quad ab + bc + ca = 15,$$

then

$$\frac{19}{175} \leq \frac{1}{b^2 + bc + c^2} + \frac{1}{c^2 + ca + a^2} + \frac{1}{a^2 + ab + b^2} \leq \frac{7}{19}.$$

(Vasile C., 2011)

Solution. From

$$(b + c)^2 \geq 4bc$$

and

$$b + c = 9 - a, \quad bc = 15 - a(b + c) = 15 - a(9 - a) = a^2 - 9a + 15,$$

we get $a \leq 7$. Since

$$b^2 + bc + c^2 = (a + b + c)(b + c) - (ab + bc + ca) = 9(9 - a) - 15 = 3(22 - 3a),$$

we may write the inequality in the form

$$\frac{57}{175} \leq f(a) + f(b) + f(c) \leq \frac{21}{19}.$$

where

$$f(u) = \frac{1}{22 - 3u}, \quad u \leq 7.$$

We have

$$g(x) = f'(x) = \frac{3}{(22-3x)^2},$$

$$g''(x) = \frac{162}{(22-3x)^4}.$$

Since $g''(x) > 0$ for $x \leq 7$, g is strictly convex on $(-\infty, 7]$. According to Corollary 1, if $a \leq b \leq c$ and

$$a + b + c = 9, \quad a^2 + b^2 + c^2 = 51,$$

then the sum $S_3 = f(a) + f(b) + f(c)$ is maximum for $a = b \leq c$, and is minimum for $a \leq b = c$.

(a) To prove the right inequality, it suffices to consider the case $a = b \leq c$. From

$$a + b + c = 9, \quad ab + bc + ca = 15,$$

we get $a = b = 1$ and $c = 7$, therefore

$$\frac{1}{b^2 + bc + c^2} + \frac{1}{c^2 + ca + a^2} + \frac{1}{a^2 + ab + b^2} = \frac{7}{19}.$$

The original right inequality is an equality for $a = b = 1$ and $c = 7$ (or any cyclic permutation).

(b) To prove the left inequality, it suffices to consider the case $a \leq b = c$, which involves $a = -1$ and $b = c = 5$, hence

$$\frac{1}{b^2 + bc + c^2} + \frac{1}{c^2 + ca + a^2} + \frac{1}{a^2 + ab + b^2} = \frac{19}{175}.$$

The original left inequality is an equality for $a = -1$ and $b = c = 5$ (or any cyclic permutation). $\square$

P 6.29. If a, b, c are real numbers so that

$$8(a^2 + b^2 + c^2) = 9(ab + bc + ca),$$

then

$$\frac{419}{175} \leq \frac{a^2}{b^2 + bc + c^2} + \frac{b^2}{c^2 + ca + a^2} + \frac{c^2}{a^2 + ab + b^2} \leq \frac{311}{19}.$$

(Vasile C., 2011)

Solution. Due to homogeneity, we may assume that

$$a + b + c = 9, \quad a^2 + b^2 + c^2 = 51.$$

Next, the proof is similar to the one of the preceding P 6.28. Write the inequality in the form

$$\frac{1257}{175} \leq f(a) + f(b) + f(c) \leq \frac{933}{19}.$$

where

$$f(u) = \frac{u^2}{22 - 3u}, \quad u \leq 7.$$

We have

$$g(x) = f'(x) = \frac{-3x^2 + 44x}{(22 - 3x)^2}, \quad g''(x) = \frac{8712}{(22 - 3x)^4}.$$

Since g is strictly convex on $(-\infty, 7]$, according to Corollary 1, the sum $S_3 = f(a) + f(b) + f(c)$ is maximum for $a = b \leq c$, and is minimum for $a \leq b = c$.

(a) To prove the right inequality, it suffices to consider the case $a = b \leq c$, which involves

$$a = b = 1, \quad c = 7,$$

and

$$\frac{a^2}{b^2 + bc + c^2} + \frac{b^2}{c^2 + ca + a^2} + \frac{c^2}{a^2 + ab + b^2} = \frac{311}{19}.$$

The original right inequality is an equality for $a = b = c/7$ (or any cyclic permutation).

(b) To prove the left inequality, it suffices to consider the case $a \leq b = c$, which involves $a = -1$ and $b = c = 5$, hence

$$\frac{a^2}{b^2 + bc + c^2} + \frac{b^2}{c^2 + ca + a^2} + \frac{c^2}{a^2 + ab + b^2} = \frac{419}{175}.$$

The original left inequality is an equality for $-5a = b = c$ (or any cyclic permutation).

□

P 6.30. Let $a_1, a_2, \dots, a_n$ be real numbers such that $a_1 + a_2 + \dots + a_n = n$. If $n \leq 10$, then

$$2(a_1^2 + a_2^2 + \dots + a_n^2)^2 - n(a_1^3 + a_2^3 + \dots + a_n^3) \geq n^2.$$

(Vasile Cîrtoaje, 2020)

Solution. Write the inequality in the homogeneous form

$$2n^2(a_1^2 + a_2^2 + \dots + a_n^2)^2 - n^2(a_1 + a_2 + \dots + a_n)(a_1^3 + a_2^3 + \dots + a_n^3) \geq (a_1 + a_2 + \dots + a_n)^4.$$

According to Corollary 2, for $a_1 + a_2 + \dots + a_n = \text{fixed} > 0$ and $a_1^2 + a_2^2 + \dots + a_n^2 = \text{fixed}$, the sum

$$S = a_1^3 + a_2^3 + \dots + a_n^3$$

is maximum when $n - 1$ of $a_1, a_2, \dots, a_n$ are equal. Therefore, it suffices to consider the case $a_2 = a_3 = \dots = a_n$. Due to homogeneity, for the nontrivial case $a_2 = a_3 = \dots = a_n \neq 0$, we may consider that $a_2 = a_3 = \dots = a_n = 1$. Thus we only need to prove that

$$2n^2(a_1^2 + n - 1)^2 - n^2(a_1 + n - 1)(a_1^3 + n - 1) \geq (a_1 + n - 1)^4,$$

which is equivalent to

$$(a_1 - 1)^2(Aa_1^2 - Ba_1 + C) \geq 0,$$

where

$$A = n(n + 1), \quad B = n(n^2 - 2n + 2), \quad C = n(n - 1)(2n - 1).$$

The inequality is true because

$$4AC - B^2 = n^4(-n^2 + 12n - 12) \geq 0.$$

The equality occurs for $a_1 = a_2 = \dots = a_n = 1$.

□

Appendix A

Glosar

1. AM-GM (ARITHMETIC MEAN-GEOMETRIC MEAN) INEQUALITY

If $a_1, a_2, \dots, a_n$ are nonnegative real numbers, then

$$a_1 + a_2 + \dots + a_n \geq n \sqrt[n]{a_1 a_2 \dots a_n},$$

with equality if and only if $a_1 = a_2 = \dots = a_n$.

2. WEIGHTED AM-GM INEQUALITY

Let $p_1, p_2, \dots, p_n$ be positive real numbers satisfying

$$p_1 + p_2 + \dots + p_n = 1.$$

If $a_1, a_2, \dots, a_n$ are nonnegative real numbers, then

$$p_1 a_1 + p_2 a_2 + \dots + p_n a_n \geq a_1^{p_1} a_2^{p_2} \dots a_n^{p_n},$$

with equality if and only if $a_1 = a_2 = \dots = a_n$.

3. AM-HM (ARITHMETIC MEAN-HARMONIC MEAN) INEQUALITY

If $a_1, a_2, \dots, a_n$ are positive real numbers, then

$$(a_1 + a_2 + \dots + a_n) \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right) \geq n^2,$$

with equality if and only if $a_1 = a_2 = \dots = a_n$.

4. POWER MEAN INEQUALITY

The power mean of order k of positive real numbers $a_1, a_2, \dots, a_n$,

$$M_k = \begin{cases} \left(\frac{a_1^k + a_2^k + \dots + a_n^k}{n} \right)^{\frac{1}{k}}, & k \neq 0 \\ \sqrt[n]{a_1 a_2 \cdots a_n}, & k = 0 \end{cases},$$

is an increasing function with respect to $k \in \mathbb{R}$. For instant, $M_2 \geq M_1 \geq M_0 \geq M_{-1}$ is equivalent to

$$\sqrt{\frac{a_1^2 + a_2^2 + \dots + a_n^2}{n}} \geq \frac{a_1 + a_2 + \dots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}}.$$

5. BERNOULLI'S INEQUALITY

For any real number $x \geq -1$, we have

- a) $(1+x)^r \geq 1+rx$ for $r \geq 1$ and $r \leq 0$;
- b) $(1+x)^r \leq 1+rx$ for $0 \leq r \leq 1$.

If $a_1, a_2, \dots, a_n$ are real numbers such that either $a_1, a_2, \dots, a_n \geq 0$ or

$$-1 \leq a_1, a_2, \dots, a_n \leq 0,$$

then

$$(1+a_1)(1+a_2) \cdots (1+a_n) \geq 1+a_1+a_2+\dots+a_n.$$

6. SCHUR'S INEQUALITY

For any nonnegative real numbers a, b, c and any positive number k , the inequality holds

$$a^k(a-b)(a-c) + b^k(b-c)(b-a) + c^k(c-a)(c-b) \geq 0,$$

with equality for $a = b = c$, and for $a = 0$ and $b = c$ (or any cyclic permutation).

For $k = 1$, we get the third degree Schur's inequality, which can be rewritten as follows

$$a^3 + b^3 + c^3 + 3abc \geq ab(a+b) + bc(b+c) + ca(c+a),$$

$$(a+b+c)^3 + 9abc \geq 4(a+b+c)(ab+bc+ca),$$

$$a^2 + b^2 + c^2 + \frac{9abc}{a+b+c} \geq 2(ab+bc+ca),$$

$$(b-c)^2(b+c-a) + (c-a)^2(c+a-b) + (a-b)^2(a+b-c) \geq 0.$$

For $k = 2$, we get the fourth degree Schur's inequality, which holds for any real numbers a, b, c , and can be rewritten as follows

$$a^4 + b^4 + c^4 + abc(a+b+c) \geq ab(a^2+b^2) + bc(b^2+c^2) + ca(c^2+a^2),$$

$$\begin{aligned}
a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2 &\geq (ab + bc + ca)(a^2 + b^2 + c^2 - ab - bc - ca), \\
(b - c)^2(b + c - a)^2 + (c - a)^2(c + a - b)^2 + (a - b)^2(a + b - c)^2 &\geq 0, \\
6abcp &\geq (p^2 - q)(4q - p^2), \quad p = a + b + c, \quad q = ab + bc + ca.
\end{aligned}$$

A generalization of the fourth degree Schur's inequality, which holds for any real numbers a, b, c and any real number m , is the following (*Vasile Cîrtoaje*, 2004)

$$\sum (a - mb)(a - mc)(a - b)(a - c) \geq 0,$$

with equality for $a = b = c$, and also for $a/m = b = c$ (or any cyclic permutation). This inequality is equivalent to

$$\begin{aligned}
\sum a^4 + m(m + 2) \sum a^2b^2 + (1 - m^2)abc \sum a &\geq (m + 1) \sum ab(a^2 + b^2), \\
\sum (b - c)^2(b + c - a - ma)^2 &\geq 0.
\end{aligned}$$

7. CAUCHY-SCHWARZ INEQUALITY

If $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ are real numbers, then

$$(a_1^2 + a_2^2 + \dots + a_n^2)(b_1^2 + b_2^2 + \dots + b_n^2) \geq (a_1b_1 + a_2b_2 + \dots + a_nb_n)^2,$$

with equality for

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \dots = \frac{a_n}{b_n}.$$

Notice that the equality conditions are also valid for $a_i = b_i = 0$, where $1 \leq i \leq n$.

8. HÖLDER'S INEQUALITY

If x_{ij} ($i = 1, 2, \dots, m; j = 1, 2, \dots, n$) are nonnegative real numbers, then

$$\prod_{i=1}^m \left(\sum_{j=1}^n x_{ij} \right) \geq \left(\sum_{j=1}^n \sqrt[m]{\prod_{i=1}^m x_{ij}} \right)^m.$$

9. CHEBYSHEV'S INEQUALITY

Let $a_1 \geq a_2 \geq \dots \geq a_n$ be real numbers.

a) If $b_1 \geq b_2 \geq \dots \geq b_n$, then

$$n \sum_{i=1}^n a_i b_i \geq \left(\sum_{i=1}^n a_i \right) \left(\sum_{i=1}^n b_i \right);$$

b) If $b_1 \leq b_2 \leq \dots \leq b_n$, then

$$n \sum_{i=1}^n a_i b_i \leq \left(\sum_{i=1}^n a_i \right) \left(\sum_{i=1}^n b_i \right).$$

10. REARRANGEMENT INEQUALITY

(1) If $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ are two increasing (or decreasing) real sequences, and $(i_1, i_2, \dots, i_n)$ is an arbitrary permutation of $(1, 2, \dots, n)$, then

$$a_1 b_1 + a_2 b_2 + \dots + a_n b_n \geq a_1 b_{i_1} + a_2 b_{i_2} + \dots + a_n b_{i_n}.$$

(2) If $a_1, a_2, \dots, a_n$ is decreasing and $b_1, b_2, \dots, b_n$ is increasing, then

$$a_1 b_1 + a_2 b_2 + \dots + a_n b_n \leq a_1 b_{i_1} + a_2 b_{i_2} + \dots + a_n b_{i_n}.$$

(3) Let $b_1, b_2, \dots, b_n$ and $c_1, c_2, \dots, c_n$ be two real sequences such that

$$b_1 + \dots + b_k \geq c_1 + \dots + c_k, \quad k = 1, 2, \dots, n.$$

If $a_1 \geq a_2 \geq \dots \geq a_n \geq 0$, then

$$a_1 b_1 + a_2 b_2 + \dots + a_n b_n \geq a_1 c_1 + a_2 c_2 + \dots + a_n c_n.$$

11. SQUARE PRODUCT INEQUALITY

Let a, b, c be real numbers, and let

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc,$$

$$s = \sqrt{p^2 - 3q} = \sqrt{a^2 + b^2 + c^2 - ab - bc - ca}.$$

From the identity

$$(a - b)^2(b - c)^2(c - a)^2 = -27r^2 + 2(9pq - 2p^3)r + p^2q^2 - 4q^3,$$

it follows that

$$\frac{-2p^3 + 9pq - 2(p^2 - 3q)\sqrt{p^2 - 3q}}{27} \leq r \leq \frac{-2p^3 + 9pq + 2(p^2 - 3q)\sqrt{p^2 - 3q}}{27},$$

which is equivalent to

$$\frac{p^3 - 3ps^2 - 2s^3}{27} \leq r \leq \frac{p^3 - 3ps^2 + 2s^3}{27}.$$

Therefore, for constant p and q , the product r is minimum and maximum when two of a, b, c are equal.

12. KARAMATA'S MAJORIZATION INEQUALITY

Let f be a convex function on a real interval $\mathbb{I}$. If a decreasingly ordered sequence

$$A = (a_1, a_2, \dots, a_n), \quad a_i \in \mathbb{I},$$

majorizes a decreasingly ordered sequence

$$B = (b_1, b_2, \dots, b_n), \quad b_i \in \mathbb{I},$$

then

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq f(b_1) + f(b_2) + \dots + f(b_n).$$

We say that a sequence $A = (a_1, a_2, \dots, a_n)$ with $a_1 \geq a_2 \geq \dots \geq a_n$ majorizes a sequence $B = (b_1, b_2, \dots, b_n)$ with $b_1 \geq b_2 \geq \dots \geq b_n$, and write it as

$$A \succ B,$$

if

$$\begin{aligned} a_1 &\geq b_1, \\ a_1 + a_2 &\geq b_1 + b_2, \\ &\dots\dots\dots \\ a_1 + a_2 + \dots + a_{n-1} &\geq b_1 + b_2 + \dots + b_{n-1}, \\ a_1 + a_2 + \dots + a_n &= b_1 + b_2 + \dots + b_n. \end{aligned}$$

13. POPOVICIU'S INEQUALITY

Theorem. If f is a convex function on a real interval $\mathbb{I}$ and $a_1, a_2, \dots, a_n \in \mathbb{I}$, then

$$\begin{aligned} f(a_1) + f(a_2) + \dots + f(a_n) + n(n-2)f\left(\frac{a_1 + a_2 + \dots + a_n}{n}\right) &\geq \\ &\geq (n-1)[f(b_1) + f(b_2) + \dots + f(b_n)], \end{aligned}$$

where

$$b_i = \frac{1}{n-1} \sum_{j \neq i} a_j, \quad i = 1, 2, \dots, n.$$

In the same conditions, the following similar inequality holds:

$$f(a_1) + f(a_2) + \dots + f(a_n) + \frac{n}{n-2}f\left(\frac{a_1 + a_2 + \dots + a_n}{n}\right) \geq \frac{2}{n-2} \sum_{1 \leq i < j \leq n} f\left(\frac{a_i + a_j}{2}\right).$$

14. CONVEX FUNCTIONS

A function f defined on a real interval $\mathbb{I}$ is said to be *convex* if

$$f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y)$$

for all $x, y \in \mathbb{I}$ and any $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. If the inequality is reversed, then f is said to be concave.

If f is differentiable on $\mathbb{I}$, then f is (strictly) convex if and only if the derivative f' is (strictly) increasing. If $f'' \geq 0$ on $\mathbb{I}$, then f is convex on $\mathbb{I}$. Also, if $f'' \geq 0$ on (a, b) and f is continuous on $[a, b]$, then f is convex on $[a, b]$.

Jensen's inequality. Let $p_1, p_2, \dots, p_n$ be positive real numbers. If f is a convex function on a real interval $\mathbb{I}$, then for any $a_1, a_2, \dots, a_n \in \mathbb{I}$, the inequality holds

$$\frac{p_1 f(a_1) + p_2 f(a_2) + \dots + p_n f(a_n)}{p_1 + p_2 + \dots + p_n} \geq f\left(\frac{p_1 a_1 + p_2 a_2 + \dots + p_n a_n}{p_1 + p_2 + \dots + p_n}\right).$$

For $p_1 = p_2 = \dots = p_n$, Jensen's inequality becomes

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq n f\left(\frac{a_1 + a_2 + \dots + a_n}{n}\right).$$

Right Half Convex Function Theorem (Vasile Cîrtoaje, 2004). Let f be a real function defined on an interval $\mathbb{I}$ and convex on $\mathbb{I}_{\geq S}$, where $S \in \text{int}(\mathbb{I})$. The inequality

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq n f\left(\frac{a_1 + a_2 + \dots + a_n}{n}\right) \quad (1)$$

holds for all $a_1, a_2, \dots, a_n \in \mathbb{I}$ satisfying $\frac{a_1 + a_2 + \dots + a_n}{n} = S$ if and only if it holds for all $a_1, a_2, \dots, a_n \in \mathbb{I}$ such that

$$a_1 \leq a_2 = a_3 = \dots = a_n.$$

Left Half Convex Function Theorem (Vasile Cîrtoaje, 2004). Let f be a real function defined on an interval $\mathbb{I}$ and convex on $\mathbb{I}_{\leq S}$, where $S \in \text{int}(\mathbb{I})$. The inequality

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq n f\left(\frac{a_1 + a_2 + \dots + a_n}{n}\right) \quad (3)$$

holds for all $a_1, a_2, \dots, a_n \in \mathbb{I}$ satisfying $\frac{a_1 + a_2 + \dots + a_n}{n} = S$ if and only if it holds for all $a_1, a_2, \dots, a_n \in \mathbb{I}$ such that

$$a_1 = a_2 = \dots = a_{n-1} \leq a_n.$$

Left Convex-Right Concave Function Theorem (Vasile Cîrtoaje, 2004). Let $a < c$ be real numbers, let $f(x)$ be a continuous function defined on $\mathbb{I} = [a, \infty)$, strictly convex for $x \leq c$ and strictly concave for $x \geq c$, and let

$$E(a_1, a_2, \dots, a_n) = f(a_1) + f(a_2) + \dots + f(a_n).$$

If $a_1, a_2, \dots, a_n \in \mathbb{I}$ so that

$$a_1 + a_2 + \dots + a_n = S = \text{constant},$$

then

- (a) E is minimum for $a_1 = a_2 = \cdots = a_{n-1} \leq a_n$;
- (b) E is maximum for either $a_1 = a$ or $a < a_1 \leq a_2 = \cdots = a_n$.

Right Half Convex Function Theorem for Ordered Variables (Vasile Cîrtoaje, 2008). Let f be a real function defined on an interval $\mathbb{I}$ and convex on $\mathbb{I}_{\geq s}$, where $s \in \text{int}(\mathbb{I})$. The inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf \left(\frac{a_1 + a_2 + \cdots + a_n}{n} \right)$$

holds for all $a_1, a_2, \dots, a_n \in \mathbb{I}$ satisfying

$$a_1 + a_2 + \cdots + a_n = ns$$

and

$$a_1 \leq a_2 \leq \cdots \leq a_m \leq s, \quad m \in \{1, 2, \dots, n-1\},$$

if and only if

$$f(x) + (n-m)f(y) \geq (1+n-m)f(s)$$

for all $x, y \in \mathbb{I}$ such that

$$x \leq s \leq y, \quad x + (n-m)y = (1+n-m)s.$$

Left Half Convex Function Theorem for Ordered Variables (Vasile Cîrtoaje, 2008). Let f be a real function defined on an interval $\mathbb{I}$ and convex on $\mathbb{I}_{\leq s}$, where $s \in \text{int}(\mathbb{I})$. The inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf \left(\frac{a_1 + a_2 + \cdots + a_n}{n} \right)$$

holds for all $a_1, a_2, \dots, a_n \in \mathbb{I}$ satisfying

$$a_1 + a_2 + \cdots + a_n = ns$$

and

$$a_1 \geq a_2 \geq \cdots \geq a_m \geq s, \quad m \in \{1, 2, \dots, n-1\},$$

if and only if

$$f(x) + (n-m)f(y) \geq (1+n-m)f(s)$$

for all $x, y \in \mathbb{I}$ such tht

$$x \geq s \geq y, \quad x + (n-m)y = (1+n-m)s.$$

Right Partially Convex Function Theorem (Vasile Cîrtoaje, 2012). Let f be a real function defined on an interval $\mathbb{I}$ and convex on $[s, s_0]$, where $s, s_0 \in \mathbb{I}$, $s < s_0$. In addition, f is decreasing on $\mathbb{I}_{\leq s_0}$ and $f(u) \geq f(s_0)$ for $u \in \mathbb{I}$. The inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf \left(\frac{a_1 + a_2 + \cdots + a_n}{n} \right)$$

holds for all $a_1, a_2, \dots, a_n \in \mathbb{I}$ satisfying

$$a_1 + a_2 + \cdots + a_n = ns$$

if and only if

$$f(x) + (n-1)f(y) \geq nf(s)$$

for all $x, y \in \mathbb{I}$ such that $x \leq s \leq y$ and $x + (n-1)y = ns$.

Left Partially Convex Function Theorem (Vasile Cîrtoaje, 2012). Let f be a real function defined on an interval $\mathbb{I}$ and convex on $[s_0, s]$, where $s_0, s \in \mathbb{I}$, $s_0 < s$. In addition, f is increasing on $\mathbb{I}_{\geq s_0}$ and $f(u) \geq f(s_0)$ for $u \in \mathbb{I}$. The inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \dots, a_n \in \mathbb{I}$ satisfying

$$a_1 + a_2 + \cdots + a_n = ns$$

if and only if

$$f(x) + (n-1)f(y) \geq nf(s)$$

for all $x, y \in \mathbb{I}$ such that $x \geq s \geq y$ and $x + (n-1)y = ns$.

Right Partially Convex Function Theorem for Ordered Variables (Vasile Cîrtoaje, 2014). Let f be a real function defined on an interval $\mathbb{I}$ and convex on $[s, s_0]$, where $s, s_0 \in \mathbb{I}$, $s < s_0$. In addition, f is decreasing on $\mathbb{I}_{\leq s_0}$ and $f(u) \geq f(s_0)$ for $u \in \mathbb{I}$. The inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \dots, a_n \in \mathbb{I}$ satisfying

$$a_1 + a_2 + \cdots + a_n = ns$$

and

$$a_1 \leq a_2 \leq \cdots \leq a_m \leq s, \quad m \in \{1, 2, \dots, n-1\},$$

if and only if

$$f(x) + (n-m)f(y) \geq (1+n-m)f(s)$$

for all $x, y \in \mathbb{I}$ such that $x \leq s \leq y$ and $x + (n-m)y = (1+n-m)s$.

Left Partially Convex Function Theorem for Ordered Variables (Vasile Cîrtoaje, 2014). Let f be a real function defined on an interval $\mathbb{I}$ and convex on $[s_0, s]$, where $s_0, s \in \mathbb{I}$, $s_0 < s$. In addition, f is increasing on $\mathbb{I}_{\geq s_0}$ and $f(u) \geq f(s_0)$ for $u \in \mathbb{I}$. The inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \dots, a_n \in \mathbb{I}$ satisfying

$$a_1 + a_2 + \cdots + a_n = ns$$

and

$$a_1 \geq a_2 \geq \cdots \geq a_m \geq s, \quad m \in \{1, 2, \dots, n-1\},$$

if and only if

$$f(x) + (n - m)f(y) \geq (1 + n - m)f(s)$$

for all $x, y \in \mathbb{I}$ such that $x \geq s \geq y$ and $x + (n - m)y = (1 + n - m)s$.

Equal Variables Theorem for Nonnegative Variables (Vasile Cîrtoaje, 2005). Let $x_1, x_2, \dots, x_n$ ($n \geq 3$) be nonnegative real numbers such that $x_1 \leq x_2 \leq \dots \leq x_n$, and let f be a real-valued function, continuous on $[0, \infty)$ and differentiable on $(0, \infty)$, so that the joined function

$$g(x) = f' \left(x^{\frac{1}{k-1}} \right)$$

is strictly convex on $(0, \infty)$. For fixed $x_1 + x_2 + \dots + x_n$ and $x_1^k + x_2^k + \dots + x_n^k$, where $k \neq 1$ ($k = 0$ means that the product $x_1 x_2 \dots x_n$ is fixed), the sum

$$S_n = f(x_1) + f(x_2) + \dots + f(x_n)$$

has the maximum value only for

$$x_1 = x_2 = \dots = x_{n-1} \leq x_n,$$

and the minimum value only for $x_1 = 0$ or $0 < x_1 \leq x_2 = x_3 = \dots = x_n$.

Equal Variables Theorem for Real Variables (Vasile Cîrtoaje, 2010). Let $x_1, x_2, \dots, x_n$ ($n \geq 3$) be real numbers such that $x_1 \leq x_2 \leq \dots \leq x_n$, let k be an even positive integer, and let f be a differentiable function on $\mathbb{R}$ so that the joined function $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$g(x) = f' \left({}^k\sqrt{x} \right)$$

is strictly convex on $\mathbb{R}$. For fixed $x_1 + x_2 + \dots + x_n$ and $x_1^k + x_2^k + \dots + x_n^k$, the sum

$$S_n = f(x_1) + f(x_2) + \dots + f(x_n)$$

has the minimum value for $x_2 = x_3 = \dots = x_n$, and the maximum value for $x_1 = x_2 = \dots = x_{n-1}$.

Best Upper Bound of Jensen's Difference Theorem (Vasile Cîrtoaje, 1990). Let $p_1, p_2, \dots, p_n$ ($n \geq 3$) be fixed positive real numbers, and let f be a convex function on $\mathbb{I} = [a, b]$. If $a_1, a_2, \dots, a_n \in \mathbb{I}$, then Jensen's difference

$$\frac{p_1 f(a_1) + p_2 f(a_2) + \dots + p_n f(a_n)}{p_1 + p_2 + \dots + p_n} - f \left(\frac{p_1 a_1 + p_2 a_2 + \dots + p_n a_n}{p_1 + p_2 + \dots + p_n} \right)$$

is maximum when all $a_i \in \{a, b\}$.

Appendix B

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